Maximal towers and ultrafilter bases in computability

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(joint work with J. Miller, Nies, M. Soskova)

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- A *tower* of sets is a collection of infinite subsets of ω linearly ordered under $\subseteq^*$.

Examples of cardinal characteristics

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ZFC proves $\aleph_1 \leq \mathfrak{t} \leq \mathfrak{a}, \mathfrak{u} \leq \mathfrak{c}$ and $\aleph_1 \leq \mathfrak{t} \leq \mathfrak{d} \leq \mathfrak{i} \leq \mathfrak{c}$ but no other relation. (CH collapses all, of course.)

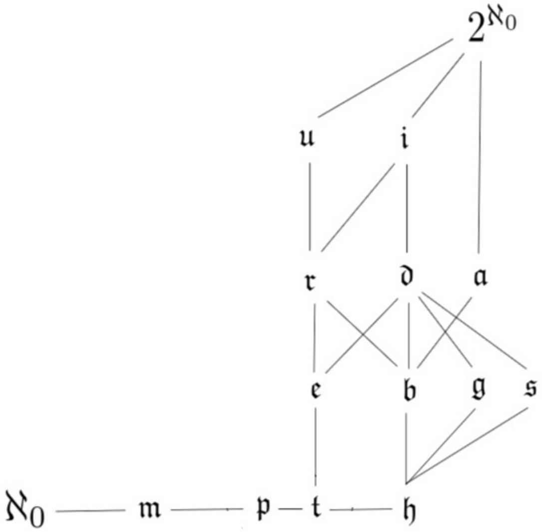

Diagram of Cardinal Characteristics

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Instead of collections of computable subsets $\mathcal{C} = \{C_i \mid i \in \omega\}$ of ω, we use single (usually noncomputable) sets $C = C_{\mathcal{C}}$ such that $\mathcal{C} = \{C^{[i]} \mid i \in \omega\}$ (where the order of columns can matter). For convenience, we will from now on freely use C_i to denote $C^{[i]}$.

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The families $\mathcal{E}$ of sets C for which the corresponding collections $\mathcal{C}_C$ have a certain property are subsets of $\mathcal{P}(\omega)$, i.e., *mass problems*.

A convenient way to compare mass problems is given by two classical reducibilities:

- the uniform *Medvedev reducibility* $\mathcal{E} \leq_s \mathcal{F}$:
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Computability theory can now be used to study mass problems arising from properties inspired by cardinal characteristics.

Two Examples of Mass Problems

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For $\geq_s$, define $\mathrm{Compl}(A) = T$ by setting $T_i = \omega - \bigcup_{j<i} A_j$.

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So 1-generic $\Delta_2^0 \implies$ index guessable $\implies$ not computing maximal towers $\implies$ low.

Two More Examples of Mass Problems

- $\mathcal{U}$ is the mass problem of ultrafilter bases (i.e., of sets U such that $\{U_i \mid i \in \omega\}$ forms an *ultrafilter base* of computable sets, namely, a (maximal) tower such that for each infinite computable set C, there is some i with $U_i \subseteq^* C$ or $U_i \subseteq^* \overline{C}$.

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Theorem

We have $\mathcal{U} \equiv_s \mathcal{I} \equiv_s$ DomFcn, the mass problem of functions dominating all total computable functions (i.e., basically, High).

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There is a co-c.e. ultrafilter base U.

Thanks!

Thanks!

Stay safe, mask up, and
get your COVID-19 booster!