## 0.1 Cellular Homology

Let us start with the following preliminary result:

**Lemma 0.1.1.** *If $X$ is a CW complex, then:*

*(a)* $H_k(X_n, X_{n-1}) = \begin{cases} 0 & \text{if } k \neq n \\ \mathbb{Z}^{\,\#\text{ n-cells}} & \text{if } k = n. \end{cases}$

*(b)* $H_k(X_n) = 0$ *if* $k > n$. *In particular, if $X$ is finite dimensional, then* $H_k(X) = 0$ *if* $k > \dim(X)$.

*(c) The inclusion* $i : X_n \hookrightarrow X$ *induces an isomorphism* $H_k(X_n) \to H_k(X)$ *if* $k < n$.

*Proof.* (a) We know that $X_n$ is obtained from $X_{n-1}$ by attaching the $n$-cells $(e^n_\lambda)_\lambda$. Pick a point $x_\lambda$ at the center of each $n$-cell $e^n_\lambda$. Let $A := X_n - \{x_\lambda\}_\lambda$. Then $A$ deformation retracts to $X_{n-1}$, so we have that

$$H_k(X_n, X_{n-1}) \cong H_k(X_n, A).$$

By excising $X_{n-1}$, the latter group is isomorphic to $\oplus_\lambda H_k(D^n_\lambda, D^n_\lambda - \{x_\lambda\})$. Moreover, the homology long exact sequence of the pair $(D^n_\lambda, D^n_\lambda - \{x_\lambda\})$ yields that

$$H_k(D^n_\lambda, D^n_\lambda - \{x_\lambda\}) \cong \widetilde{H}_{k-1}(S^{n-1}_\lambda) \cong \begin{cases} \mathbb{Z} & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}$$

So the claim follows.

(b) Consider the following portion of the long exact sequence of the pair for $(X_n, X_{n-1})$:

$$\to H_{k+1}(X_n, X_{n-1}) \to H_k(X_{n-1}) \to H_k(X_n) \to H_k(X_n, X_{n-1}) \to$$

If $k+1 \neq n$ and $k \neq n$, we have from part (a) that $H_{k+1}(X_n, X_{n-1}) = 0$ and $H_k(X_n, X_{n-1}) = 0$. Thus $H_k(X_{n-1}) \cong H_k(X_n)$. Hence if $k > n$ (so in particular, $n \neq k+1$ and $n \neq k$), we get by iteration that

$$H_k(X_n) \cong H_k(X_{n-1}) \cong \cdots \cong H_k(X_0).$$

Note that $X_0$ is just a collection of points, so $H_k(X_0) = 0$. Thus when $k > n$ we have $H_k(X_n) = 0$ as desired.

(c) We only prove the statement for finite dimensional CW complexes. Let $k < n$ and consider the long exact sequence for the pair $(X_{n+1}, X_n)$:

$$\to H_{k+1}(X_{n+1}, X_n) \to H_k(X_n) \to H_k(X_{n+1}) \to H_k(X_{n+1}, X_n) \to$$

Since $k < n$ we have $k+1 \neq n+1$ and $k \neq n+1$, so by part (a) we get that $H_{k+1}(X_{n+1}, X_n) = 0$ and $H_k(X_{n+1}, X_n) = 0$. Thus $H_k(X_n) \cong H_k(X_{n+1})$. By repeated iterations, we obtain:

$$H_k(X_n) \cong H_k(X_{n+1}) \cong H_k(X_{n+2}) \cong \cdots \cong H_k(X_{n+l}) = H_k(X),$$

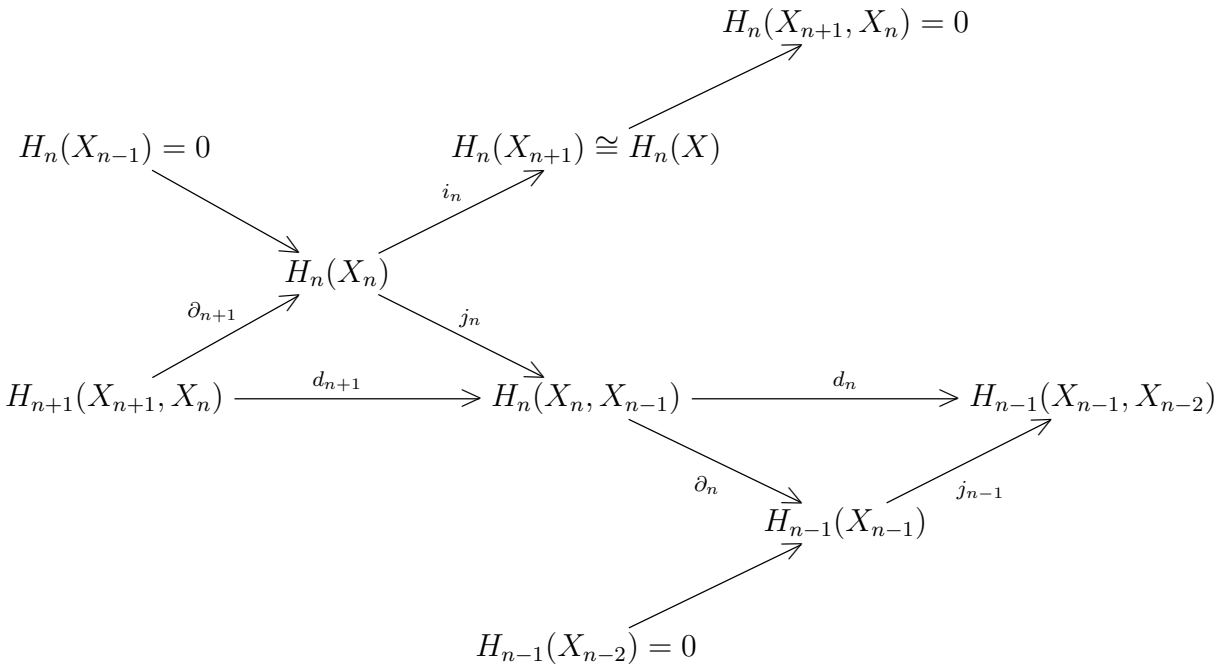
where $l$ is so that $X_{n+l} = X$. Since $X$ is finite dimensional we know that $X = X_{n+l}$ for some $l$. This proves the claim. □

In what follows we defined the *cellular homology of a CW complex $X$* in terms of a given cell structure, then we show that it coincides with the singular homology, so it is in fact independent on the cell structure. Cellular homology is very useful for computations.

**Definition 0.1.2.** *The cellular homology $H_*^{CW}(X)$ of a CW complex $X$ is the homology of the cellular chain complex $(\mathcal{C}_*(X), d_*)$ indexed by the cells of $X$, i.e.,*

$$\mathcal{C}_n(X) := H_n(X_n, X_{n-1}) = \mathbb{Z}^{\#n-cells}, \tag{0.1.1}$$

*and with differentials $d_n : \mathcal{C}_n(X) \to \mathcal{C}_{n-1}(X)$ defined by the following diagram:*




The diagonal arrows are induced from long exact sequences of pairs, and we use Lemma 0.1.1 for the identifications $H_n(X_{n-1}) = 0$, $H_{n-1}(X_{n-2}) = 0$ and $H_n(X_{n+1}) \cong H_n(X)$ in the diagram. In the notations of the above diagram, we now set:

$$d_n = j_{n-1} \circ \partial_n : \mathcal{C}_n(X) \to \mathcal{C}_{n-1}(X), \tag{0.1.2}$$

and note that we have

$$d_n \circ d_{n+1} = 0. \tag{0.1.3}$$

Indeed,

$$d_n \circ d_{n+1} = j_{n-1} \circ \partial_n \circ j_n \circ \partial_{n+1} = 0,$$

since $\partial_n \circ j_n = 0$ as the composition of two consecutive maps in a long exact sequence. So $(\mathcal{C}_*(X), d_*)$ is a chain complex.

The following result asserts that cellular homology is independent on the cell structure used for its definition:

**Theorem 0.1.3.** *There are isomorphisms*

$$H_n^{CW}(X) \cong H_n(X)$$

*for all $n$, where $H_n(X)$ is the singular homology of $X$.*

*Proof.* Since $H_n(X_{n+1}, X_n) = 0$ and $H_n(X) \cong H_n(X_{n+1})$, we get from the diagram above that

$$H_n(X) \cong H_n(X_n)/\ker i_n \cong H_n(X_n)/\text{Image } \partial_{n+1}.$$

Now, $H_n(X_n) \cong \text{Image } j_n \cong \ker \partial_n \cong \ker d_n$. The first isomorphism comes from $j_n$ being injective, while the second follows by exactness. Finally, $\ker \partial_n = \ker d_n$ since $d_n = j_{n-1} \circ \partial_n$ and $j_{n-1}$ is injective. Also, we have Image $\partial_{n+1} =$ Image $d_{n+1}$. Indeed, $d_{n+1} = j_n \circ \partial_{n+1}$ and $j_n$ is injective.

Altogether, we have

$$H_n(X) \cong H_n(X_n)/\text{Image } \partial_{n+1} = \ker d_n/\text{Image } d_{n+1} = H_n^{CW}(X)$$

So we have proved the theorem. □

Let us now discuss some **immediate consequences** of the above theorem.

(a) *If $X$ has no $n$-cells, then $H_n(X) = 0$.*
Indeed, in this case we have $\mathcal{C}_n = H_n(X_n, X_{n-1}) = 0$, so $H_n^{CW}(X) = 0$.

(b) *If $X$ is connected and has a single 0-cell then $d_1 : \mathcal{C}_1 \to \mathcal{C}_0$ is the zero map.*
Indeed, since $X$ contains only a single 0-cell, $\mathcal{C}_0 = \mathbb{Z}$. Also, since $X$ is connected, $H_0(X) = \mathbb{Z}$. So by the theorem above $\mathbb{Z} = H_0(X) = \ker d_0/\text{Image } d_1 = \mathbb{Z}/\text{Image } d_1$. This implies that Image $d_1 = 0$, so $d_1$ is the zero map as desired.

(c) *If $X$ has no two cells in adjacent dimensions then $d_n = 0$ for all $n$ and $H_n(X) \cong \mathbb{Z}^{\#n-cells}$ for all $n$.*
Indeed, in this case all maps $d_n$ vanish. So for any $n$, $H_n^{CW}(X) \cong \mathcal{C}_n \cong \mathbb{Z}^{\#n-cells}$.

**Example 0.1.4.** Recall that $\mathbb{C}P^n$ has one cell in each dimension $0, 2, 4, \ldots, 2n$. So $\mathbb{C}P^n$ has no two cells in adjacent dimensions, meaning we can apply Consequence (c) above to obtain:

$$H_i(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & \text{if } i = 0, 2, 4, \ldots, 2n \\ 0 & \text{otherwise.} \end{cases}$$

**Example 0.1.5.** When $n > 1$, $S^n \times S^n$ has one 0-cell, two $n$-cells, and one $2n$-cell. Since $n > 1$, these cells are not in adjacent dimensions so again Consequence (c) above applies to give:

$$H_i(S^n \times S^n) = \begin{cases} \mathbb{Z} & i = 0, 2n \\ \mathbb{Z}^2 & i = n \\ 0 & \text{otherwise.} \end{cases}$$

In the remaining of this section, we discuss how to compute in general the maps

$$d_n : \mathcal{C}_n(X) = \mathbb{Z}^{\#n-cells} \to \mathcal{C}_{n-1}(X) = \mathbb{Z}^{\#(n-1)-cells}$$

of the cellular chain complex. Let us consider the $n$-cells $\{e^n_\alpha\}_\alpha$ as the basis for $\mathcal{C}_n(X)$ and the $(n-1)$-cells $\{e^{n-1}_\beta\}_\beta$ as the basis for $\mathcal{C}_{n-1}(X)$. In particular, we can write:

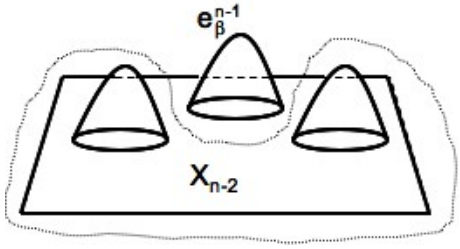
$$d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta} \cdot e^{n-1}_\beta,$$

with $d_{\alpha\beta} \in \mathbb{Z}$. The following result provides a way of computing the coefficients $d_{\alpha\beta}$:

**Theorem 0.1.6.** *The coefficient $d_{\alpha\beta}$ is equal to the degree of the map $\Delta_{\alpha\beta} : S^{n-1}_\alpha \to S^{n-1}_\beta$ defined by the composition:*

$$S^{n-1}_\alpha = \partial D^n_\alpha \xrightarrow{\varphi^n_\alpha} X_{n-1} = X_{n-2} \amalg_\gamma e^{n-1}_\gamma \xrightarrow{collapse} X_{n-1}/(X_{n-2} \sqcup_{\gamma\neq\beta} e^{n-1}_\gamma) = S^{n-1}_\beta,$$

*where $\varphi^n_\alpha$ is the attaching map of $e^n_\alpha$, and the collapsing map sends $X_{n-2} \coprod_{\gamma\neq\beta} e^{n-1}_\gamma$ to a point.*




*Proof.* We will proceed with the proof by chasing the following diagram:

$$\begin{array}{ccccc}
H_n(D^n_\alpha, S^{n-1}_\alpha) & \xrightarrow[\simeq]{\partial} & \widetilde{H}_{n-1}(S^{n-1}_\alpha) & \xrightarrow{(\Delta_{\alpha\beta})_*} & \widetilde{H}_{n-1}(S^{n-1}_\beta) \\
\downarrow{\scriptstyle (\Phi^n_\alpha)_*} & & \downarrow{\scriptstyle (\varphi^n_\alpha)_*} & & \uparrow{\scriptstyle q_{\beta *}} \\
H_n(X_n, X_{n-1}) & \xrightarrow{\partial_n} & \widetilde{H}_{n-1}(X_{n-1}) & \xrightarrow{q_*} & \widetilde{H}_{n-1}(X_{n-1}/X_{n-2}) \\
& \searrow^{d_n} & \downarrow{\scriptstyle j_{n-1}} & & \downarrow{\scriptstyle \simeq} \\
& & H_{n-1}(X_{n-1}, X_{n-2}) & \xrightarrow{\simeq} & H_n(\frac{X_{n-1}}{X_{n-2}}, \frac{X_{n-2}}{X_{n-2}})
\end{array}$$

where:

- $\Phi^n_\alpha$ is the characteristic map of the cell $e^n_\alpha$ and $\phi^n_\alpha$ is its attaching map.
- $q_* : \widetilde{H}_{n-1}(X_{n-1}) \to \widetilde{H}_{n-1}(X_{n-1}/X_{n-2}) = \oplus_\beta \widetilde{H}_{n-1}(D^{n-1}_\beta/\partial D^{n-1}_\beta)$ is induced by the quotient map $q : X_{n-1} \to X_{n-1}/X_{n-2}$.

- $q_\beta : X_{n-1}/X_{n-2} \to S^{n-1}_\beta$ collapses the complement of the cell $e^{n-1}_\beta$ to a point, the resulting quoting sphere being identified with $S^{n-1}_\beta = D^{n-1}_\beta/\partial D^{n-1}_\beta$ via the characteristic map $\Phi^{n-1}_\beta$.

- $\Delta_{\alpha\beta} : S^{n-1}_\alpha = \partial D^n_\alpha \to S^{n-1}_\beta$ is the composition $q_\beta \circ q \circ \phi^n_\alpha$, i.e., the attaching map of $e^n_\alpha$ followed by the quotient map $X_{n-1} \to S^{n-1}_\beta$ collapsing the complement of $e^{n-1}_\beta$ in $X_{n-1}$ to a point.

Note that $(\Delta_{\alpha\beta})_*$ is defined so that the top right square commutes.

Recall that our goal is to compute $d_n(e^n_\alpha)$. The upper left square is natural and therefore commutes (it is induced by the characteristic map $\Phi : (D^*, S^{*-1}) \to (X_*, X_{*-1})$ of a cell), while the lower left triangle is part of the exact diagram defining the chain complex $\mathcal{C}_*(X)$ and is defined to commute as well. The map $(\Phi^n_\alpha)_*$ takes the generator $[D^n_\alpha] \in H_n(D^n_\alpha, S^{n-1}_\alpha)$ to a generator of the $\mathbb{Z}$-summand of $H_n(X_n, X_{n-1})$ corresponding to $e^n_\alpha$, i.e.,

$$(\Phi^n_\alpha)_*([D^n_\alpha]) = e^n_\alpha.$$

Since the top left square and the bottom left triangle both commute, this gives that

$$d_n(e^n_\alpha) = d_n \circ (\Phi^n_\alpha)_*([D^n_\alpha]) = j_{n-1} \circ (\varphi^n_\alpha)_* \circ \partial([D^n_\alpha]).$$

Looking to the bottom right square, recall that since $X$ is a $CW$ complex, $(X_n, X_{n-1})$ is a good pair. This gives the isomorphism $H_{n-1}(X_{n-1}, X_{n-2}) \simeq \widetilde{H}_{n-1}(X_{n-1}/X_{n-2})$ Moreover, we also have that $\widetilde{H}_{n-1}(X_{n-1}/X_{n-2}) \simeq H_{n-1}(X_{n-1}/X_{n-2}, X_{n-2}/X_{n-2})$. The bottom right square commutes by the definition of $j_{n-1}$ and $q_*$, which combined with the commutativity of the top left square yields that

$$d_n(e^n_\alpha) = q_* \circ \partial_n \circ (\Phi^n_\alpha)_*([D^n_\alpha]) = q_* \circ (\varphi^n_\alpha)_* \circ \partial([D^n_\alpha]),$$

where formally we should precompose on the left hand side with the isomorphism between $H_{n-1}(X_{n-1}, X_{n-2})$ and $\widetilde{H}_{n-1}(X_{n-1}/X_{n-2})$ so that everything is in the same space. This last map takes the generator $[D^n_\alpha]$ to a linear combination of generators in $\oplus_\beta \widetilde{H}_{n-1}(D^{n-1}_\beta/\partial D^{n-1}_\beta)$. To see which generators it maps to, we project down to the respective $\beta$ summands to obtain

$$d_n(e^n_\alpha) = \sum_\beta q_{\beta *} \circ q_* \circ (\varphi^n_\alpha)_* \circ \partial([D^n_\alpha]).$$

As noted before, we have defined $(\Delta_{\alpha\beta})_* = q_{\beta *} \circ q_* \circ (\varphi^n_\alpha)_*$. So writing

$$d_n(e^n_\alpha) = \sum_\beta (\Delta_{\alpha\beta})_* \partial([D^n_\alpha]),$$

we see from the definition of the above maps and the fact that $\partial([D^n_\alpha])$ is a generator of $\widetilde{H}_{n-1}(S^{n-1}_\alpha)$, that $(\Delta_{\alpha\beta})_*$ is multiplication by $d_{\alpha\beta}$. □

**Example 0.1.7.** Let $M_g$ be the closed oriented surface of genus $g$, with its usual CW structure: one 0-cell, $2g$ 1-cells $\{a_1, b_1, \cdots a_g, b_g\}$, and one 2-cell attached by product of commutators $[a_1, b_1] \cdots [a_g, b_g]$. The associated cellular chain complex of $M_g$ is:

$$0 \xrightarrow{d_3} \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

Since $M_g$ is connected and has only one 0-cell, we get that $d_1 = 0$. We claim that $d_2$ is also the zero map. This amounts to showing that $d_2(e) = 0$, where $e$ denotes the 2-cell. Indeed, let us compute the coefficients $d_{ea_i}$ and $d_{eb_i}$ in our degree formula. As the attaching map sends the generator to $a_1 b_1 a_1^{-1} b_1^{-1} ... a_g b_g a_g^{-1} b_g^{-1}$, when we collapse all 1-cells (except $a_i$, resp. $b_i$) to a point, the word defining the attaching map $a_1 b_1 a_1^{-1} b_1^{-1} ... a_g b_g a_g^{-1} b_g^{-1}$ reduces to $a_i a_i^{-1}$ and resp. $b_i b_i^{-1}$. Hence $d_{ea_i} = 1 - 1 = 0$. Similarly, $d_{eb_i} = 1 - 1 = 0$, for each $i$. Altogether,

$$d_2(e) = a_1 + b_1 - a_1 - b_1 + \cdots a_g + b_g - a_g - b_g = 0.$$

So the homology groups of $M_g$ are given by

$$H_n(M_g) = \begin{cases} \mathbb{Z} & \text{i=0, 2} \\ \mathbb{Z}^{2g} & \text{i=1} \\ 0 & \text{otherwise.} \end{cases}$$

**Example 0.1.8.** Let $N_g$ be the closed nonorientable surface of genus $g$, with its cell structure consisting of one 0-cell, $g$ 1-cells $\{a_1, \cdots, a_g\}$, and one 2-cell $e$ attached by the word $a_1^2 \cdots a_g^2$. The cellular chain complex of $N_g$ is given by

$$0 \xrightarrow{d_3} \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^g \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

As before, $d_1 = 0$ since $N_g$ is connected and there is only one cell in dimension zero. To compute $d_2 : \mathbb{Z} \to \mathbb{Z}^g$ we again apply the cellular boundary formula, and obtain

$$d_2(1) = (2, 2, \cdots, 2)$$

since each $a_1$ appears in the attaching word with total exponent 2, which means that each map $\Delta_{\alpha\beta}$ is homotopic to the map $z \mapsto z^2$ of degree 2. In particular, $d_2$ is injective, hence $H_2(N_g) = 0$. If we change the standard basis for $\mathbb{Z}^g$ by replacing the last standard basis element $e_n = (0, \cdots, 0, 1)$ by $e'_n = (1, \cdots, 1)$, then $d_2(1) = 2 \cdot e'_n$, so

$$H_1(N_g) \cong \mathbb{Z}^g / \text{Image } d_2 \cong \mathbb{Z}^g / 2\mathbb{Z} \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2.$$

Altogether,

$$H_n(N_g) = \begin{cases} \mathbb{Z} & \text{i=0} \\ \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2 & \text{i=1} \\ 0 & \text{otherwise.} \end{cases}$$

**Example 0.1.9.** Recall that $\mathbb{RP}^n$ has a CW structure with one cell $e^k$ in each dimension $0 \leq k \leq n$. Moreover, the attaching map of $e^k$ in $\mathbb{RP}^n$ is the two-fold cover projection $\varphi : S^{k-1} \to \mathbb{RP}^{k-1}$. The cellular chain complex for $\mathbb{RP}^n$ looks like:

$$0 \xrightarrow{d_{n+1}} \mathbb{Z} \xrightarrow{d_n} ...\mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

To compute the differential $d_k$, we need to compute the degree of the composite map

$$\Delta : S^{k-1} \xrightarrow{\varphi} \mathbb{RP}^{k-1} \xrightarrow{q} \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} = S^{k-1}.$$

The map $\Delta$ is a homeomorphism when restricted to each component of $S^{k-1} \setminus S^{k-2}$, and these homeomorphisms are obtained from each other by precomposing with the antipodal map $a$ of $S^{k-1}$, which has degree $(-1)^k$. Hence, by our local degree formula, we get that:

$$\deg \Delta = \deg id + \deg a = 1 + (-1)^k.$$

In particular,

$$d_k = \begin{cases} 0 & \text{if k is odd} \\ 2 & \text{if k is even,} \end{cases}$$

and therefore we obtain that

$$H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z}_2 & \text{if k is odd , } 0 < k < n \\ \mathbb{Z} & k = 0, \text{ and } k = n \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

Finally, note that an equivalent definition of the above map $\Delta$ is obtained by first collapsing the equatorial $S^{k-2}$ to a point to get $S^{k-1} \vee S^{k-1}$, and then mapping the two copies of $S^{k-1}$ onto $S^{k-1}$, the first one by the identity map, and the second by the antipodal map.

