Mathematics 221, Lecture 1 **Name:** ____________________
Instructor: L. Maxim **TA's Name:** ____________________

EXAM I – ANSWER KEY

Do all six of the following problems.
Show all your work, justify your answers: answers without supporting work will not receive full credit.
It is not necessary to simplify your answers.

No.	Points		Score
1	15		
2	20		
3	15		
4	20		
5	15		
6	15		
	100	TOTAL POINTS	

Problem I (15 points) Show that the equation

$$x + 2\cos x = 0$$

has at least one solution.

Solution: Consider the function

$$f(x) = x + 2\cos x$$

f is continuous as a sum of continuous functions. Moreover, since $-1 \leq \cos x \leq 1$ for all x, we see that f takes only positive values for $x > 2$, and f takes only negative values for $x < -2$. So by the intermediate value theorem, f must take the value 0 at least once. $\square$

Problem II (20 points) The position at time $t \geq 0$ of a particle moving along a coordinate line is

$$s = 10\cos(t + \frac{\pi}{4})$$

a) What is the particle's starting position ($t = 0$) ?

Solution: $s(0) = 10\cos(\frac{\pi}{4}) = 10 \cdot \frac{\sqrt{2}}{2} = 5\sqrt{2}$.

b) What are the points farthest to the left and right of the origin reached by the particle ?

Solution: At each of the farthest points, the velocity is zero, i.e., $v(t) = 0$. Since

$$v(t) = s'(t) = -10\sin(t + \frac{\pi}{4}),$$

the farthest points are reached at times t given by solving: $\sin(t + \frac{\pi}{4}) = 0$, i.e., $t + \frac{\pi}{4}$ must be an integer multiple of π,

$$t + \frac{\pi}{4} = k\pi \ , \ k \in \mathbb{Z},$$

or, $t \in \{k\pi - \frac{\pi}{4} \mid k \in \mathbb{Z}\}$. Since we only care about $t \geq 0$, we restrict k to $k \geq 1$. So the furthest points reached by the particle are:

$$s(k\pi - \frac{\pi}{4}) = 10\cos(k\pi) = 10 \cdot (-1)^k.$$

For each $k \geq 1$ an *even* integer, $s(k\pi - \frac{\pi}{4}) = 10$ is the furthest point on the right (reached multiple times). Similarly, for each $k \geq 1$ an *odd* integer, $s(k\pi - \frac{\pi}{4}) = -10$ is the furthest point on the left.

c) Find the particle's velocity and acceleration at the points in part (b).

Solution: As already mentioned above, at each of the furthest points reached by the particle, the velocity is zero. The acceleration $a(t) = v'(t) = -10\cos(t + \frac{\pi}{4})$ at each of these points is:

$$a(k\pi - \frac{\pi}{4}) = -10\cos(k\pi) = -10 \cdot (-1)^k.$$

So the acceleration is negative at each of the furthest points on the right (for k even), and is positive at each of the furthest points on the left (for k odd).

Problem III (15 points) Evaluate the following limits:.

a) $\lim_{x\to\infty} \frac{x+\sin x+2\sqrt{x}}{x+\sin x}$

Solution:

$$\lim_{x\to\infty} \frac{x+\sin x+2\sqrt{x}}{x+\sin x} = \lim_{x\to\infty} \frac{x(1+\frac{\sin x}{x}+\frac{2}{\sqrt{x}})}{x(1+\frac{\sin x}{x})} = \lim_{x\to\infty} \frac{1+\frac{\sin x}{x}+\frac{2}{\sqrt{x}}}{1+\frac{\sin x}{x}} = 1,$$

where we use the fact that $\lim_{x\to\infty} \frac{\sin x}{x} = 0$, which is an easy application of the sandwich theorem.

b) $\lim_{x\to 0} \frac{\sin(1-\cos x)}{x}$. (Hint: use the fact proved in class that $\lim_{z\to 0} \frac{\sin(z)}{z} = 1$.)

Solution:

$$\lim_{x\to 0} \frac{\sin(1-\cos x)}{x} = \lim_{x\to 0} \frac{\sin(1-\cos x)}{1-\cos x} \cdot \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{\sin(1-\cos x)}{1-\cos x} \cdot \lim_{x\to 0} \frac{1-\cos x}{x}$$

Let $z = 1-\cos x$ and note that $z \to 0$ as $x \to 0$. So,

$$\lim_{x\to 0} \frac{\sin(1-\cos x)}{1-\cos x} = \lim_{z\to 0} \frac{\sin z}{z} = 1$$

Also,

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)} = \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{\sin x}{1+\cos x} = 1\cdot 0 = 0$$

Altogether, $\lim_{x\to 0} \frac{\sin(1-\cos x)}{x} = 0$.

c) $\lim_{x\to 0} x^{100} \cos(\frac{1}{\sqrt{x}})$. (Hint: use the sandwich theorem.)

Solution: Since $-1 \leq \cos(\frac{1}{\sqrt{x}}) \leq 1$, by multiplying by x^{100} we get that

$$-x^{100} \leq x^{100} \cos(\frac{1}{\sqrt{x}}) \leq x^{100}.$$

Finally, since $\lim_{x\to 0} -x^{100} = \lim_{x\to 0} x^{100} = 0$, we get by the *sandwich theorem* that

$$\lim_{x\to 0} x^{100} \cos(\frac{1}{\sqrt{x}}) = 0$$

Problem IV (20 points) Find the asymptotes of the graph of

$$f(x) = \frac{\sqrt{x^2+4}}{x} \ ,$$

then sketch the graph.

Solution: The domain of f is: $(-\infty, 0) \cup (0, \infty)$.

In order to find vertical asymptotes, we compute the side limits of f at $x = 0$.

$$\lim_{x\to 0^+} f(x) = \frac{2}{0^+} = \infty$$

$$\lim_{x\to 0^-} f(x) = \frac{2}{0^-} = -\infty$$

So, $x = 0$ is a vertical asymptote.

In order to find horizontal asymptotes, we compute the limits of f at $x = \infty$ and $x = -\infty$, respectively.

$$\lim_{x\to\infty} f(x) = \frac{|x|\sqrt{1+\frac{4}{x^2}}}{x} = \lim_{x\to\infty} \sqrt{1+\frac{4}{x^2}} = 1$$

$$\lim_{x\to-\infty} f(x) = \frac{|x|\sqrt{1+\frac{4}{x^2}}}{x} = - \lim_{x\to-\infty} \sqrt{1+\frac{4}{x^2}} = -1$$

So $y = 1$ is a horizontal asymptote at ∞, and $y = -1$ is a horizontal asymptote at $-\infty$..

There are no oblique asymptotes. □

Problem V (15 points) For what value (or values) of the constant m, if any, is

$$f(x) = \begin{cases} \sin 2x \ , & x \leq 0 \\ mx \ , & x > 0 \end{cases}$$

a) continuous at $x = 0$?
b) differentiable at $x = 0$?
Give reasons for your answers.

Solution:
a) f is continuous at $x = 0$ if, and only if,

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = f(0).$$

We have:

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} mx = 0,$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} \sin 2x = 0,$$

$$f(0) = \sin 0 = 0.$$

So f is continuous at $x = 0$ for all m.

b) f is differentiable at $x = 0$ if $f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$ exists. For this we need that:

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{f(h) - f(0)}{h}$$

We have:

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{mh}{h} = m.$$

$$\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{\sin 2h}{h} = \lim_{h \to 0^-} \frac{2 \sin h \cos h}{h} = 2 \cdot \lim_{h \to 0^-} \frac{\sin h}{h} \cdot \lim_{h \to 0^-} \cos h = 2.$$

So, f is differentiable at $x = 0$ if, and only if, $m = 2$. $\square$

Problem VI (15 points) Show that the tangents to the curve

$$y = \frac{\pi \sin x}{x}$$

at $x = \pi$ and $x = -\pi$ intersect at right angles.

Solution: We have to show that the slopes of the tangents at $x = \pi$ and $x = -\pi$ are *negative reciprocals*. The slope of the tangent at $x = \pi$ is $y'(\pi)$ and the slope of the tangent at $x = -\pi$ is $y'(-\pi)$, so we need to show that $y'(\pi) \cdot y'(-\pi) = -1$.

An application of the quotient rule yields:

$$y'(x) = \pi \cdot \frac{x \cos x - \sin x}{x^2}$$

So, $y'(\pi) = \pi \cdot \frac{\pi \cdot (-1) - 0}{\pi^2} = -1$ and, similarly, $y'(-\pi) = \pi \cdot \frac{-\pi \cdot (-1) - 0}{(-\pi)^2} = 1$. Thus,

$$y'(\pi) \cdot y'(-\pi) = -1.$$

Note that there is no need to write the equations of the two tangent lines explicitly. $\square$