Mathematics 221, Lecture 1 **Name:** ____________________
Instructor: L. Maxim **TA's Name:** ____________________

EXAM II

Do all five of the following problems. Show all your work, and write neatly.

No.	Points		Score
1	20		
2	20		
3	20		
4	20		
5	20		
	100	TOTAL POINTS	

Problem I (20 points)
A highway patrol plane flies 3 miles above a level, straight road at a steady 120 mi/h. The pilot sees an oncoming car and with radar determines that at the instant the line-of-sight distance from plane to car is 5 miles, the line-of-sight distance is decreasing at the rate of 160 mi/h. Find the car's speed along the highway.

Solution: Let x denote the horizontal distance between the car and plane, and y the line-of-sight distance between the car and plane. Then, at any instance:

$$y^2 = x^2 + 3^2,$$

hence, after differentiating with respect to time t, we get:

$$\frac{dx}{dt} = \frac{y}{x} \cdot \frac{dy}{dt}. \tag{1}$$

When $y = 5$, we know that $\frac{dy}{dt} = -160$, and the Pythagorean theorem gives $x = 4$. Hence, at this instance of time, equation (1) yields:

$$\frac{dx}{dt} = -200.$$

So x is decreasing at a rate of 200 mph, but as a consequence of the two objects, plane and car, moving towards each other. So, speed of car + speed of plane = 200, whence the speed of car is 80 mph. □

Problem II (20 points)
Determine where the curve $y = \frac{x}{x^2-1}$ is increasing, decreasing, concave up and concave down. Where are its local extrema and inflection points? Use this information to sketch the curve.

Solution: This was part of your Homework #8. I only include the essential calculations here.

First, the domain is $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$. Secondly, $y=0$ is a horizontal asymptote, and $x=1$ and $x=-1$ are vertical asymptotes. Indeed, we have,

$$\lim_{x\to\pm\infty} y = 0$$

and

$$\lim_{x\to 1^+} y = \infty, \lim_{x\to 1^-} y = -\infty, \lim_{x\to -1^+} y = \infty, \lim_{x\to -1^-} y = -\infty.$$

$$y' = -\frac{x^2+1}{(x^2-1)^2} < 0$$

so f is decreasing over its domain.

$$y'' = \frac{2x(x^2+3)}{(x^2-1)^3}$$

so $y''=0$ only if $x=0$. Also, $y''>0$ for $x\in(-1,0)\cup(1,\infty)$, so y is concave up on this part of the domain. Similarly, $y''<0$ on $(-\infty,-1)\cup(0,1)$, where y is concave down.

So $x=0$ is an inflection point. There are no local extrema. The graph of y looks like....any guess?

Problem III (20 points)
Show that the function

$$f(x) = x + \sin^2\left(\frac{x}{3}\right) - 8$$

has exactly one zero.

Solution:

$$f'(x) = 1 + \frac{2}{3}\sin\left(\frac{x}{3}\right)\cos\left(\frac{x}{3}\right) = 1 + \frac{1}{3}\sin\left(\frac{2x}{3}\right) > 0$$

So $f(x)$ is increasing over its domain $x \in (-\infty, +\infty)$. Also, $f(0) = -8 < 0$ and $f(8) = \sin^2\left(\frac{8}{3}\right) > 0$. Hence the graph of $f(x)$ crosses the x-axis exactly once. $\square$

Problem IV (20 points)
The sum of two non-negative numbers is 36. Find the numbers if the sum of their square roots is to be as large as possible.

Solution: Let x and y denote the two numbers. Then $x \leq 0$, $y \leq 0$ and $x + y = 36$. We need to maximize $\sqrt{x} + \sqrt{y}$, or, writing $y = 36 - x$, this is the same as maximizing the function

$$f(x) = \sqrt{x} + \sqrt{36 - x}$$

over the interval $x \in [0, 36]$.

We have:

$$f'(x) = \frac{1}{2\sqrt{x}} + \frac{-1}{2\sqrt{36 - x}} = \frac{\sqrt{36 - x} - \sqrt{x}}{2\sqrt{x}\sqrt{36 - x}}.$$

So critical points are $x = 18$ and the points where f' is not defined: $x = 0$ and $x = 36$. By evaluating f at these points, we get: $f(0) = 6$, $f(18) = 2\sqrt{18} = 6\sqrt{2}$ and $f(36) = 6$. So the maximum of f is achieved for $x = 18$ (hence $y = 18$ as well).

The numbers are 18 and 18. □

Problem V (20 points)
Show that the value of the integral $\int_0^1 \sqrt{x+8}\, dx$ lies between $2\sqrt{2}$ and 3.
Solution: The function $f(x) = \sqrt{x+8}$ is increasing on $[0, 1]$ (since $f' > 0$ on this interval), so $\max(f) = f(1) = 3$ and $\min(f) = f(0) = 2\sqrt{2}$. Therefore,

$$(1-0)\min(f) \leq \int_0^1 \sqrt{x+8}\, dx \leq 1-0)\max(f)$$

which translates into:

$$2\sqrt{2} \leq \int_0^1 \sqrt{x+8}\, dx \leq 3.$$

□