# On the Order Dimension of and Embeddings into the Turing Degrees

Steffen Lempp, University of Wisconsin-Madison

http://www.math.wisc.edu/∼lempp

(Joint work with Kojiro Higuchi, Dilip Raghavan and Frank Stephan)

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**Question (Sacks 1963)**

Are the Turing degrees universal as a locally countable partial order of size continuum? I.e., does every locally countable partial order embed into the Turing degrees?

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**Question (Higuchi 2016)**

What is the order dimension of the Turing degrees viewed as a partial ordering?

The answer to Higuchi's question will turn out to be independent of ZFC (combining our work with work of A. Kumar and Raghavan).

**Definition**

A partial order $\mathcal{P}$ is *locally finite/countable* if for all $a \in P$, the set $\{b \in P \mid b <_P a\}$ is finite/countable, respectively.

Locally finite/countable partial orders are sometimes also said to have the *finite/countable predecessor property*, respectively.

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Let $\mathcal{P} = (P, <_P)$ be a partial order. The *order dimension* of $\mathcal{P}$ is the size of the smallest family $\{<_i \mid i < \kappa\}$ of linearizations (linear extensions) of $<_P$ whose intersection is $<_P$.

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Clearly, the order dimension of an infinite partial order $\mathcal{P}$ is at most $|[P]^2| = |P|$.
But it can be much lower, e.g., 1 (for a linear order $\mathcal{P}$) or 2 (for an antichain $\mathcal{P}$ of size $> 1$).

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**Theorem (Sacks 1961)**

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**Theorem (Sacks 1961)**

1. Any locally countable partial order of size at most $\aleph_1$ embeds into the Turing degrees.
2. Any locally countable partial order of size at most continuum such that any element has at most $\aleph_1$ successors embeds into the Turing degrees.
3. Any locally *finite* partial order of size at most continuum embeds into the Turing degrees.

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3. Any locally *finite* partial order of size at most continuum embeds into the Turing degrees.

Sacks (1963) conjectured the answer to his original question to be yes, but there has been no real progress since 1961!

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But, to showcase our ignorance, we don't even know whether every locally countable *well-founded* partial order of size continuum embeds into the Turing degrees.

A very weak result in this direction is the following

**Observation (Higuchi)**

The universal locally countable partial order of size continuum in which any chain contains at most two elements embeds into the Turing degrees.

To my knowledge, this was not known before, but his proof does not seem to generalize to well-founded partial orders of higher height.

For initial segment embeddings, a bit more is known; the best known positive result, after many earlier results, is the following

**Theorem (Abraham, Shore 1986)**

Every locally countable upper semilattice of size at most $\aleph_1$ embeds into the Turing degrees as an initial segment.

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**Theorem (Abraham, Shore 1986)**

Every locally countable upper semilattice of size at most $\aleph_1$ embeds into the Turing degrees as an initial segment.

However, without the Continuum Hypothesis, this result does not generalize from $\aleph_1$ to size continuum by the following

**Theorem (Groszek, Slaman 1983)**

It is consistent with ZFC that $2^{\aleph_0} > \aleph_1$ and that there is a locally finite upper semilattice of size $\aleph_2$ which does not embed into the Turing degrees as an initial segment.

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Proof: Call a subset $S$ of a partial order $\mathcal{P}$ a *strongly independent antichain* if for any subset $T \subset S$ with $|T| < |S|$ and any $x \in S \setminus T$, there is an upper bound $y \in P$ of $T$ such that $x \not\leq_P y$.

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$$\mathcal{P} = \Big(\{\{\alpha\} \mid \alpha < \omega_1\} \cup \{\{\gamma : \gamma < \beta\} \setminus \{\alpha\} \mid \alpha, \beta < \omega_1\}, \subset\Big),$$

in which $S = \{\{\alpha\} \mid \alpha < \omega_1\}$ forms a strongly independent antichain.

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in which $S = \{\{\alpha\} \mid \alpha < \omega_1\}$ forms a strongly independent antichain.
His technical lemma states that the existence of a strongly independent antichain $S$ implies that the order dimension is $\geq |S|$.

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Thus there is some $j_0 < \rho$ such that $y_i <_{j_0} x$ for unboundedly many $i < \rho^+$, and so $z <_{j_0} x$ for all $z \in U \setminus \{x\}$.
Similarly, we can do this for distinct $x, x' \in U$, obtaining a single $j_0 < \rho$ such that $x <_{j_0} x' <_{j_0} x$, a contradiction.

Our main result on the order dimension of the Turing degrees is the following

**Theorem (Higuchi, Lempp, Raghavan, Stephan)**

The order dimension of the Turing degrees is at least $\aleph_1$ but consistently $< 2^{\aleph_0}$: It can equal $2^{\aleph_0}$ only if $2^{\aleph_0}$ is $\aleph_1$, a limit cardinal, or the successor of a cardinal of countable cofinality.

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More specifically, they built a forcing extension of a model of ZFC where $\kappa$ is a measurable cardinal before forcing, and after forcing, with all cardinals preserved, $\kappa$ is weakly inaccessible and equals both $2^{\aleph_0}$ and the order dimension of the Turing degrees.

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More specifically, they built a forcing extension of a model of ZFC where $\kappa$ is a measurable cardinal before forcing, and after forcing, with all cardinals preserved, $\kappa$ is weakly inaccessible and equals both $2^{\aleph_0}$ and the order dimension of the Turing degrees.

Thus whether the order dimension of the Turing degrees equals the continuum, or $\aleph_1$, or lies in between, is independent of ZFC.

We also considered other degree structures:

**Theorem (Higuchi, Lempp, Raghavan, Stephan)**

1. The order dimension of the Muchnik degrees is $2^{\aleph_0}$.
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The order dimension of the collection of downward closed subsets of a partial order $\mathcal{P}$ is the *chain covering number* of $\mathcal{P}$, i.e., the least size of a collection of chains in $\mathcal{P}$ covering all of $P$.

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The chain covering number of the Turing degrees is clearly $2^{\aleph_0}$.

Our main result on the order dimension is really a set-theoretic result about the order dimension of any locally countable partial order:

**Theorem (Higuchi, Lempp, Raghavan, Stephan)**

Let $\kappa$ be an uncountable cardinal of uncountable cofinality, and let $\mathcal{P} = (P, <_P)$ be a locally countable partial order of size $\kappa^+$. Then $\mathcal{P}$ has order dimension at most $\kappa$.

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We can now apply this theorem to give the upper bound on the order dimension of the Turing degrees stated before.

The proof of this theorem requires some techniques of Erdős-Rado and Hausdorff, which we will try to sketch now.

We restrict ourselves here to the case of a regular cardinal $\kappa$:

**Lemma 1**

There is a sequence $\langle f_\alpha : \alpha < \kappa \rangle$ of functions $f_\alpha : P \to 2$ such that for any countable subset $A \subseteq P$ and any $y \in P \setminus A$, there is $\alpha < \kappa$ such that $f''_\alpha A = \{0\}$ and $f_\alpha(y) = 1$.

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**Lemma 2**

Let $\alpha < \kappa$. Then there is a linearization $<_\alpha$ of $<_P$ such that for any $x, y \in P$, if $f_\alpha(u) = 1$ for some $u \in P$ with $u \leq_P x$ and $f_\alpha(z) = 0$ for all $z \in P$ with $z \leq_P y$, then $y <_\alpha x$.

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Proof of Theorem: Suppose $x \not\leq_P y$. Apply Lemma 1 to the countable set $A = \{z \in P \mid z \leq_P y\}$ to obtain a function $f_\alpha$; then apply Lemma 2 to obtain a linearization $<_\alpha$ with $y <_\alpha x$.

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Claim 1: There is a sequence $\langle E_\xi : \xi < \kappa^+ \rangle$ of subsets of $\kappa$ such that for any countable set $B \subseteq \kappa^+$ and any $\zeta \in \kappa^+ \setminus B$, there is a finite set $F \subseteq \kappa$ such that for all $\xi \in B$, $E_\xi \cap F \neq E_\zeta \cap F$.

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Claim 3: The sequence $\langle f_\alpha : \alpha < \kappa \rangle$ exists as claimed.

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So toward a contradiction, define $x <'' y$ iff $x <_P y$ or $x <'_\alpha y$, and assume that there is a finite sequence $x_0 <'' x_1 <'' \cdots <'' x_n = x_0$.

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$<_P$ is irreflexive, so by a cyclic permutation assume that $x_0 <'_\alpha x_1$.

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Note that $<'_\alpha$ is irreflexive. We now only need to show that the transitive closure of $<_P \cup <'_\alpha$ on $P$ is irreflexive.
So toward a contradiction, define $x <'' y$ iff $x <_P y$ or $x <'_\alpha y$, and assume that there is a finite sequence $x_0 <'' x_1 <'' \cdots <'' x_n = x_0$.
$<_P$ is irreflexive, so by a cyclic permutation assume that $x_0 <'_\alpha x_1$.
If $x_i <'_\alpha x_{i+1} <_P x_{i+2}$ then $x_i <'_\alpha x_{i+2}$, so we can delete $x_{i+1}$.

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Now $<'_\alpha$ is irreflexive, so $x_0 <'_\alpha x_1 <'_\alpha x_2$. So on the one hand, $f_\alpha(u) = 1$ for some $u \leq_P x_1$ by $x_0 <'_\alpha x_1$; on the other hand, $f_\alpha(u) = 0$ for all $u \leq_P x_1$ by $x_1 <'_\alpha x_2$, a contradiction.

Thanks!

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By well-known arguments, there is an almost disjoint family $\langle E_\xi : \xi < \kappa^+ \rangle$ of subsets of $\kappa$, namely, a family such that for all $\xi < \kappa^+$, $E_\xi \in [\kappa]^\kappa$, and for all distinct $\xi, \zeta < \kappa^+$, $|E_\xi \cap E_\zeta| < \kappa$ (using the regularity of $\kappa$).

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Suppose $B \subseteq \kappa^+$ is countable and $\zeta \in \kappa^+ \setminus B$. Then $|E_\zeta \cap E_\xi| < \kappa$ for all $\xi \in B$, and since $\mathrm{cf}(\kappa) > \omega$, $|\bigcup_{\xi \in B} E_\zeta \cap E_\xi| < \kappa$.

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Suppose $B \subseteq \kappa^+$ is countable and $\zeta \in \kappa^+ \setminus B$. Then $|E_\zeta \cap E_\xi| < \kappa$ for all $\xi \in B$, and since $\mathrm{cf}(\kappa) > \omega$, $|\bigcup_{\xi \in B} E_\zeta \cap E_\xi| < \kappa$.
So choose $\alpha \in E_\zeta \setminus \bigcup_{\xi \in B} E_\xi$. Then $F = \{\alpha\}$ is a finite subset of $\kappa$, and for any $\xi \in B$, $F \cap E_\xi = \emptyset$, while $F \cap E_\zeta = \{\alpha\}$.

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Let $L = \{\langle F, H \rangle \mid F \in [\kappa]^{<\omega} \wedge H \subseteq \mathcal{P}(F)\}$. Clearly $|L| = \kappa$, so it suffices to find a sequence $\langle h_\xi : \xi < \kappa^+ \rangle$ of functions $h_\xi : L \to 2$ instead of the sequence $\langle g_\xi : \xi < \kappa^+ \rangle$ above.

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Fix $\xi < \kappa^+$ and define $h_\xi : L \to 2$ such that for all $\langle F, H \rangle \in L$, $h_\xi(\langle F, H \rangle) = 1$ iff $E_\xi \cap F \in H$. Let $B \subseteq \kappa^+$ be countable and suppose $\zeta \in \kappa^+ \setminus B$.

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By assumption, there is a finite set $F \subseteq \kappa$ such that for all $\xi \in B$, $E_\xi \cap F \neq E_\zeta \cap F$. Let $H = \{F \cap E_\zeta\}$. Then $\langle F, H\rangle \in L$.

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Fix $\xi < \kappa^+$ and define $h_\xi : L \to 2$ such that for all $\langle F, H \rangle \in L$, $h_\xi(\langle F, H \rangle) = 1$ iff $E_\xi \cap F \in H$. Let $B \subseteq \kappa^+$ be countable and suppose $\zeta \in \kappa^+ \setminus B$.
By assumption, there is a finite set $F \subseteq \kappa$ such that for all $\xi \in B$, $E_\xi \cap F \neq E_\zeta \cap F$. Let $H = \{F \cap E_\zeta\}$. Then $\langle F, H \rangle \in L$.
Since $E_\zeta \cap F \in H$, $h_\zeta(\langle F, H \rangle) = 1$. On the other hand, for any $\xi \in B$, $E_\xi \cap F \notin H$ because $E_\xi \cap F \neq E_\zeta \cap F$. Hence $h_\xi(\langle F, H \rangle) = 0$. So we have for all $\xi \in B$, $h_\xi(\langle F, H \rangle) = 0$.

## Claim 3

*There is a sequence* $\langle f_\alpha : \alpha < \kappa \rangle$ *of functions* $f_\alpha : P \to 2$ *such that for any countable subset* $A \subseteq P$ *and any* $y \in P \setminus A$*, there is* $\alpha < \kappa$ *such that* $f_\alpha'' A = \{0\}$ *and* $f_\alpha(y) = 1$*.*

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Let $P = \{x_\xi \mid \xi < \kappa^+\}$. Fix $\langle g_\xi : \xi < \kappa^+ \rangle$ as in Claim 2. Fix $\alpha < \kappa$ and define $f_\alpha : P \to 2$ such that $f_\alpha(x_\xi) = g_\xi(\alpha)$ for each $\xi < \kappa^+$. Suppose that $A \subseteq P$ is countable and that $y \in P \setminus A$.

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So there is $\alpha < \kappa$ so that for all $\xi \in B$, $g_\xi(\alpha) = 0$ and $g_\zeta(\alpha) = 1$.
If $x \in A$, then $x = x_\xi$ for some $\xi \in B$, and so we have $f_\alpha(x) = f_\alpha(x_\xi) = g_\xi(\alpha) = 0$. Therefore, for all $x \in A$, $f_\alpha(x) = 0$, but $f_\alpha(y) = f_\alpha(x_\zeta) = g_\zeta(\alpha) = 1$.