HOMEWORK #1

1. Show that if $h, h' : X \to Y$ are homotopic and $k, k' : Y \to Z$ are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic.

2. Let x_0 and x_1 be points of the path-connected space X. Show that $\pi_1(X, x_0)$ is abelian if and only if for every pair α and β of paths from x_0 to x_1, we have $\alpha_\# = \beta_\# : \pi_1(X, x_0) \to \pi_1(X, x_1)$. (Recall that $\alpha_\# : \pi_1(X, x_0) \to \pi_1(X, x_1)$ is the group isomorphism defined by $\alpha_\#([\gamma]) := [\alpha^{-1} * \gamma * \alpha]$.)

3. Let A be a subspace of $\mathbb{R}^n$; let $h : (A, a_0) \to (Y, y_0)$ be a continuous map of pointed spaces. Show that if h is extendable to a continuous map of $\mathbb{R}^n$ into Y, then h induces the trivial homomorphism on fundamental groups (i.e., h_* maps everything to the identity element).

4. Show that any two maps from an arbitrary space to a contractible space are homotopic. As a consequence, prove that if X is a contractible space, then any point in X is a deformation retract of X.

5. Show that if X and Y are path-connected spaces, and $x \in X$, $y \in Y$, then $\pi_1(X \times Y, (x, y))$ is isomorphic to $\pi_1(X, x) \times \pi_1(Y, y)$.

6. Let $A \subset X$; suppose $r : X \to A$ is a continuous map so that $r(a) = a$ for each $a \in A$. (The map r is called a *retraction* of X onto A.) If $a_0 \in A$, show that $r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$ is surjective.

7. Using the fact that the fundamental group of the circle S^1 is $\mathbb{Z}$, show that there are no retractions $r : X \to A$ in the following cases:
(1) $X = \mathbb{R}^3$, with A any subspace homeomorphic to S^1.
(2) $X = S^1 \times D^2$, with A its boundary torus $S^1 \times S^1$.
(3) X is the Möbius band and A its boundary circle.

8. Let A be a real 3×3 matrix, with all entries positive. Show that A has a positive real eigenvalue. (Hint: Use Brower's fixed point theorem.)

9. Show that if $f : \mathbb{D}^2 \to \mathbb{D}^2$ is a continuous map so that the restriction $f_{|S^1}$ is a homeomorphism $S^1 \to S^1$, then f is surjective.

10. Let V be a finite dimensional vector space and W a subspace. Compute $\pi_1(V \setminus W)$.

11. What is the homotopy type of $\mathbb{RP}^2$ minus a point?