Some Recent Results on Possible Spectra Of Computable Models of Strongly Minimal Theories in Infinite Binary Relational Languages

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(Joint work with Uri Andrews)

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Recall that a theory is *strongly minimal* if all subsets definable (with parameters) in all its models are finite or cofinite, and that any strongly minimal theory is $\aleph_1$-categorical.

The following theorem describes all the countable models of T:

Theorem (Baldwin/Lachlan 1971)

The countable models of any $\aleph_1$-categorical but not totally categorical theory T in any countable language form an elementary chain

$$\mathcal{M}_0 \prec \mathcal{M}_1 \prec \ldots \prec \mathcal{M}_\omega$$

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Definition

The *spectrum of computable models* of an $\aleph_1$-categorical but not totally categorical theory T in any computable language is

$$\mathrm{SCM}(T) = \{\alpha \leq \omega \mid \mathcal{M}_\alpha \text{ is computable}\}.$$

Theorem

The following is a complete list of all known spectra of computable models of strongly minimal (and indeed all $\aleph_1$-categorical) theories:

- $\emptyset$ and $\omega + 1$ (trivial)
- $\{0\}$ (Goncharov 1978) and $[0, n]$ ($n \in \omega$, Kudaibergenov 1980)

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Theorem (Nies 1999)

Any spectrum of computable models of a strongly minimal (or indeed any $\aleph_1$-categorical) theory is a $\Sigma^0_{\omega+3}$-set.

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- $\{0\}$ (Herwig/Lempp/Ziegler 1999)
- $[0, \alpha)$ ($\alpha \leq \omega$) and $\{\omega\}$ (Andrews 2011)

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Definition

A theory T is *disintegrated* (for strongly minimal T) if for any subset A of any model $\mathcal{M}$,

$$\mathrm{acl}(A) = \bigcup_{a \in A} \mathrm{acl}(\{a\}).$$

Theorem (Goncharov/Harizanov/Laskowski/Lempp/McCoy 2003)

A strongly minimal disintegrated theory T is model complete in the language $\mathcal{L}_M$ (expanded by constants for any model $\mathcal{M}$ of T).

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For any strongly minimal disintegrated theory T, if one countable model of T is computable then all are $\mathbf{0}''$-decidable.

Theorem (Khoussainov/Laskowski/Lempp/Solomon 2007)

There is a strongly minimal disintegrated theory T with a computable prime model such that (the open diagrams of) all the other countable models of T compute $\mathbf{0}''$.

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If T is a strongly minimal disintegrated theory in a *finite* language $\mathcal{L}$, then the only possibilities for the spectrum of computable models are $\omega + 1$, $\emptyset$ and $\{0\}$.

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In addition to disintegrated theories, the result of Andrews/Medvedev also extends to locally modular expansions of a group and, by Poizat (1988), to field-like theories.

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Theorem (Andrews/Lempp, in progress)

If T is a strongly minimal disintegrated theory in a (possibly infinite) *binary relational* language $\mathcal{L}$, then the only possibilities for the spectrum of computable models are the following seven sets: $\omega + 1$, $\emptyset$, $\{0\}$, $\{1\}$, $\{0, 1\}$, $\{\omega\}$, and $[1, \omega]$.

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Conjecture

If T is a strongly minimal disintegrated theory in a (possibly infinite) relational language $\mathcal{L}$ of arity at most n, then there are only finitely many possible spectra of computable models.

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We have some preliminary partial results in this direction for ternary relational languages.

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Next, define $A_i = \{m \in M_k \mid \exists^\infty x\, R_i(m, x)\}$. Using the above, we have that each A_i is a finite subset of $\mathrm{acl}_{\mathcal{M}_k}(\emptyset)$. Also, $m \in A_i$ iff $\mathcal{M}_k \models R_i(m, a_1) \wedge R_i(m, a_2)$; so each A_i is uniformly computable.

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Now, $\mathcal{M}_k$ is the disjoint union of $\mathrm{acl}_{\mathcal{M}_k}(\emptyset)$ and k many "connected components" $\mathrm{iacl}(a_i)$ (neglecting R_i-relations to A_i) for a generic k-tuple $\overline{a}$, and each $\mathrm{iacl}(a_i)$ is Σ^0_1;

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We are now down to ten possible spectra: $\omega + 1$, $\emptyset$, $\{0\}$, $\{1\}$, $\{0, 1\}$, $\{\omega\}$, $[1, \omega]$, as well as $\{0, \omega\}$, $\{1, \omega\}$, and $\{0, 1, \omega\}$.

Step 3: If $\mathcal{M}_\omega$ and one of $\mathcal{M} = \mathcal{M}_0$ or $\mathcal{M} = \mathcal{M}_1$ are computable, then $\mathcal{M}_2$ is computable (showing the last three spectra to be impossible):

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Otherwise, $\mathrm{acl}_{\mathcal{M}_\omega}(\emptyset)$ is just the "connected component" of the corresponding c' in M_ω, so we can build $\mathcal{M}_2$ as the disjoint union of the Σ^0_1-sets $\mathrm{acl}_{\mathcal{M}_\omega}(\emptyset)$, $\mathrm{iacl}_{\mathcal{M}_\omega}(a_1)$, and $\mathrm{iacl}_{\mathcal{M}_\omega}(a_2)$.

Thanks!