Math 751 - Week 2 notes

September 25, 2014

1 Homotopy equivalences

Definition 1. Let $f : X \to Y$ and $g : X \to Y$ be continuous maps. If $f \circ g : Y \to Y$ is homotopic to e_y and $g \circ f : X \to X$ is homotopic to e_x then we say that f (and g) is a homotopy equivalence (abbreviated h.e) and say that g is the homotopy inverse of f.

It is easy to check that homotopy equivalence is an equivalence relation. If X and Y are homotopy equivalent, we say that they have the same homotopy type.

As we will see below, the fundamental group π_1 is not only a homeomorphism invariant, but is invariant under homotopy equivalence as well.

Theorem 1. *If $f : X \to Y$ is a h.e., then $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ is an isomorphism.*

Proof. We need the following lemma to prove the theorem.
Lemma (58.4 Munkres): Let $h : X \to Y$ and $k : X \to Y$ be continuous maps, $x_0 \in X,\ y_0 = h(x_0),\ y_1 = k(x_0)$. If $h \underset{k}{\sim}$, there exists a path $\alpha \in Y$ joining y_0 to y_1, s.t. $k_* = \alpha_\# \circ h_*$ i.e., the following diagram commutes,

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{h_*} & \pi_1(Y, y_0) \\ & \searrow^{k_*} & \downarrow \hat{\alpha} \\ & & \pi_1(Y, y_1) \end{array}$$

In fact, if $H : X \times I = Y$ is a homotopy between h and k, we can take $\alpha = H(x_0, t)$.

Proof. Refer to Munkres. □

Now to prove the theorem, let $g : Y \to X$ be a homotopy inverse for f. Consider the maps

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (X, x_1) \xrightarrow{f} (Y, y_1)$$

where $y_0 = f(x_0),\ x_1 = g(y_0)$ and $y_1 = f(x_1)$. Since f and g are homotopy equivalences, by lemma we have that $(g \circ f)_* = \alpha_\# \circ (e_x)_* = \alpha_\#$ is an isomorphism. Now, $(g \circ f)_* = g_* \circ (f_{x_0})_*$ is an isomorphism which implies that g_* is surjective. Similarly, $(f \circ g)_* = (f_{x_1})_* \circ g_*$ is an isomorphism which implies that g_* is injective. Hence g_* is an isomorphism. Using $(g \circ f)_* = \alpha_\#$ we conclude that,

$$(f_{x_0}) = (g_*)^{-1} \circ \alpha_\#$$

so that $(f_{x_0})_*$ is also an isomophism.
Note that although g is a homotopy inverse for f, the homomorphism g_* is not an inverse for the homomorphism $(f_{x_0})_*$. □

2 Deformation Retracts, $\pi_1\left(S^2\right)$ and Brower's fixed point theorem

We assume that all the spaces that we consider are path connected.

Definition 2. X is contractible if $id_X \underset{h.e}{\sim}$ const.

Example 1. *$\mathbb{R}^n$, D^n, $\{x\}$ are contractible, while S^1, S^2 are not.*

Proposition 1. *If X is contractible, then $\pi_1(X,x)$ is trivial.*

Proof. exercise. □

Definition 3. A space X is called simply-connected if $\pi_1(X,x)$ is trivial for any $x \in X$.

Remark 1. So a contractible space is also simply-connected. The converse is not true, e.g., S^2 is simply-connected, but not contractible (why not?)

Proposition 2. *X is simply-connected if, and only if, there is a unique homotopy class of paths connecting any points in X.*

Proof. ($\Longrightarrow$) If $x, y \in X$ and $f, g : I \to X$ are paths from x to y in X, then we have the following homotopy equivalences:

$$f \underset{h.e}{\sim} f * e_y \underset{h.e}{\sim} f * \bar{g} * g \underset{h.e}{\sim} e_x * g \underset{h.e}{\sim} g,$$

where we use the fact that $\bar{g} * g$ and $f * \bar{g}$ are loops in X at y and x, resp., hence homotopic to the respective constant paths.

($\Longleftarrow$) Take $x = y$. So, by hypothesis, any loop γ at $x \in X$ is in the homotopy class of e_x. □

Definition 4. $A \subset X$ is called a *retract* if there is a map $r : X \to A$, so that $r_{|A} = \mathrm{id}_A$ (i.e., if $i : A \hookrightarrow X$ is the inclusion map, then $r \circ i = \mathrm{id}_A$). $A \subset X$ is called a *deformation retract* if in addition $i \circ r \sim id_X$.

Remark 2. If A is a deformation retract of X, then A is homotopy equivalent to X.

Lemma 1. *S^n is a deformation retract of $\mathbb{R}^{n+1}\backslash\{0\}$.*

Proof. Let $r : X = \mathbb{R}^{n+1} - \{0\} \to S^n$ be defined as $r(\vec{x}) = \vec{x}/\|\vec{x}\|$. By definition, we have $r_{|S^n} = \mathrm{id}_{S^n}$. Also, $i \circ r \sim \mathrm{id}_X$ via $H : X \times [0,1] \to X$ defined as,

$$H(\vec{x}, t) = \frac{\vec{x}}{(1-t) + t\|\vec{x}\|}$$

We can easily see that $H(\vec{x}, 0) = \vec{x} = \mathrm{id}_X(\vec{x})$ and $H(\vec{x}, 1) = \frac{\vec{x}}{\|\vec{x}\|} = i \circ r(\vec{x})$. □

Proposition 3. *S^n is simply-connected if $n \geq 2$.*

Proof. Let $\gamma : [0,1] \to S^n$ be a loop at $x \in S^n$. If we can show that a loop η in the homotopy class of γ is not onto, i.e., there exists $y (\neq x) \notin \mathrm{Im}(\eta)$, then η factors as $[0,1] \to S^n\backslash\{y\} \cong \mathbb{R}^n \hookrightarrow S^n$. Since $\mathbb{R}^n$ is contractible, it follows that $\eta \underset{h.e}{\sim} e_x$. By transitivity, we get that $\gamma \underset{h.e}{\sim} e_x$.

This can be achieved in several different ways:

1. We need the following standard fact from differential topology: Any continuous map between differentiable manifolds contains a smooth map in its homotopy class.
   Using this fact we have, $\gamma \underset{h.e}{\sim} \eta : [0,1] \to S^n$ where η is smooth. Since $\dim([0,1]) < \dim(S^n) = n \geq 2$, every value of η is critical. By Sard's theorem, η has a regular value. If $y \in S^n$ is such a regular value of η, then $y \notin \mathrm{Im}(\eta)$.

2. Point-set topology approach: Let $y \neq x$. We will homotop γ away from y. This can be done as follows. Let B_y be an open ball in S^n around y. Note that $\gamma^{-1}(B_y)$ is open in $(0,1)$. Hence $\gamma^{-1}(B_y) = \sqcup_{i\in I}(a_i, b_i)$, where $|I|$ is possibly infinity. $\gamma^{-1}(y)$ is compact, so only finitely many (a_i, b_i) cover $\gamma^{-1}(y)$. Let $(a_j, b_j), j \in I$ be so that $(a_j, b_j) \cap \gamma^{-1}(y) \neq \emptyset$. Let $\gamma_j := \gamma_{|[a_j,b_j]} \subset \bar{B}_y$. So $\gamma(a_j), \gamma(b_j) \in \partial\bar{B}_y = S_y^{n-1}$. As S_y^{n-1} is path connected, there is a path δ_j in S_y^{n-1} from $\gamma(a_j)$ to $\gamma(b_j)$. Since $\bar{B}_y$ is contractible, we then obtain that $\delta_j \underset{h.e}{\sim} \gamma_j$ in $\bar{B}_y$. Note that $y \notin \mathrm{Im}\,(\delta_j)$. Homotop γ by deforming γ_j to δ_j, and keeping the rest unchanged. Repeat for all j's such that $(a_j, b_j) \cap \gamma^{-1}(y) \neq \emptyset$. Get a loop $\eta \underset{h.e}{\sim} \gamma$ with $\mathrm{Im}(\eta) \cap \{y\} = \emptyset$.

□

Corollary 1. *$\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ if $n \neq 2$.*

Proof. If $n = 1$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is a homeomorphism then we have $\mathbb{R}^2\backslash\{0\} \cong \mathbb{R}\backslash\{f(0)\}$. But $\mathbb{R}^2\backslash\{0\}$ is path connected whereas $\mathbb{R}\backslash\{f(0)\}$ is not path connected. Hence $\mathbb{R}^2 \ncong \mathbb{R}$.
Now let $n \geq 2$ and $f : \mathbb{R}^2 \to \mathbb{R}^n$ be a homeomorphism. Then we have,

$$\mathbb{R}^2\backslash\{0\} \cong \mathbb{R}^n\backslash\{f(0)\}$$

hence

$$\pi_1\left(\mathbb{R}^2\backslash\{0\}\right) \cong \pi_1\left(\mathbb{R}^n\backslash\{f(0)\}\right)$$

But we know that,

$$\pi_1\left(\mathbb{R}^n\backslash\{a\}\right) \cong \pi_1\left(S^{n-1}\right) = \begin{cases} \mathbb{Z}, & n = 2 \\ 0, & n > 2 \end{cases}$$

Hence f cannot be a homeomorphism. □

2.1 Brower's fixed point theorem

Theorem 2. *Any continuous map $f : D^2 \to D^2$ has a fixed point.*

Proof. Assume $f(x) \neq x$ for all $x \in D^2$. Let $r : D^2 \to S^1$ be defined such that $r(x)$ is intersection of the line joining $f(x)$ and x with S^1 (with x between $f(x)$ and $r(x)$). We have $r_{|S^1} = \mathrm{id}_{S^1}$, i.e., $r \circ i = \mathrm{id}_{S^1}$ for $S^1 \overset{i}{\hookrightarrow} D^2$ the inclusion map. We have the following commutative diagram,

$$\begin{array}{ccc} S^1 & \xrightarrow{\ i\ } & D^2 \\ & \searrow^{\mathrm{id}} & \downarrow r \\ & & S^1 \end{array}$$

which, on the level of fundamental groups yields:

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\ i_*\ } & 0 \\ & \searrow^{\mathrm{id}} & \downarrow r_* \\ & & \mathbb{Z} \end{array}$$

But this is not possible since the identity map cannot factor through the zero map. □

As an application of Brower's fixed point theorem, we have the following:

Proposition 4. *Let $A \in \mathcal{M}_3(\mathbb{R})$, $a_{ij} \geq 0$ for all $i, j \in \{1, 2, 3\}$. Assume $det(A) \neq 0$. Then A has a positive real eigenvalue.*

Proof. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map corresponding to A. Let

$$B = S^2 \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1, x_2, x_3 \geq 0\} \cong D^2.$$

If $x \in B$, then all coordinates of $Tx = Ax$ are nonnegative, and not all zero since A is nonsingular (so $Ax = 0 \iff x = 0$). So $Tx/\|Tx\| \in B$. We get a map $f : B \to B$ defined as $f(x) = Tx/\|Tx\|$. By Brower's fixed point theorem, there exists $x_0 \in B$ so that $f(x_0) = x_0$, i.e., $Tx_0 = \|Tx\| x_0$. Take $\lambda = \|Tx_0\|$, then λ is an eigenvalue of A, so that $\lambda \in \mathbb{R}$ and $\lambda > 0$. □