Math 751 Week 6 Notes

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Definition 1.1. A map $p\colon E \to B$ is called a covering if

1. P is continuous and onto.
2. For all $b \in B$, there exists an open neighborhood U of b which is evenly covered, i.e., $p^{-1}(U) = \sqcup_\alpha V_\alpha$, where the v_α are disjoint and open, and $p|_{V_\alpha}\colon V_\alpha \to U$ is a homeomophism.

Example 1.2.

1. $p\colon \mathbb{R} \to S^1, t \mapsto e^{2\pi it}$
2. $id_X\colon X \to X$
3. $p\colon X \times \{1, \ldots, n\} \to X, (x, k) \mapsto x$
4. $p\colon S^1 \to S^1, z \mapsto z^n$
5. $p\colon S^n \to \mathbb{RP}^n, x \mapsto [x] = \{\pm x\}$
6. $p\colon \mathbb{C} \to \mathbb{C}^*, z \mapsto e^x$
7. Products of covering maps: If $p_i\colon E_i \to B_i, i = 1, 2$ are coverings, then $p_1 \times p_2\colon E_1 \times E_2 \to B_1 \times B_2$ is a covering.

Remark 1.3.

1. Being a covering implies the map is open and locally a homeomorphism.
2. Not any local homeomorphism is a covering.
3. $p^{-1}(b)$ is discrete, for all $b \in B$ (by disjointness.)

Definition 1.4. Let $p_1\colon E_1 \to B, p_2\colon E_2 \to B$ be two coverings. We say p_1 and p_2 are equivalent if there exists a homemorphism $f\colon E_1 \to E_2$ such that $p_2 \circ f = p_1$. Note: This is an equivalence relation .

Problem: Find all coverings of a space B (up to equivalence.)

Lemma 1.5. *If $p\colon E \to B$ is a covering, $B_0 \subset B$, and $E_0 := p^{-1}(B_0)$, then $p|_{E_0}\colon E_0 \to B_0$ is a covering.*

Example 1.6. Let $p\colon \mathbb{R}^2 \to T^2$ be a covering. Overlay the integer lattice on $\mathbb{R}^2$, and identity each square with a torus in the usual way. Let $p_0 = (1, 0) \in S^1$, and let $B_0 = S^1 \times \{p_0\} \cup \{p_0\} \times S^1$. Then $p^{-1}(B_0) = \mathbb{R} \times \mathbb{Z} \cup \zeta \times \mathbb{R}$ (so it gets rid of the inside of the squares.)

Theorem 1.7 (Path lifting property). *Let $P\colon E \to B$ be a covering, $b_0 \in B$, and $e_0 \in p^{-1}(b_0)$. If $\gamma\colon I \to B$ is a path in B starting at b_0, then there is a unique lift $\widetilde{\gamma}_{e_0}\colon I \to E$ such that $\widetilde{\gamma}_{e_0}(0) = e_0$.*

The proof of this theorem follows from the previous lemma.

Theorem 1.8 (Homotopy lifting property). *Let $F\colon I \times I \to B$ be a homotopy with $b_0 := F(0, 0)$. Then there is a unique lift $\widetilde{F}\colon I \times I \to E$ of F such that $\widetilde{F}(0, 0) = e_0$.*

Corollary 1.9. *If γ_1, γ_2 are paths in B which are homotopic by some F, then $\gamma_1(0) = \gamma_2(0) = b_0$, then $(\widetilde{\gamma_1})_{e_0} \sim^{\tilde{F}} (\widetilde{\gamma_2})_{e_0}$. In particular, these lifts have the same endpoints: $(\widetilde{\gamma_1})_{e_0}(1) = (\widetilde{\gamma_2})_{e_0}(1)$.*

Definition 1.10. Let $b_0 \in B$. For $e_0 \in p^{-1}(b_0)$, define

$$\begin{aligned} \Phi_{e_0}\colon \pi_1(B, b_0) &\to p^{-1}(b_0) \\ [\gamma] &\mapsto \widetilde{\gamma}_{e_0}(1) \end{aligned}$$

Note that by corollary 1.9, this map is well-defined.

Theorem 1.11. *Let Φ_{e_0} be defined as above. Then Φ_{e_0} is onto if E is path-connected, and it is injective if E is simply connected.*

Proof. First suppose that E is path-connected. Let $e_1 \in p^{-1}(b_0)$, and let δ be a path in E from e_0 to e_1. Then $p \circ \delta$ is a path in B. Further, $\gamma := p \circ \delta\colon I \to B$ is a loop in B with base point b_0. Then δ is a lift of γ starting at e_0. Then we have $\phi_{e_0}([\gamma]) = \widetilde{\gamma}_{e_0}(1) = \delta(1) = \delta_1$, so ϕ_{e_0} is surjective. Note that the equality $\widetilde{\gamma}_{e_0}(1) = \delta(1)$ comes from the uniqueness of lifts.

Now suppose E is simply connected. Let γ_1, γ_2 be loops in B with base point b_0 such that $\phi_{e_0}([\gamma_1]) = \phi_{e_0}([\gamma_2]) = e_1$. By definition, this means that $(\widetilde{\gamma_1})_{e_0}(1) = (\widetilde{\gamma_2})_{e_0}(1)$. To show that ϕ_{e_0} is injective, we must show $\gamma_1 \sim \gamma_2$. Since E is simply connected, there is a unique homotopy class of paths from e_0 to e_1, so $(\widetilde{\gamma_1})_{e_0} \sim (\widetilde{\gamma_2})_{e_0}$ by some homotopy F. This gives a homotopy $p \circ F\colon I \times I \to B$ from $p \circ (\widetilde{\gamma_1})_{e_0} = \gamma_1$ to $p \circ (\widetilde{\gamma_2})_{e_0} = \gamma_2$, which shows that ϕ_{e_0} is injective. □

Example 1.12. Let $p\colon S^n \to \mathbb{RP}^n$ be a covering. For $n \geq 2$, S^n is path-connected and simply connected. Then by theorem 1.11, $\Phi_{e_0}\colon \pi_1(\mathbb{RP}^n, b_0) \to (p^{-1}(b_0)$ is a bijection. Since $\#p^{-1}(b_0) = 2$, it must be that $\pi_1(\mathbb{RP}^n, b_0) \cong \mathbb{Z}/2\mathbb{Z}$.

Example 1.13. Let $p\colon \mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$. Since $\mathbb{R}$ is both simply connected and path-econnected, theorem 1.11 tell us that $\phi_{e_0}\colon \pi_1(S^1, b_0) \to \mathbb{Z}$ is a bijection. To show that the groups are isomorphic, we now need to show that ϕ_{e_0} is a homomorphism. Let $\gamma, \delta \in \pi_1(S^1, b_0)$, and let $\widetilde{\gamma}_0, \widetilde{\delta}_0$ be their lifts in $\mathbb{R}$. Let $\widetilde{\gamma}_0(1) = n \in \mathbb{Z}, \widetilde{\delta}_0(1) = m \in \mathbb{Z}$. By definition, $\phi_{e_0}([g]) = n$, $\phi_{e_0}([\delta]) = m$.

Claim 1.14. $\phi_{e_0}([g] \cdot [\delta]) = n + m$ (i.e., it is a homomorphism.)

Proof. We have

$$\phi_{e_0}([\delta] \cdot [\gamma]) = \phi_{e_0}([\delta * \gamma]) = \widetilde{(\gamma * \delta)}_0(1) = (\widetilde{\gamma}_0 * \widetilde{\delta}^*)(1) = \widetilde{\delta}^*(1) = n + m$$

Now set $\widetilde{\delta}^*(t) = n + \widetilde{\delta}_0(t)$ so that $\widetilde{\delta}^*(0) = n$, $\widetilde{\delta}^*(1) = n + m$. Thus, ϕ_{e_0} is a homomorphism and therefore an isomorphism. □

Proposition 1.15. If $p\colon E \to B$ is a covering and B is path-connected, and $b_0, b_1 \in B$, there there is a bijection $p^{-1}(b_0) \to p^{-1}(b_1)$.

Proof. Let γ be a path in B from b_0 to b_1 (which exists because B is path-connected.) Define the bijection $f_\gamma\colon p^{-1}(b_0) \to p^{-1}(b_1)$ by $e_0 \mapsto \widetilde{\gamma}_{e_0}(1)$. It has inverse $(f_\gamma)^{-1} = f_{\overline{\gamma}}$. □

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Proposition 2.1. Let E be path connected, $p\colon E \to B$ a covering, and $p(e_0) = b_0$. Then $p_*\colon \pi_1(E, e_0) \to \pi_1(B, b_0)$ is injective. Further, if e_0 is changed to some other $e_1 \in p^{-1}(b_0)$, then the images of p_* are conjugate in $\pi_1(B, b_0)$.

Proof. Let $p_*([\gamma_1]) = p_*([\gamma_2])$. Then $\mathcal{P} \circ \gamma_1 \sim p \circ \gamma_2$ by some homotopy F. By homotopy lifting, we have that $\widetilde{(p \circ \gamma_1)}_{e_0} \sim \widetilde{(p \circ \gamma_2)}_{e_0}$, which implies that $\gamma_1 \sim \gamma_2$, by the uniqueness of lifts. Thus, p_* is injective.

Now let e_1 be a different point in the fiber of p over b_0. Define $H_1 = P_*\pi_1(E, e_0), H_2 = p_*\pi_1(E, e_1)$. We want to show these are conjugate. First let δ be a path in E from $e_0 \to e_1$. Then the following diagram commutes:

$$\begin{array}{ccc} \pi_1(E, e_0) & \xrightarrow{p_*} & \pi_1(B, b_0) \\ \downarrow{\scriptstyle \delta_\#} & & \downarrow{\scriptstyle (p\circ\delta)_\#} \\ \pi_1(E, e_1) & \longrightarrow & \pi_1(B, b_0) \end{array}$$

Note that $\delta_\#$ is an isomorphism. So we have $H_1 \cong (p \circ \delta)_\# H_2$, by conjugation with $[p \circ \delta]$. □

Theorem 2.2. *Let E be path-connected, $p\colon E \to B$ a covering map, and $e_0 \in p^{-1}(b_0)$. Let $H := p_*\pi_1(E, e_0) \leq \pi_1(B, b_0)$. Then:*

a *A closed path γ in B based at b_0 lifts to a loop in E at e_0 iff $[\gamma] \in H$.*

b $\phi_{e_0}\colon H\backslash\pi_1(B, b_0) \to p^{-1}(b_0), [\gamma] \mapsto \widetilde{\gamma}_{e_0}(1)$ *is a bijection. In particular* $\#p^{-1}(b_0) = [\pi_1(B, b_0)\colon p_*\pi_1(E, e_0)]$.

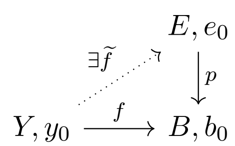
Proof of (b). First show that ϕ_{e_0} is well-defined, i.e., if $[\delta] \in H$, then $\phi_{e_0}([\delta[\cdot[\gamma]) = \phi_{e_0}([\gamma])$. We have

$$\phi_{e_0}([\gamma] \cdot [\delta]) = \phi_{e_0}([\delta * \gamma]) = (\widetilde{\delta * \gamma})_{e_0}(1) = (\widetilde{\delta}_{e_0} * \widetilde{\gamma}_{\tilde{\delta}_{e_0}(1)})(1) = \widetilde{\gamma}_{\tilde{\delta}_{e_0}(1)}$$

By part (a), since $[\delta] \in H$, we have that $\widetilde{\delta}_{e_0}(1) = e_0$. Thus, $\widetilde{\gamma}_{\tilde{\delta}_{e_0}(1)} = \phi_{e_0}([\gamma])$, so it's well defined. From last class we know that ϕ_{e_0} is onto, so it remains to show that it's injective.

Suppose that $\phi_{e_0}([\gamma_1]) = \phi_{e_0}([\gamma_2])$. By definition, this means that $(\widetilde{\gamma_1})_{e_0}(1) = (\widetilde{\gamma_2})_{e_0}(1)$. Thus, $(\widetilde{\gamma_1})_{e_0} * (\widetilde{\gamma_2})_{e_0}$ is a loop in E based at e_0, which is in turn a lift of $\gamma_1 * \overline{\gamma_2}$. By (a), $[\gamma_1 * \overline{\gamma_2}] \in H$. Finally, $[\gamma_1] = [\gamma_1 * \overline{\gamma_2} * \gamma_2] = [\gamma_1 * \overline{\gamma_2}] \cdot [\gamma_2]$. Note that $[\gamma_1 * \overline{\gamma_2}] \in H$, so γ_1, γ_2 are equivalent in the set of cosets. Thus, the function is injective. □

Theorem 2.3 (Lifting Lemma)**.** *Let E, B, Y be path-connected and locally path-connected (i.e., for all $x \in X$ and for all neighborhoods U_x of x, there exists a V_k which is path connected-connected, contains x, and is contained in U_x.) Let $p\colon E \to B$ be a cover, $b_0 \in B$, $e_0 \in p^{-1}(b_0)$, and $f\colon Y \to B$ such that $f(y_0) = b_0$. Then there exists a lift $\widetilde{f}\colon Y \to E$ such that $\widetilde{f}(y_0) = e_0, p \circ \widetilde{f} = f$ iff $f_*(\pi_1(Y, y_0) \subset p_*\pi_1(E, e_0)$.*



Proof. The $\Rightarrow$ direction is clear. Let $y \in Y$. How should we define $\widetilde{f}(y)$? Let α be a path in$Y$ from y_0 to y_1. Then $f \circ \gamma$ is a path in B starting at b_0. Define $\widetilde{f}(y) := (\widetilde{f \circ \alpha})(1)$. We have $(p \circ \widetilde{f})(y) = p \circ (\widetilde{f \circ \alpha})_{e_0}(1) = f \circ \alpha(1) = f(y)$. Thus, $\widetilde{f}$ is actually a lift.

Now we need to show $\widetilde{f}$ is well-defined (i.e., independent of α). If β is another path in$Y$ from y_0 to y, then $\alpha * \overline{\beta} \in \pi_1(Y, y_0)$, so $f \circ (\alpha * \overline{\beta}) \in f_*\pi_1(Y, y_0)$. Now by assumption, we have $f_*\pi_1(Y, y_0) \subset p_*\pi_1(E, e_0)$. This means that $(f \circ (\alpha * \overline{\beta}))_{e_0}$ is a loop at e_0. Then we have

$$(\widetilde{f \circ \alpha})_{e_0} * (\widetilde{f \circ \overline{\beta}})_{(\widetilde{f \circ \alpha})_{e_0}} = (\widetilde{f \circ \alpha}) * (\widetilde{\overline{f \circ \beta}}) = (\widetilde{f \circ \alpha}) * \overline{(\widetilde{f \circ \beta})}$$

This means that $(\widetilde{f \circ \alpha})_{e_0} = (\widetilde{f \circ \beta})_{e_0}$, which is what we wanted to show.

Now we need to show that $\widetilde{f}$ is continuous. Let $y \in Y$, and let U be a path connected neighborhood of $f^{-1}(y_1) \in B$ (which exists by the locally-connected assumption). Let V be the slice in $p^{-1}(U)$ which contains $\widetilde{f}(y_1)$. By the continuity of f, there is some path-connected neighborhood of y, say W, in Y such that $f(W) \subset U$. Then $\widetilde{f} = (p|_V)^{-1} \circ f|_W$ is continuous. □

Corollary 2.4. *If Y is simply connected, then such a lift always exists.*

Proposition 2.5. (Lift uniqueness) If Y is connected and $\widetilde{f}_1, \widetilde{f}_2 \colon Y \to E$ are two lifts as in the previous theorem, then $\widetilde{f}_1 = \widetilde{f}_2$.

Proof. Let $A = \{y \colon \widetilde{f}_1(y) = \widetilde{f}_2(y)\} \neq \emptyset$. We will show $A = Y$ by showing that A is both open and closed. Let $y \in Y$, and let U be an evenly covered neighborhood of $f(y)$ in B. Then we have $p^{-1}(u) = \sqcup_\alpha \widetilde{U}_\alpha$ such that $p|_{\widetilde{U}_\alpha} : \widetilde{U}_\alpha \to U$ is a homeomorphism. Let $\widetilde{U}_1, \widetilde{U}_2$ be the $\widetilde{U}_\alpha$ containing $\widetilde{f}_1(y)$ and $\widetilde{f}_2(y)$, respectively. Note that the $\widetilde{f}_i$ are continuous, so there is a neighborhood N of y such that $\widetilde{f}_1(N) \subset \widetilde{U}_1$ and $\widetilde{f}_2(N) \subset \widetilde{U}_2$. If $\widetilde{f}_1(y) \neq \widetilde{f}_2(y)$, then $\widetilde{U}_1 \neq \widetilde{U}_2$, so $\widetilde{U}_1 \cap \widetilde{U}_2 = \emptyset$. This means that $\widetilde{f}_1 \neq \widetilde{f}_2$ on N, so A is closed (as the complement of an open set). On the other hand, if $\widetilde{f}_1(y) = \widetilde{f}_2(y)$, then $\widetilde{U}_1 = \widetilde{U}_2$, which implies that $\widetilde{f}_1 = \widetilde{f}_2$ on N (since $p\widetilde{f}_1 = p\widetilde{f}_2 = f$, and p is injective on $\widetilde{U}_1 = \widetilde{U}_2$). Thus, A is open, which proves the proposition. □

Exercise 1. Any continuous map $\mathbb{RP}^2 \to S^1$ is null-homotopic.