Smoothing estimates for nonlinear dispersive PDE

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Joint work with N. Tzirakis
For manuscripts see http://www.math.uiuc.edu/∼berdogan/

Consider the nonlinear Schrödinger equation (NLS) on $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$:

$$\begin{aligned} &iu_t + u_{xx} \pm |u|^2 u = 0, \quad x \in \mathbb{T}, t \in \mathbb{R}, \\ &u(\cdot, 0) = g(\cdot) \in H^s(\mathbb{T}). \end{aligned}$$

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$$u(\cdot, 0) = g(\cdot) \in H^s(\mathbb{T}).$$

• Satisfies the mass and energy conservation:

$$\|u\|_{L^2} = \|g\|_{L^2},$$

$$E(t) := \frac{1}{2}\|\partial_x u\|_{L^2}^2 \mp \frac{1}{4}\|u\|_{L^4}^4 = E(0).$$

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• Closes the gap between scaling $q = \infty$ and wellposedness $q = 2$.

• In addition, the solutions exhibit the following smoothing property:
For any $p > q/3$ and $p \geq 1$,

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• H^s smoothing on $\mathbb{R}^n$ setting: Bourgain '98, 2d cubic NLS: For H^s data, $s_0 < s < 1$, the nonlinear part of the evolution is in H^1. Extension to other dimensions by Keraani-Vargas '09.

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- Ignoring the resonant terms: $n^2 - j^2 + k^2 - \ell^2 = 0$

$$= \frac{1}{2} \sum_{j-k+\ell=n}^{*} \frac{a_j(0)\overline{a_k(0)}a_\ell(0)}{(n-j)(n-\ell)} (e^{i(n^2-j^2+k^2-\ell^2)t} - 1) + \cdots .$$

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- If $a_j \in \ell^q$ for some $q < 3$, say, then the sum is in ℓ^1_n.
- "$\cdots$" is the real difficulty. Repeat the process infinitely many times!

Smoothing properties of nonlinear dispersive PDE on $\mathbb{T}$ or on $\mathbb{R}^+$

$$\begin{cases} u_t + L(u) + N(u) = 0, \quad x \in \mathbb{T}, \quad t \in \mathbb{R}, \\ u(x,0) = g(x) \in H^s(\mathbb{T}). \end{cases}$$

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• Applications of smoothing: Growth bounds for H^s norms, existence of global attractors, Talbot effect (fractal solutions).

Theorem (E., Tzirakis '13)

Fix $s > 0$. Assume that $g \in H^s(\mathbb{T})$. Then for any $a < \min(2s, 1/2)$,

$$u(x,t) - e^{it(\partial_{xx}+P)}g \in C^0_{t\in\mathbb{R}} H^{s+a}_{x\in\mathbb{T}}, \quad P = \|g\|_2^2/\pi.$$

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- The proof uses $X^{s,b}$ spaces:

$$\|u\|_{X^{s,b}} = \Big\| \langle k\rangle^s \langle \tau + k^2\rangle^b \, \widehat{u}(k,\tau) \Big\|_{\ell^2_k L^2_\tau} = \|e^{-it\partial_{xx}} u\|_{H^s_x H^b_t}.$$

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- For $b > \frac{1}{2}$, $X^{s,b} \subset C^0_t H^s_x$.
- For fixed $s > 0$, $a < \min(2s, \frac{1}{2})$, $0 < b - \frac{1}{2}$ (sufficiently small)

$$\Big\| \int_0^t e^{i(t-t')\partial_{xx}} R(u(t')) dt' \Big\|_{X^{s+a,b}} \lesssim \|R(u)\|_{X^{s+a,b-1}} \lesssim \|u\|^3_{X^{s,b}},$$

where $R(u) = \big(|u|^2 - \frac{1}{\pi}\|u\|^2_{L^2}\big) u$.

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$$iu_t + (-\Delta)^\alpha u + |u|^2 u = 0, \quad u(x,0) = g(x) \in H^s(\mathbb{T}).$$

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• E.–Tzirakis '13: $\min(2s+1, 1)$-smoothing for KdV on $\mathbb{T}$:

$$u_t + u_{xxx} + uu_x = 0, \;\; u(0,\cdot) = g(\cdot) \in H^s(\mathbb{T}), \;\; s > -\frac{1}{2}.$$

Requires a normal form transform and a trilinear $X^{s,b}$ space estimate for the modified nonlinearity.

Inverse scattering methods

• Kappeler–Schaad–Topalov '15: defocusing NLS on $\mathbb{T}$. For $g \in H^k$, $k \geq 1$ integer,

$$\|u - e^{itL} g\|_{H^{k+1}} \leq C(\|g\|_{H^k}), \;\; t \in \mathbb{R}$$

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• Implies that if $g \in H^s$ for some $s \geq 1$, then $\|u\|_{H^s} \lesssim 1$ for all times.

• They also have 1-smoothing for $u(x,t) - e^{i(\partial_{xx}+P)t}g$ when $k \geq 2$.

NLS on $\mathbb{R}^+$

Consider the following initial-boundary value problem

$$
\begin{aligned}
&iu_t + u_{xx} \pm |u|^2 u = 0, \quad x \in \mathbb{R}^+, t \in \mathbb{R}^+, \\
&u(x,0) = g(x), \quad u(0,t) = h(t).
\end{aligned} \tag{1}
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Here $g \in H^s(\mathbb{R}^+)$ and $h \in H^{\frac{2s+1}{4}}(\mathbb{R}^+)$, $s \in [0, \frac{5}{2}) \backslash \{\frac{1}{2}, \frac{3}{2}\}$, with the additional compatibility condition $g(0) = h(0)$ for $s > \frac{1}{2}$.

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• Kato smoothing:

$$\left\|\eta(t)e^{it\partial_{xx}} g\right\|_{L^\infty_x H_t^{\frac{2s+1}{4}}} \lesssim \|g\|_{H^s(\mathbb{R})}.$$

Brief history

• Carroll–Bu '91: Unique global $H^2(\mathbb{R}^+)$ solutions for $g \in H^2$ and $h \in C^2$.

• Fokas '02: complete integrability methods for smooth g and h.

• Colliander–Kenig '02, Holmer '05: KdV using $X^{s,b}$, converting boundary data to forcing.

• Holmer '05: NLS on $\mathbb{R}^+$, $s \geq 0$, using Colliander–Kenig method and Strichartz.

• Bona–Sun–Zhang '15: Laplace transform method and Strichartz for NLS on $\mathbb{R}^+$, $s \geq 0$. Global in H^1.

To construct the solutions of (1), first consider the linear problem:

$$iv_t + v_{xx} = 0, \quad x \in \mathbb{R}^+, t \in \mathbb{R}^+, \tag{2}$$
$$v(x,0) = g(x) \in H^s(\mathbb{R}^+), \quad v(0,t) = h(t) \in H^{\frac{2s+1}{4}}(\mathbb{R}^+).$$

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$$v = W_0^t(g,h) = W_0^t(0, h-p) + e^{it\partial_{xx}} g_e,$$

where g_e is an H^s extension of g to $\mathbb{R}$, and $p(t) = e^{it\partial_{xx}} g_e\big|_{x=0}$, which is locally in $H^{\frac{2s+1}{4}}(\mathbb{R}^+)$ by Kato smoothing.

Here $W_0^t(0,h)$ denotes the solution of (2) when $g = 0$.

Using Laplace transform (c.f. Bona–Sun–Zhang), we have $W_0^t(0,h) = W_1 h + W_2 h$, where

$$W_1 h(x,t) = \frac{1}{\pi}\int_0^\infty e^{-i\beta^2 t + i\beta x}\beta \widehat{h}(-\beta^2)d\beta,$$

$$W_2 h(x,t) = \frac{1}{\pi}\int_0^\infty e^{i\beta^2 t - \beta x}\rho(\beta x)\beta \widehat{h}(\beta^2)d\beta,$$

$\rho(x)$: a smooth function supported on $(-2,\infty)$, $\rho(x) = 1$ for $x > 0$.

$$\widehat{h}(\xi) = \mathcal{F}\big(\chi_{(0,\infty)}h\big)(\xi) = \int_0^\infty e^{-i\xi t}h(t)dt.$$

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$W_0^t(0,h)$ is well-defined for $x \in \mathbb{R}$, but satisfies linear Schrödinger for $x > 0$.

Duhamel's formula for (1).

$$u(t) = W_0^t(g,h) + \int_0^t e^{i(t-t')\partial_{xx}} |u|^2 u \, dt' - W_0^t(0,q)(t), \tag{3}$$

$$q(t) = \int_0^t e^{i(t-t')\partial_{xx}} |u|^2 u \, dt' \Big|_{x=0}.$$

Solve (3) on $\mathbb{R}$. The restriction of the solution to $\mathbb{R}^+$ satisfies NLS.

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Fixed point argument on $X^{s,b}$ on $\mathbb{R}$ for $b < \frac{1}{2}$, $s > 0$.

In addition, the solution is in

$$C_t^0 H_x^s \cap C_x^0 H_t^{\frac{2s+1}{4}}.$$

$X^{s,b}$ estimates ($0 \le b < \frac{1}{2}$)

$$\left\| \eta(t) \int_0^t e^{i(t-t')\partial_{xx}} F(t') dt' \right\|_{X^{s,b}} \lesssim \|F\|_{X^{s,-b}}, \quad s \in \mathbb{R},$$

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$$\left\| \eta(t) W_0^t(g,h) \right\|_{X^{s,b}} \lesssim \|g\|_{H^s(\mathbb{R}^+)} + \|h\|_{H_t^{\frac{2s+1}{4}}(\mathbb{R}^+)}, \quad s \ge 0,$$

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$$\left\|\eta(t)\int_0^t e^{i(t-t')\partial_{xx}}Fdt'\right\|_{C_x^0 H_t^{\frac{2s+1}{4}}} \lesssim \begin{cases} \|F\|_{X^{s,-b}} & 0 \le s \le 1/2, \\ \|F\|_{X^{\frac{1}{2},\frac{2s-1-4b}{4}}} + \|F\|_{X^{s,-b}} & 1/2 \le s \le 5/2. \end{cases}$$

• For fixed $0 < s < \frac{5}{2}$, and $0 \leq a < \min(2s, \frac{1}{2}, \frac{5}{2} - s)$, there exists $\epsilon > 0$ such that for $\frac{1}{2} - \epsilon < b < \frac{1}{2}$, we have

$$\big\| |u|^2 u \big\|_{X^{s+a,-b}} \lesssim \|u\|^3_{X^{s,b}},$$
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Theorem (E., Tzirakis '15)

Fix $s \in (0, \frac{5}{2})$, $g \in H^s(\mathbb{R}^+)$, and $h \in H^{\frac{2s+1}{4}}(\mathbb{R}^+)$, with the additional compatibility condition $g(0) = h(0)$ for $s > \frac{1}{2}$. Then,

$$u(x,t) - W_0^t(g,h) \in C_t^0 H_x^{s+a}, \quad a < \min(2s, \frac{1}{2}, \frac{5}{2} - s).$$

Energy identities

Recall that on $\mathbb{R}$: $\|u\|_{L^2} = \|g\|_{L^2}$, and

$$E(t) := \frac{1}{2}\|\partial_x u\|_{L^2}^2 \mp \frac{1}{4}\|u\|_{L^4}^4 = E(0).$$

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The following provide a priori bounds for the H^1 norm of the solution, bounded in the defocusing, and exponential in the focusing case.

$$\partial_t |u|^2 = -2\Im(u_x \overline{u})_x,$$

$$\partial_t (|u_x|^2 \mp \frac{1}{2}|u|^4) = 2\Re(u_x \overline{u_t})_x,$$

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• Bona–Sun–Zhang '15. Solution is global in H^1 if $g, h \in H^1$.

Theorem (E., Tzirakis '15)

In the case $s \in [1, \frac{5}{2})$, $g \in H^s(\mathbb{R}^+)$, and $h \in H^{\frac{2s+1}{4}}(\mathbb{R}^+) \cap H^1(\mathbb{R}^+)$, the solution u is global and the smoothing statement holds for all times. Moreover, in the defocusing case $\|u\|_{H^s(\mathbb{R}^+)}$ grows at most polynomially, whereas in the focusing case it grows at most exponentially.

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- In preparation: analogous results for the Zakharov system on $\mathbb{R}^+$.

Talbot effect

Berry et al, starting from 80s:

Linear Schrödinger on $\mathbb{T}$

$$iu_t + u_{xx} = 0, \quad u(0,x) = g(x) \in L^2(\mathbb{T}).$$

$$u(t,x) = e^{it\partial_{xx}} g = \sum_{k=-\infty}^{\infty} e^{-itk^2} \widehat{g}(k) e^{ikx}.$$

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Observations/conjectures:

• Quantization: At $t = 2\pi\frac{p}{q}$, the solution is a linear combination of up to q translates of the initial data.

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• The fractal dimension is $\frac{3}{2}$ even if there is a nonlinear perturbation.

- Quantization: Berry and Klein (Also M. Taylor): For $t = 2\pi\frac{p}{q}$

$$e^{it\partial_{xx}} g = \frac{1}{q} \sum_{j=0}^{q-1} G_{p,q}(j) g\big(x - 2\pi \frac{j}{q}\big).$$

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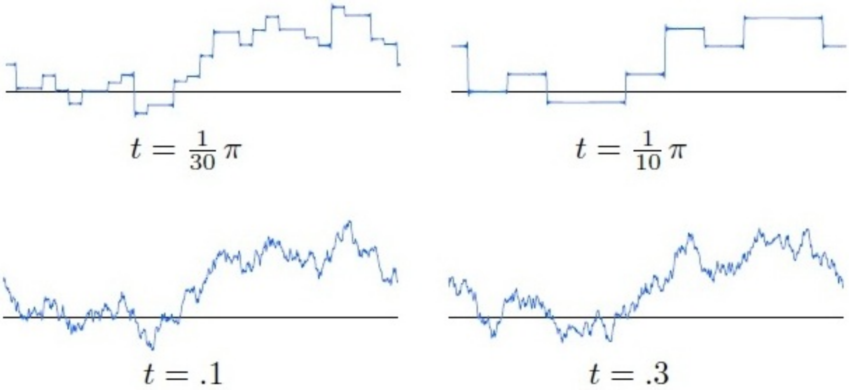
- Follows from the distributional identity:

$$e^{2\pi i\frac{p}{q}\partial_{xx}}\delta = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{-2\pi i k^2\frac{p}{q}} e^{ikx} = \frac{1}{2\pi}\sum_{l=0}^{q-1} e^{-2\pi i l^2\frac{p}{q}} e^{ilx}\sum_{j=-\infty}^{\infty} e^{iqjx}$$

$$= \sum_{l=0}^{q-1} e^{-2\pi i l^2\frac{p}{q}} e^{ilx}\frac{1}{q}\sum_{j=0}^{q-1}\delta\big(x - 2\pi\frac{j}{q}\big)$$

$$= \frac{1}{q}\sum_{j=0}^{q-1}\Big[\sum_{l=0}^{q-1} e^{-2\pi i l^2\frac{p}{q}} e^{2\pi i l\frac{j}{q}}\Big]\delta\big(x - 2\pi\frac{j}{q}\big).$$

Talbot effect for linear KdV on the torus



Evolved data at the given time for the initial data $\chi_{[0,\pi]}$.
Figure from Chen–Olver, *Dispersion of discontinuous periodic waves* '12, http://www.math.umn.edu/∼olver

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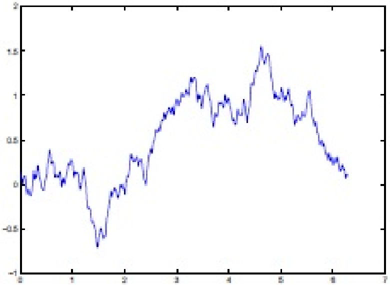
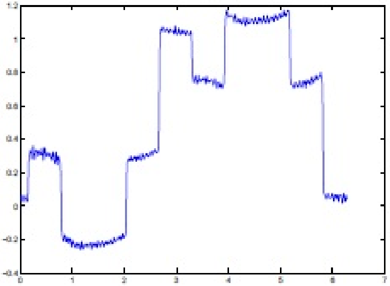


$t = .3$



$t = .1\pi$

Evolved data for KdV on $[-\pi, \pi]$ with initial data $\chi_{[0,\pi]}$.
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• Choussionis–E.–Tzirakis '15: Dimension of the graph of linear KdV and KdV evolution on $\mathbb{T}$ is in $[5/4, 7/4]$ for almost every t.

Vortex Filaments (Choussionis–E.–Tzirakis ’15):

• Approximation of the dynamics of a vortex filament under the Euler equations:

$$\gamma_t = \gamma_x \times \gamma_{xx} = \kappa b, \quad x \text{ arclength parameter}, t \text{ time}.$$

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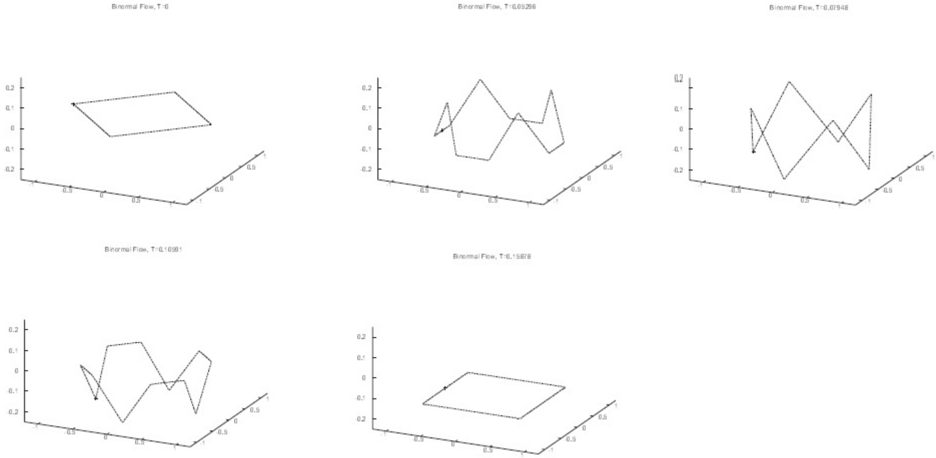
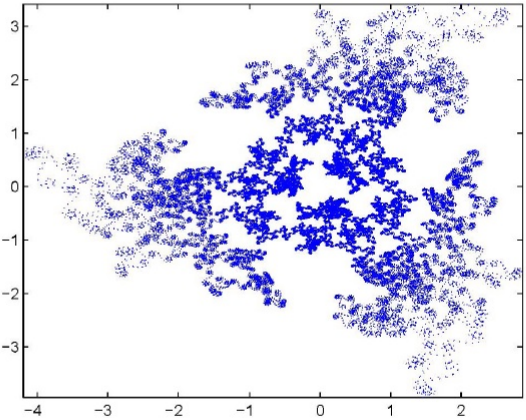


Vortex filaments at rational times (initial data = square)
Figure from Jerrard and Smets, *On the motion of a curve by its binormal curvature*, arXiv:1109.5483v1



Stereographic projection of the unit tangent vector at an irrational time (initial data = equilateral triangle)
Figure from de la Hoz and Vega, *Vortex filament equation for a regular polygon*, arXiv:1304.5521v1

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q satisfies NLS:

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• We have fractal solutions of NLS in $H^{\frac{1}{2}-}$ or in H^s level, $s \in (\frac{1}{2}, \frac{3}{4})$.

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Following Chang-Shatah-Uhlenbeck, Nahmod-Shatah-Vega-Zeng we obtain:

• Global strong solutions of the SM for $u(0,\cdot) \in H^s(\mathbb{T})$, $s > \frac{3}{2}$, and identity holonomy.

• Talbot effect gives fractal curves that evolve for smoother initial curves; $u(0,\cdot) \in H^s(\mathbb{T})$, $s \in (\frac{3}{2}, \frac{7}{4})$. In particular the components of the curvature vector u_x in the frame are fractal curves.

Global attractors

Smoothing results can be extended to forced and weakly damped equations. Forced and weakly damped KdV on $\mathbb{T}$:

$$u_t + u_{xxx} + \gamma u + uu_x = f, \quad t \in \mathbb{R},\ x \in \mathbb{T},$$
$$u(x,0) = g(x) \in \dot{L}^2(\mathbb{T}) := \{h \in L^2(\mathbb{T}) : \widehat{h}(0) = 0\},$$

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$\gamma > 0$ and $f \in \dot{L}^2$.

For $t > T = T(\gamma, \|g\|, \|f\|)$, we have $\|u(t)\| < 2\|f\|/\gamma$.

$B(0, 2\|f\|/\gamma) \subset L^2(\mathbb{T})$ is called an absorbing set.

Definition

A Global Attractor for a semigroup $\{U(t)\}_{t\geq 0}$ on a Hilbert space H is a compact set $\mathcal{A} \subset H$ which is invariant under the flow and which attracts all solutions:

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• Existence and regularity of the global attractor for forced damped KdV: Ball, Ghidaglia, Goubet, Rosa, Tsugawa.

Theorem (E., Tzirakis, '13)

Fix $s \in (0,1)$. Consider the forced and weakly damped KdV equation on $\mathbb{T} \times \mathbb{R}$ with $u(x,0) = g(x) \in \dot{L}^2$. Then

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Gives a simple proof of the existence of the attractor in $\dot{L}^2$. Analogously for the Zakharov system in the energy space $H^1 \times L^2 \times H^{-1}$.