

HOMEWORK #3

1. There are six ways to obtain a compact surface by identifying pairs of sides in a square. In each case determine what surface one obtains.

2. The following labeling schemes describe two dimensional surfaces:

- $abc^{-1}b^{-1}a^{-1}c$
- $abc^{-1}c^{-1}ba$
- $a_1a_2 \cdots a_n a_1^{-1} a_2^{-1} \cdots a_n^{-1}$

In each case determine what standard surface it is homeomorphic to.

3. Consider the space X obtained from a seven-sided polygonal region by means of the labeling scheme $abaaab^{-1}a^{-1}$. Show that $\pi_1(X)$ is the free product of two cyclic groups.

4. Let X be the quotient space obtained from an eight-sided polygonal region P by means of the labeling scheme $abcdad^{-1}cb^{-1}$. Let $\pi : P \rightarrow X$ be the quotient map.

- Show that P does not map all the vertices of P to the same point of X .
- Determine the space $A = \pi(\text{Bd } P)$ (the boundary of P), and calculate its fundamental group.
- Calculate the fundamental group of X . (Hint: first transform the labeling scheme into a standard one by cutting and pasting operations.)
- What surface is X homeomorphic to?

5. Let X be a space obtained by pasting the edges of a polygonal region together in pairs.

- Show that X is homeomorphic to exactly one of the spaces in the following list: S^2 , \mathbb{P}^2 , K , T_n , $T_n \# \mathbb{P}^2$, $T_n \# K$, where K is the Klein bottle and $n \geq 1$.
- Show that X is homeomorphic to exactly one of the spaces in the following list: S^2 , \mathbb{P}^2 , K_m , T_n , $\mathbb{P}^2 \# K_m$, where K_m is the m -fold connected sum of K with itself and $m \geq 1$.

6. Let A be the annulus in the plane consisting of the set

$$A := \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 4\}.$$

Let S denote the surface obtained from A by identifying antipodal points of the inner circle and by identifying antipodal points of the outer circle. Compute $\pi_1(S)$ and write S as a connected sum of tori and projective planes.

7. Let X be the topological space obtained by identifying by parallel translation the opposite edges of a solid regular hexagon. Calculate the fundamental group of X .