HOMEWORK #5

1. Show that if X is a path-connected topological space and $f : X \to X$ is a continuous function, then the induced map $f_* : H_0(X) \to H_0(X)$ is the identity map.

2. Show that $H_0(X, A) = 0$ iff A meets each path-component of X.

3. Show that $H_1(X, A) = 0$ iff $H_1(A) \to H_1(X)$ is surjective and each path-component of X contains at most a path-component of A.

4. Let A be a retract of X, i.e., there exists a map $r : X \to A$ whose restriction to A is the identity. Let $i : A \to X$ be the inclusion map. Show that $i_* : H_*(A) \to H_*(X)$ is a monomorphism and r_* is an epimorphism.

5. If X is path-connected and A is a finite set of points in X, compute the relative homology groups $H_n(X, A)$ in terms of the homology groups of X.

6. Let $f : (X, A) \to (Y, B)$ be a map such that both $f : X \to Y$ and $f : A \to B$ are homotopy equivalences. Show that $f_* : H_n(X, A) \to H_n(Y, B)$ is an isomorphism for all n.

7. A graded abelian group is a sequence of abelian groups $A_\bullet := (A_n)_{n\geq 0}$. We say that $A_\bullet$ is of *finite type* if

$$\sum_{n\geq 0} \text{rank} A_n < \infty.$$

The *Euler characteristic* of a finite type graded abelian group $A_\bullet$ is the integer

$$\chi(A_\bullet) := \sum_{n\geq 0} (-1)^n \cdot \text{rank} A_n.$$

A short exact sequence of graded groups $A_\bullet$, $B_\bullet$, $C_\bullet$, is a sequence of short exact sequences

$$0 \to A_n \to B_n \to C_n \to 0, \quad n \geq 0.$$

Prove that if $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ is a short exact sequence of graded abelian groups of finite type, then

$$\chi(B_\bullet) = \chi(A_\bullet) + \chi(C_\bullet).$$

8. Suppose

$$\cdots \to C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \to 0$$

is a chain complex such that the graded abelian group $C_\bullet$ is of finite type. Denote by H_n the n-th homology group of this complex and form the corresponding graded group $H_\bullet = (H_n)_{n\geq 0}$. Show that $H_\bullet$ is of finite type and

$$\chi(H_\bullet) = \chi(C_\bullet).$$

9. Suppose we are given three finite type graded abelian groups $A_\bullet$, $B_\bullet$, $C_\bullet$, which are part of a long exact sequence

$$\cdots \to A_k \xrightarrow{i_k} B_k \xrightarrow{j_k} C_k \xrightarrow{\partial_k} A_{k-1} \to \cdots \to A_0 \to B_0 \to C_0 \to 0.$$

Show that

$$\chi(B_\bullet) = \chi(A_\bullet) + \chi(C_\bullet).$$