NAME: ——————————————————

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PRACTICE MIDTERM EXAM

INSTRUCTIONS: *You have exactly* 75 *minutes to complete the exam. You must show all your work and be clear in your reasoning in order to receive full credit. No calculators, phones or tablets are allowed. You must obey the principles of academic integrity. You must include this sheet with your exam in order to receive a grade.*

1. Let $L(S)$ be the subspace spanned by a subset S of a vector space V. Prove that:
(a) $S \subseteq L(S)$.
(b) S is a subspace of V if and only if $L(S) = S$.

2. Consider the Euclidian space
$$V = \{p(x) \mid p(x) = a + bx + cx^2, x \in [-1, 1], a, b, c \in \mathbb{R}\},$$
on which the inner product is defined as:
$$(f, g) = \int_{-1}^{1} x^2 f(x) g(x) \; dx, \quad \text{for any } f, g \in V.$$
Derive an orthogonal basis $\{e_1(x), e_2(x), e_3(x)\}$ for V using the Gram-Schmidt process.

3. In the real linear space $C(-1, 1)$ with inner product $(f, g) = \int_{-1}^{1} f(x) g(x) \; dx$, let $f(x) = e^x$. Find the linear polynomial g nearest to f, and compute $||g - f||^2$ for this g.

4. Let $T : V \to W$ be a linear transformation.
(a) Prove that $N(T) = \{0\}$ implies that T is one-to-one.
(b) Consider the linear transformation $T = \frac{d^2}{dx^2}$, where
$$V = \{f(x) \mid f \text{ is a polynomial of deg } \leq n\}$$
and
$$W = \{f(x) \mid f \text{ is a polynomial of deg } \leq n - 2\}.$$
What is the dimension of $N(T)$? What is the rank of T?