Math 221 Spring 2010
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Solutions to Practice Exam 2

1. First, we have to show that f does have a zero in the given interval. There are two ways to go about this. One way is to 'guess' that $f(-2) = 0$. The other way is to observe that the function is continuous on the given interval: notice that f is a sum of rational functions, all of which are defined on $(-\infty, 0)$, so f itself is continuous on this interval.

   So we can apply the intermediate value theorem. Try to get a positive value out of f (remember that we can only use values of x in the domain!): plugging in -1 for x yields $f(-1) = -1 + 4 + 7 = 10$. Next try to get a negative value out of f. Plug in -3 for x: $f(-3) = -27 + (4/9) + 7$, which is negative. The IVT says that somewhere between -3 and 1 there is a value c such that $f(c) = 0$.

   Second, we have to show that f cannot have more than one zero in the given interval. Differentiating, we find that

   $$f'(x) = 3x^2 - \frac{8}{x^3}$$

   On the interval $(-\infty, 0)$, f' is always positive, since both $3x^2$ and $-8/x^3$ are positive whenever $x < 0$. So f is always increasing - it never gets to turn down towards the x axis once it crosses it, which immediately implies that f can only have one zero on this interval.

2. For convenience we'll say $y = f(x) = \frac{(x+1)^2}{1+x^2}$.

   Domain: f is defined on the whole real line, since the denominator is never 0.
   Intercepts: First note that f can't be negative, because of all the squared terms. But it does touch the x-axis when $x = -1$. Instead of crossing the axis, the curve bounces back up.
   Limits as $x \to \pm\infty$: f is a rational function where the degree of the polynomial in the numerator is the same as the degree in the bottom, so the limit in either case will be the quotient of the leading coefficients, which is 1. So $y = 1$ is a horizontal asymptote.
   First derivative: Use the quotient rule:

   $$\begin{aligned} f'(x) &= \frac{(1+x^2)2(x+1) - (x+1)^2 2x}{(1+x^2)^2} \\ &= \frac{(x+1)(2(1+x^2) - (x+1)2x)}{(1+x^2)^2} \\ &= \frac{2(x+1)(1-x)}{(1+x^2)^2} \end{aligned}$$

From this it is easy to see that the first derivative is 0 when $x = 1$ or $x = -1$.

Second derivative: Again, use the quotient rule:

$$
\begin{aligned}
f''(x) &= 2\left(\frac{(1+x^2)^2(-2x) - (1-x^2)2(1+x^2)2x}{(1+x^2)^4}\right) \\
&= 2\left(\frac{2x(1+x^2)(-(1+x^2) - 2(1-x^2))}{(1+x^2)^4}\right) \\
&= \frac{4x(1+x^2)(x^2-3)}{(1+x^2)^4} \\
&= \frac{4x(x^2-3)}{(1+x^2)^3}
\end{aligned}
$$

So the second derivative is 0 when $x = 0$ or $x = \pm\sqrt{3}$.

Table of sign patterns:

	$-\infty$		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$		∞
f	1		$\frac{(1-\sqrt{3})^2}{4}$		0		1		2		$\frac{(1+\sqrt{3})^2}{4}$		1
f'	0	-	-	-	0	+	+	+	0	-	-	-	0
f''	0	-	0	+	+	+	0	-	-	-	0	+	0

Inflection points, local and absolute extrema: From the table it's clear that there are three inflection points: $x = 0$, $x = -\sqrt{3}$ and $x = \sqrt{3}$.
The second derivative test shows us that $x = -1$ is a local minimum and $x = 1$ is a local maximum. In fact, $x = -1$ is also an absolute minimum, since $f(-1) = 0$ and we know that f can't be negative. Further, $x = 1$ is an absolute maximum: the limits at infinity are both 1 and $f(1) = 2$.
Graph: Try http://www.wolframalpha.com

3. (a) Naively taking the limit gives you the $\infty - \infty$ indeterminate form, so some algebraic manipulation is needed:

$$
\begin{aligned}
\lim_{x\to\infty}(x - \sqrt{x^2+x}) &= \lim_{x\to\infty}(x - \sqrt{x^2+x})\frac{x+\sqrt{x^2+x}}{x+\sqrt{x^2+x}} \\
&= \lim_{x\to\infty}\frac{x^2-(x^2+x)}{x+\sqrt{x^2+x}} \\
&= \lim_{x\to\infty}\frac{-x}{x+\sqrt{x^2+x}}
\end{aligned}
$$

If you try to take the limit now, you get an ∞/∞ indeterminate form. You could use L'Hospital's rule to find out what this limit is, or you could just divide throughout by x:

$$\begin{aligned}
\lim_{x\to\infty} \frac{-x}{x+\sqrt{x^2+x}} &= \lim_{x\to\infty} \frac{-\frac{x}{x}}{\frac{x}{x}+\frac{1}{x}\sqrt{x^2+x}} \\
&= \lim_{x\to\infty} \frac{-1}{1+\sqrt{\frac{x^2+x}{x^2}}} \\
&= \lim_{x\to\infty} \frac{-1}{1+\sqrt{1+\frac{1}{x}}} \\
&= -\frac{1}{2}
\end{aligned}$$

(b) Recall that cot is $1/\tan$, so you can rewrite this limit as:

$$\lim_{x\to\frac{\pi}{2}} \frac{x-\frac{\pi}{2}}{\tan(x-\frac{\pi}{2})}$$

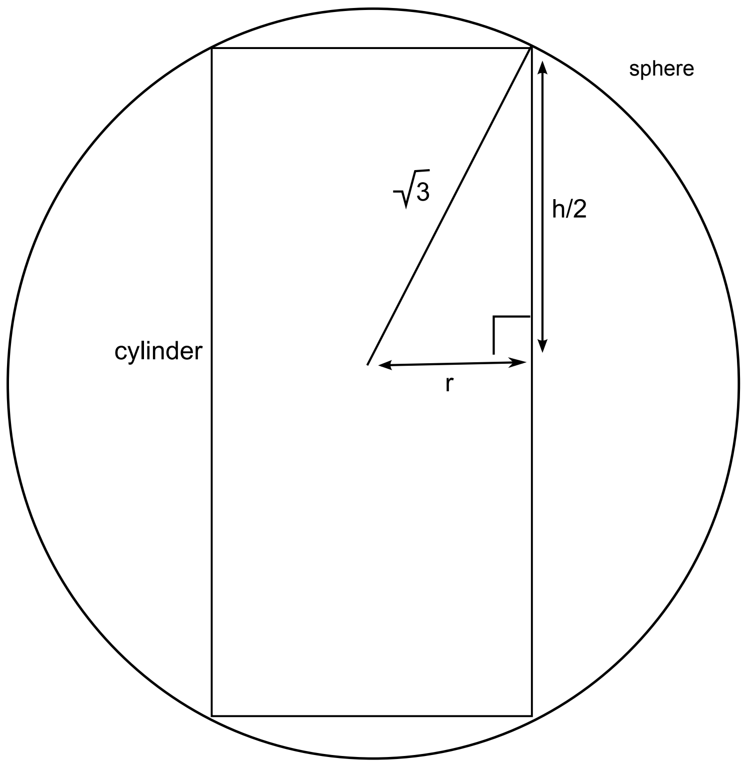
This is a $0/0$ indeterminate form, so you can apply L'Hospital's rule to obtain:

$$\lim_{x\to\frac{\pi}{2}} \frac{1}{\sec^2(x-\frac{\pi}{2})} = \lim_{x\to\frac{\pi}{2}} \cos^2(x-\frac{\pi}{2}) = 1$$

(c) It helps to remember that $\sin(\frac{\pi}{4}) = \cos(\frac{\pi}{4}) = \sqrt{2}/2$, so this is again a $0/0$ indeterminate form. Apply L'Hospital's rule to get:

$$\lim_{x\to\frac{\pi}{4}} \frac{\cos(x)+\sin(x)}{1} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$$

4. The volume of a right circular cylinder is given by the formula $V = \pi r^2 h$ where r is the radius of the base of the cylinder and h is its height. Because this cylinder is being inscribed in a sphere of radius $\sqrt{3}$, we have the following diagram:



Using the pythagorean formula we have $r^2 + \frac{h^2}{4} = 3$, so $r^2 = 3 - \frac{h^2}{4}$. Plugging this back into the formula for the volume of the cylinder, we obtain:

$$V = \pi(3 - \frac{h^2}{4})h = \pi(3h - \frac{h^3}{4})$$

Before we proceed we need to clarify what the domain for V is. Now, h can't be negative or 0 and it also has to be strictly smaller than the diameter of the sphere (otherwise, our cylinder would have 0 volume) so the domain is $(0, 2\sqrt{3})$.
Next, we differentiate V with respect to h:

$$V' = \pi(3 - \frac{3h^2}{4}) = \frac{3}{4}\pi(4 - h^2)$$

Setting $V' = 0$, we find that the only solution in our domain is $h = 2$. Checking that this is indeed a maximum is straightforward: When $h < 2$, V' is positive, and so V is increasing on the interval $(0, 2)$. When $h > 2$. V' is negative, so V is decreasing on the interval $(2, 2\sqrt{3})$. This shows that $h = 2$ yields an absolute maximum for the volume on our domain. Plugging this value into the equation relating h and r, we find that $r = \sqrt{2}$.

5. (a) Let $u = 1 + x^{3/2}$. Then $du = \frac{3}{2}x^{\frac{1}{2}}\,dx$ and we can rewrite the integral as:

$$\int_{x=0}^{x=1} \frac{10(\frac{2}{3})}{u^2}\,du$$

Changing the limits so that they're now in terms of u we get:

$$\int_{u=1}^{u=2} \frac{20}{3u^2}\,du = \left[\frac{-20}{3u}\right]_1^2 = \frac{20}{3} - \frac{20}{6} = \frac{10}{3}$$

(b) Let $u = 3 + 2\cos(x)$. Then $du = -2\sin(x)dx$. The integral can now be rewritten as:

$$\int_{x=0}^{x=\pi/2} -\frac{1}{2u^2}\,du$$

Changing the limits so that they're now in terms of u we get:

$$\int_{u=5}^{u=3} -\frac{1}{2u^2}\,du = \left[\frac{1}{2u}\right]_5^3 = \frac{1}{6} - \frac{1}{10} = \frac{1}{15}$$

(c) Let $f(x) = x\sqrt{1-x^2}$. Then $f(-x) = -x\sqrt{1-x^2} = -f(x)$, so f is an odd function. We're integrating it on a symmetric interval, so the result will be equal to 0. If you fail to notice this about f, then you can get the same answer using a substitution:
Let $u = 1 - x^2$. Then $du = -2x\,dx$ and the integral can be rewritten as:

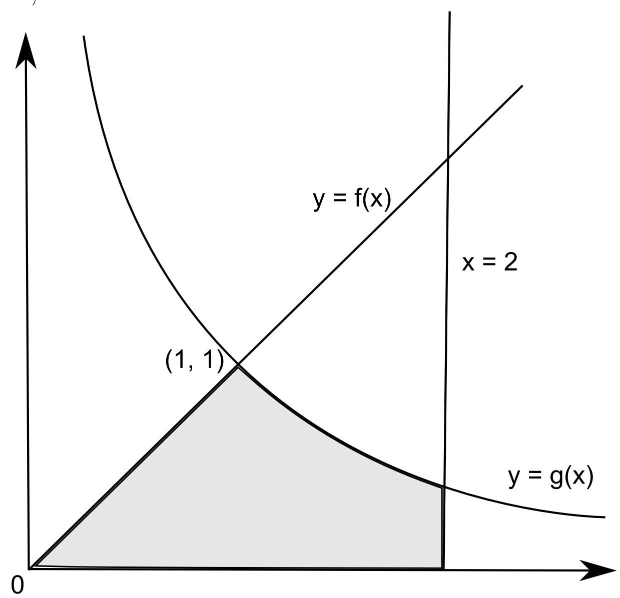
$$\int_{x=-1}^{x=1} -\frac{1}{2}\sqrt{u}\,du$$

Changing the limits so that they're in terms of u we get:

$$\int_0^0 -\frac{1}{2}\sqrt{u}\,du = 0$$

since the upper and lower limits of integration are the same.

6. It's helpful to sketch a picture of what's going on with these curves in the first quadrant. Let $f(x) = x$ and $g(x) = 1/x^2$. Setting $f(x) = g(x)$ and solving for x, we find that the two curves intersect at only one point, $(1, 1)$.



To compute the area of the shaded region, we need to evaluate two integrals:

$$\int_0^1 f(x)\,dx + \int_1^2 g(x)\,dx = \left[\frac{x^2}{2}\right]_0^1 + \left[-\frac{1}{x}\right]_1^2 = \frac{1}{2} + \frac{1}{2} = 1$$