# ON THE ROLE OF THE COLLECTION PRINCIPLE FOR $\Sigma_2^0$-FORMULAS IN SECOND-ORDER REVERSE MATHEMATICS

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ABSTRACT. We show that the principle PART from Hirschfeldt and Shore [7] is equivalent to the $\Sigma_2^0$-Bounding principle $B\Sigma_2^0$ over RCA$_0$, answering one of their open questions.

Furthermore, we also fill a gap in a proof in Cholak, Jockusch and Slaman [1] by showing that $D_2^2$ implies $B\Sigma_2^0$ and is thus indeed equivalent to Stable Ramsey's Theorem for Pairs (SRT$_2^2$). This also allows us to conclude that the combinatorial principles IPT$_2^2$, SPT$_2^2$ and SIPT$_2^2$ defined by Dzhafarov and Hirst [5] all imply $B\Sigma_2^0$, and thus that SPT$_2^2$ and SIPT$_2^2$ are both equivalent to SRT$_2^2$ as well.

Our proof uses the notion of a bi-tame cut, the existence of which we show to be equivalent, over RCA$_0$, to the failure of $B\Sigma_2^0$.

## 1. INTRODUCTION AND RESULTS

Let $\mathcal{M}$ be a model of RCA$_0$. In their paper on combinatorial principles implied by Ramsey's Theorem for pairs (RT$_2^2$), Hirschfeldt and Shore [7, section 4] introduced the following combinatorial principle:

**Definition 1.1.** The Principle PART states: Let $\langle M, \prec \rangle$ be a recursive (i.e., $\Delta_1^0$-definable in $\mathcal{M}$) linear ordering with least and greatest

element. Assume that for any $x \in M$, exactly one of $\{y \in M : y \prec x\}$ and $\{y \in M : x \prec y\}$ is $\mathcal{M}$-finite. Then for any $\mathcal{M}$-finite partition $\{a_i : i \leq k\}$ of $\langle \mathcal{M}, \prec \rangle$, $\{y \in M : a_i \prec y \prec a_{i+1}\}$ is not $\mathcal{M}$-finite for exactly one $i < k$. (For simplicity, we will assume here that the endpoints of $\mathcal{M}$ under $\prec$ are $a_0$ and $a_k$. Note that we will always denote the universe of the model $\mathcal{M}$ by $M$.)

Note that the conclusion of PART clearly implies the hypothesis, which in turn implies that there is at most one $i < k$ for which $\{y \in M : a_i \prec y \prec a_{i+1}\}$ is not $\mathcal{M}$-finite. PART was introduced by Hirschfeldt and Shore [7] as one in a series of principles shown to be strictly weaker than Ramsey's Theorem for pairs. In fact, they studied the Chain Antichain Principle (CAC), the Ascending and Descending Sequence Principle (ADS), and their *stable* versions, denoted respectively as SCAC and SADS, and proved that CAC, and hence SCAC, is strictly weaker than $\mathsf{RT}_2^2$, and that SCAC implies SADS, which strictly implies PART.

A general problem that was discussed quite extensively in [7] is the strength of the first-order theory of these principles. Hirst [8] has shown that $\mathsf{RT}_2^2$ implies the $\Sigma_2^0$-Bounding principle $B\Sigma_2^0$, and in [7, Proposition 4.1], $B\Sigma_2^0$ was proved to be strictly weaker than SCAC. On the other hand, while PART does not follow from Recursive Comprehension (RCA$_0$, see [7, Corollary 4.7]), it is a consequence of $B\Sigma_2^0$ over RCA$_0$ (see [7, Proposition 4.4]). Question 6.4 of [7] asks whether SADS or indeed PART implies, or is weaker than, $B\Sigma_2^0$. In the latter case, PART would have been the first "natural" principle of reverse mathematics strictly between RCA$_0$ and $B\Sigma_2^0$. However, we show in this paper that the former case holds, namely, that PART and $B\Sigma_2^0$ are indeed equivalent, adding further evidence to $B\Sigma_2^0$ being a very robust proof-theoretic principle:

**Theorem 1.2.** *The Principle* PART *is equivalent to* $B\Sigma_2^0$ *over* RCA$_0$.

The key idea of the proof is to show that PART does not hold in any model of RCA$_0$ in which $B\Sigma_2^0$ fails. It turns out that the failure of $B\Sigma_2^0$ in a model of RCA$_0$ where, by definition, the $\Sigma_1^0$-induction scheme $I\Sigma_1^0$ holds, is captured by the existence of cuts with a property we call "bitameness". The notion of a *tame* $\Sigma_2^0$-function was introduced by Lerman in $\alpha$-recursion theory (Lerman [9], Chong [2]), and later adapted to models of fragments of Peano arithmetic to study the complexity of infinite injury priority arguments in the context of reverse recursion theory (Chong and Yang [3]).

A *cut* is a nonempty bounded subset of $M$ that is closed downwards and under the successor function. A $\Sigma_2^0$-*cut* is a cut that is $\Sigma_2^0$-definable.

The existence of a $\Sigma_2^0$-cut characterizes models of $B\Sigma_2^0$ in which the $\Sigma_2^0$ induction scheme $I\Sigma_2^0$ fails. Let $I$ be a $\Sigma_2^0$-cut bounded by $k$. Then $I$ is *bi-tame* if both it and $[0, k] \setminus I$ are tame $\Sigma_2^0$ (to be defined below). As we shall see, the existence of a bi-tame cut characterizes precisely the models of $\mathsf{RCA}_0$ that do not satisfy $B\Sigma_2^0$. This fact will then be used to establish the failure of $\mathsf{PART}$. Recall here that the existence of a $\Sigma_2^0$-cut characterizes the failure of $I\Sigma_2^0$ over the base theory $I\Sigma_1^0$. Thus bi-tameness separates the $\Sigma_2^0$-cuts $I$ in models satisfying only $I\Sigma_1^0$ from those in models also satisfying $B\Sigma_2^0$.

We conclude this paper by proving Theorem 1.4, which implies some other consequences of Ramsey type combinatorial principles. To elaborate, our method allows one to fill a gap in Cholak, Jockusch and Slaman [1]. They define the following principle:

**Definition 1.3.** The Principle $D_2^2$ states: For any $\Delta_2^0$-definable subset $A$ of $M$, there is an infinite subset $B$ in $\mathcal{M}$ which is either contained in, or disjoint from, $A$.

It was claimed in [1, Lemma 7.10] that $D_2^2$ is equivalent to Stable Ramsey's Theorem for Pairs ($\mathsf{SRT}_2^2$) over $\mathsf{RCA}_0$. However, the argument that $D_2^2$ implies $\mathsf{SRT}_2^2$ implicitly assumes $B\Sigma_2^0$ and thus contains a gap. We close this gap in the following

**Theorem 1.4.** *The Principle $D_2^2$ implies $B\Sigma_2^0$, and is therefore equivalent to Stable Ramsey's Theorem ($\mathsf{SRT}_2^2$), over $\mathsf{RCA}_0$.*

Dzhafarov and Hirst [5] introduced the following "polarized" versions of Ramsey's Theorem, based on similar notions in Erdős and Rado [4, §9]:

**Definition 1.5.** Let $n, k \geq 1$ and $f : [M]^n \to k$.
  (1) A *p-homogeneous* set for $f$ is a sequence $\langle H_1, \ldots, H_n \rangle$ of $\mathcal{M}$-infinite sets such that for some $c < k$, $f(\{x_1, \ldots, x_n\}) = c$ for every $\langle x_1, \ldots, x_n \rangle \in H_1 \times \cdots \times H_n$.
  (2) Such a sequence $\langle H_1, \ldots, H_n \rangle$ of $\mathcal{M}$-infinite sets is called *increasing p-homogeneous* if (1) is required to hold only for increasing tuples $\langle x_1, \ldots, x_n \rangle$.
  (3) $\mathsf{PT}_2^2$ (or $\mathsf{IPT}_2^2$, respectively, for "(Increasing) Polarized Theorem") is the statement that every $f : [M]^2 \to 2$ has an (increasing) p-homogeneous set.
  (4) $\mathsf{SPT}_2^2$ (or $\mathsf{SIPT}_2^2$, respectively, for "Stable (Increasing) Polarized Theorem") is the statement that every $f : [M]^2 \to 2$ has an (increasing) p-homogeneous set.

(There are natural extensions to $k$ many colors and, for $(\mathsf{I})\mathsf{PT}_2^2$, to $n$-tuples.)

The following was proved in [5]:

**Theorem 1.6** (Dzhafarov, Hirst [5])**.**

(1) *Over* $\mathsf{RCA}_0$, $\mathsf{PT}_2^2$ *implies* $B\Sigma_2^0$.
(2) *Over* $\mathsf{RCA}_0$, $\mathsf{PT}_2^2$ *implies* $\mathsf{IPT}_2^2$ *and* $\mathsf{SPT}_2^2$.
(3) *Over* $\mathsf{RCA}_0$, *both* $\mathsf{IPT}_2^2$ *and* $\mathsf{SPT}_2^2$ *imply* $\mathsf{SIPT}_2^2$.
(4) *Over* $\mathsf{RCA}_0$, $\mathsf{SIPT}_2^2$ *implies* $D_2^2$.
(5) *Over* $\mathsf{RCA}_0 + B\Sigma_2^0$, $\mathsf{IPT}_2^2$ *implies* $\mathsf{SPT}_2^2$.
(6) *Over* $\mathsf{RCA}_0 + B\Sigma_2^0$, *all of* $\mathsf{SRT}_2^2$, $\mathsf{SPT}_2^2$ *and* $\mathsf{SIPT}_2^2$ *are equivalent.*

Since all these principles imply $D_2^2$, Theorem 1.4 considerably simplifies the picture (and answers Questions 5.1 (part 2) and 5.2 in [5]):

**Theorem 1.7.**

(1) *Over* $\mathsf{RCA}_0$, *all of* $\mathsf{IPT}_2^2$, $\mathsf{SPT}_2^2$ *and* $\mathsf{SIPT}_2^2$ *imply* $B\Sigma_2^0$.
(2) $\mathsf{SRT}_2^2$, $\mathsf{SPT}_2^2$ *and* $\mathsf{SIPT}_2^2$ *are equivalent over* $\mathsf{RCA}_0$.
(3) *Over* $\mathsf{RCA}_0$, $\mathsf{IPT}_2^2$ *implies* $\mathsf{SPT}_2^2$.                                  $\square$

The question of whether either or both of the middle two implications in

$$\mathsf{RT}_2^2 \Leftrightarrow \mathsf{PT}_2^2 \Rightarrow \mathsf{IPT}_2^2 \Rightarrow \mathsf{SPT}_2^2 \Leftrightarrow \mathsf{SRT}_2^2$$

are strict remains open.

The rest of this paper is devoted to the proofs of Theorems 1.2 and 1.4.

## 2. The proof of Theorem 1.2

We work in models of $\mathsf{RCA}_0$. Since $I\Sigma_1^0$ is the most important consequence of $\mathsf{RCA}_0$ we use, we can work as if we were in first-order Peano arithmetic. (We refer the reader to Hájek/Pudlák [6] for background on first-order arithmetic and to Simpson [10] for background on second-order arithmetic and reverse mathematics.)

We will show the equivalence of $\mathsf{PART}$ and $B\Sigma_2^0$ in two steps, after introducing the notion of bi-tame cuts.

**Definition 2.1.** Suppose $\mathcal{M}$ is a model of $I\Sigma_1^0$. We say a set $I$ is a *bi-tame* cut in $\mathcal{M}$ iff

(1) $I$ is a *cut*, i.e., closed under successor and closed downward.
(2) There are a point $k \notin I$ and a $\Sigma_2^0$-function $g : [0, k] \to M$ with recursive approximation $h(i, s) : [0, k] \times M \to M$ such that:
    (a) The domain of $g$ is the whole interval $[0, k]$.

(b) The range of $g$ is unbounded in $\mathcal{M}$.
(c) (Tame $\Sigma_2^0$ on $I$) For any $i \in I$, there is an $s$ such that for all $j < i$, for all $t > s$, $h(j,t) = h(j,s)$, i.e., $g$ settles down on all initial segments, and so, in particular, $g \upharpoonright [0,i)$ is $\mathcal{M}$-finite.
(d) (Tame $\Sigma_2^0$ on $\bar{I}$) For any $i < k$ not in $I$, there is an $s$ such that for all $j$ with $i < j \leq k$ and all $t > s$, $h(j,t) = h(j,s)$, i.e., $g$ also settles down on all final segments, and so, in particular, $g \upharpoonright (i,k]$ is $\mathcal{M}$-finite.

**Remark.** Throughout this paper, we use boldface definability, so, e.g., $\Sigma_2^0$ is really $\Sigma_2^0(\mathcal{M})$, i.e., with parameters from $M$. Observe that any bi-tame cut $I$ is $\Delta_2^0$, as both $I$ and $[0,k] \setminus I$ are $\Sigma_2^0$.

First we show that the failure of PART is equivalent to the existence of bi-tame cuts.

The failure of PART can be stated as: There is a recursive linear ordering $\mathcal{M} = (M, \prec)$ together with an $\mathcal{M}$-finite partition $\{a_i : i \leq k\}$ (which we refer to as *landmarks*) such that for any $x \in M$, exactly one of $\{y : y \prec x\}$ and $\{y : x \prec y\}$ is $\mathcal{M}$-finite, but that for all $i < k$, the interval $\{y : a_i \prec y \prec a_{i+1}\}$ is $\mathcal{M}$-finite.

**Lemma 2.2.** *The existence of a linear ordering witnessing the failure of* PART *is equivalent to the existence of a bi-tame cut.*

A pictorial version of the proof proceeds as follows: Imagine the graph of the function $g$ which witnesses the bi-tameness of $I$ as consisting of $k$ many vertical columns, the $i$-th one of which is of height $g(i)$. Now "push the columns from both ends" as in Domino, to produce a linear ordering. The resulting horizontal picture is more or less the linear order.

For the converse, just "un-Domino" the horizontal picture. We get the bi-tameness from the condition that either the initial segment or the final segment is $\mathcal{M}$-finite.

*Proof.* ($\Longleftarrow$) Let $I$ be a bi-tame cut with witness $\Sigma_2^0$-function $g : [0,k] \to M$ and recursive approximation $h(x,s)$ of $g(x)$ as in Definition 2.1. We recursively enumerate the linear order $\prec$ as follows.

Stage 0 (laying out the landmarks): Set $a_i = i$ for $i \leq k$ and $a_i \prec a_j$ iff $i < j$.

Stage $s + 1$: Suppose we have specified the order up to $m \in M$ at the end of the stage $s$. Then, for each $i < k$ in increasing order, if $h(i, s+1) > h(i,s)$, then set $k = h(i, s+1) - h(i,s)$ and insert the next $k$ many elements of $\mathcal{M}$ between the landmarks $a_i$ and $a_{i+1}$, to the right of all the elements previously inserted between these landmarks.

We check that the linear order $\prec$ works: It is a recursive linear ordering because a linear order $\prec$ is recursive iff it has an r.e. copy. Once the approximation $h(i,s)$ of $g(i)$ has settled down on the initial segment $[0, i+1]$ or the final segment $[i, k]$, depending on whether $i \in I$ or not, no elements will enter the interval $\{y : a_i \prec y \prec a_{i+1}\}$, hence this interval is $\mathcal{M}$-finite. Finally, a one-point partition leaves either an initial segment or a final segment $\mathcal{M}$-finite because of the bi-tameness, and it cannot leave both $\mathcal{M}$-finite as $g$ is unbounded.

($\Rightarrow$) Suppose $\prec$ is such a recursive linear order with landmarks $\{a_i : i \le k\}$. Fix a recursive enumeration of $\mathcal{M}$. Define a recursive approximation $h(i,s) = x$ of $g : [0, k] \to M$ by taking $x$ to be the maximal element enumerated into the interval $\{y : a_i \prec y \prec a_{i+1}\}$ up to stage $s$. $g$ is unbounded since every element of $M$ appears in the enumeration. Define the $\Sigma_2^0$-cut $I$ by $i \in I$ iff $i < k$ and the initial segment of the one-point partition by $a_i$ is $\mathcal{M}$-finite. We check that $I$ is bi-tame. We only need to show that if $i \in I$ then $g$ settles down on the initial segment $[0, i]$, as the other case for the final segment is symmetric. Since the interval $[0, a_i] = \{y : y \prec a_i\}$ is $\mathcal{M}$-finite, apply $B\Sigma_1^0$ to the formula $\forall y \in [0, a_i]\, \exists s\, [y \prec a_i \text{ at stage } s]$ to obtain a uniform upper bound $t$ such that no element enters the interval $[0, a_i]$ after stage $t$. Thus $h(i,s) = h(i,t)$ for all $s > t$.                                    $\square$

The second part of the proof is to link the existence of bi-tame cuts to $B\Sigma_2^0$. The essential idea is based on the proof of the equivalence of $B\Sigma_2^0$ and $I\Delta_2^0$ by Slaman [11].

**Lemma 2.3.** *Suppose $\mathcal{M} \models I\Sigma_1^0$. Then $\mathcal{M} \not\models B\Sigma_2^0$ iff there exists a bi-tame cut in $\mathcal{M}$.*

*Proof.* Clearly, if $\mathcal{M} \models B\Sigma_2^0$ then there is no $\Delta_2^0$-cut and thus a fortiori no bi-tame cut.

On the other hand, suppose that $\mathcal{M} \not\models B\Sigma_2^0$. We need to construct a bi-tame cut. We start by proving two claims:

**Claim 2.4.** *Suppose that $\mathcal{M} \models I\Sigma_1^0$ and $\mathcal{M} \not\models B\Sigma_2^0$. Then there are an element $k \in M$ and a function $f : [0, k) \to M$ such that*

*(a) $f$ is injective;*
*(b) the domain of $f$ is $[0, k)$ and the range of $f$ is unbounded; and*
*(c) the graph of $f$ is $\Pi_1^0$.*

*Proof.* Let $\forall t\, \psi(x, y, t)$ be a $\Pi_1^0$-formula which witnesses the failure of $B\Pi_1^0$ (which is equivalent to $B\Sigma_2^0$) on some interval $[0, k)$. We define a $\Pi_1^0$-function $f : [0, k) \to M$ by setting $f(x) = \langle x, \langle y, s \rangle \rangle$ iff

$$\forall t\, \psi(x, y, t) \land \forall z < y\, \exists t < s\, \neg\psi(x, z, t) \land \forall t < s\, \exists z < y\, \forall v < t\, \psi(x, z, v).$$

Intuitively, the first coordinate $x$ of $f(x)$ just makes $f$ injective. So, $f$ essentially maps $x$ to the least $y$ such that $\forall t\, \psi(x, y, t)$, but this alone would give us only a $\Delta_2^0$-graph. To make the graph $\Pi_1^0$, we observe furthermore that for any $z < y$, $\exists t_z\, \neg\psi(x, z, t_z)$. The number $s$ in $f(x)$ is the least upper bound on all such $t_z$, which exists by $B\Sigma_1^0$. It is now easy to check that $f$ works, concluding the proof of Claim 2.4. $\qquad \square$

We may think of $f(x)$ as the stage at which $x$ is enumerated into $[0, k)$. We now construct a "tame" $\Sigma_2^0$-function $g$ which enumerates the interval $[0, k)$. Here, "tameness" means that $g$ settles down on all initial segments. This tameness constitutes the essential difference between $g$ and $f$. More precisely, we have the following

**Claim 2.5.** *Let $k$ and $f$ be as in Claim 2.4. Then there are a $\Sigma_2^0$-cut $I$ and a $\Sigma_2^0$-function $g : I \to [0, k)$, together with a recursive approximation $h$ to $g$, such that*

*(a) $g$ is 1-1 from $I$ onto $[0, k)$;*
*(b) $g$ is "tame", i.e., for all $i \in I$, there is a stage $s$ such that for all $j < i$ and all $t > s$, $h(j, t) = h(j, s) = g(j)$; so, in particular, $g \upharpoonright i$ is $\mathcal{M}$-finite; and*
*(c) $g$ is not "coded" on $I \times [0, k)$, i.e., $g \neq X \cap (I \times [0, k))$ for any $\mathcal{M}$-finite set $X$. (Informally, there is no $\mathcal{M}$-finite "end-extension" of (the graph of) $g$).*

*Proof.* We start with the definition of a function $F$. Let $\theta(x, y, u)$ be a $\Sigma_0^0$-formula such that $(x, y) \in f$ iff $\forall u\, \theta(x, y, u)$. For each $s \in M$, we will define $F(s)$ as the approximation to $f$ at stage $s$; since $f$ is a 1-1 function, $F(s)$ can be made a 1-1 function as well (possibly with a smaller domain). $F(s)$ is defined as follows: Set $(x, y) \in F(s)$ iff

$$x < k \land y \leq s \land \forall u < s\, \theta(x, y, u) \land \forall y' < y\, \exists u' < s\, \neg\theta(x, y', u')$$
$$\land \neg\exists x' < x\, [\forall u < s\, \theta(x', y, u) \land \forall y' < y\, \exists u' < s\, \neg\theta(x', y, u')].$$

Since $F(s)$ is an $\mathcal{M}$-finite set of pairs, we can list all its elements $(x_0, y_0), \ldots, (x_e, y_e)$ (for some $e = e_s < k$, say) ordered by their second coordinates, i.e., such that $y_i < y_j$ iff $i < j$. We define $h(i, s) = x_i$ for all $i \leq e$. Formally, we define $h(i, s) = x$ iff there is $c < 2^{\langle k, s\rangle}$ which is a code of an $\mathcal{M}$-finite sequence $\langle c_0, \ldots, c_i\rangle$ of length $i + 1$, say, such that

(1) for each $j \leq i$, $c_j$ is a pair $\langle x_j, y_j\rangle$;
(2) $x = x_i$;
(3) $\forall j \leq i\, ((x_j, y_j) \in F(s))$;
(4) $\forall j < k \leq i\, (y_j < y_k)$; and
(5) $\forall j < i\, \forall z < y_{j+1}\, [y_j < z \to \forall x < k\, ((x, z) \notin F(s))]$.

Let $I = \{i : \exists s \, \forall j \leq i \, \forall t > s \, [h(j,s) = h(j,t)]\}$ and, for each $i \in I$, let $g(i) = \lim_s h(i,s)$. We first note that since $F$ and $h$ are $\Delta_0^0$, both $I$ and the graph of $g$ are $\Sigma_2^0$.

We now check that $I$, $g$ and $h$ satisfy statements (a)-(c) from the claim.

(a) We first show that $g$ is 1-1. Observe that if $g(i) = x$ then there is $s$ such that $\forall t > s \, (h(i,t) = x)$; so, in particular, there is some $y$ such that for all $t > s$, $(x,y) \in F(t)$, i.e., $f(x) = y$. Suppose that $i_1 < i_2$ are two elements in $I$, and that $g(i_1) = x_1$ and $g(i_2) = x_2$. By definition of $I$, there is a stage $s$ such that $g$ settles down at both $i_1$ and $i_2$. Thus there are $y_1$ and $y_2$ such that $(x_1, y_1), (x_2, y_2) \in F(t)$ for all $t > s$. By the choice of $F(t)$, $y_1 \neq y_2$ and thus $x_1 \neq x_2$.

Next we show that $g$ is onto $[0, k)$. For any $m < k$, $f(m)$ is defined, thus $f \upharpoonright [0, k) \times [0, f(m))$ is a bounded $\Pi_1^0$-set and hence coded. Therefore, its complement $[0, k) \times [0, f(m)) \setminus f$ is an $\mathcal{M}$-finite $\Sigma_1^0$-set. Now, by $B\Sigma_1^0$, there is a uniform bound $s$ such that for each $(x, y)$ in this complement, $\exists u < s \, \neg \theta(x, y, u)$. Hence for all $t > s$, $(x, y) \notin F(t)$. Hence, if $(m, f(m))$ is the $e$-th pair in $F(s)$, $g(e) = m$. This establishes (a).

(b) follows from the definition of $g$ and $h$.

(c) Observe that $I$ is indeed a cut. Suppose that $i \in I$. Let $s$ be the (least) stage by which $g$ settles down on $[0, i]$. At stage $s$, we will see the $\mathcal{M}$-finite set $F(s)$, say, $\{(x_0, y_0), \ldots, (x_i, y_i)\}$, listed with increasing $y$-coordinates. Since $f$ is unbounded, let $z$ be the (least) number in the range of $f$ such that $z > y_i$. Then $g(i+1) = z$. Finally, we show that $g$ is not coded on $I \times [0, k)$. Suppose otherwise, say, $X \cap (I \times [0, k)) = g$ for some $\mathcal{M}$-finite set $X$. Then

$$g = \{(i, m) \in X : m < k \text{ and } i \text{ is the least } j \text{ such that } (j, m) \in X\},$$

which is $\mathcal{M}$-finite. Hence its domain $I$ would be $\mathcal{M}$-finite, a contradiction. This concludes the proof of Claim 2.5. $\qquad\square$

Finally, we use $g$ and $h$ to obtain a bi-tame cut $J$ with its approximation $l(j, s)$.

We start with the interval $[0, k^k]$ and initially place two markers $l$ and $r$ at $0$ and $k^k$, respectively. At each stage $s$, the construction is performed in $e$ many steps, where $e$ is the least number not in the domain of $h(\cdot, s)$. At the end of each step, we shrink the gap between $l$ and $r$ by a factor of $k$.

Step 0. Calculate $h(0, s)$. Set $l(0, s) = h(0, s)k^{k-1}$ and $r(0, s) = l(0, s) + k^{k-1}$.

Step $i$. Suppose $l(i-1,s)$ and $r(i-1,s)$ are the current positions of the markers and $r(i-1,s) - l(i-1) = k^{k-i}$. Calculate $h(i,s)$, and let $l(i,s) = l(i-1,s) + h(i,s)k^{k-i-1}$ and $r(i,s) = l(i,s) + k^{k-i-1}$.

Now let $J = \{x : \exists s\, \exists i\, \forall t > s\, \forall j < i\, [l(j,s) = l(j,t) \wedge x < l(i,s)]\}$ and $\bar{J} = \{x : \exists s\, \exists i\, \forall t > s\, \forall j < i\, [r(j,s) = r(j,t) \wedge x > r(i,s)]\}$. Then $J$ and $\bar{J}$ are both $\Sigma_2^0$, and when $h$ settles down on the initial segment $[0,i]$, then both $l(i,s)$ and $r(i,s)$ settle down as well. Clearly, $J$ and $\bar{J}$ are disjoint, so it remains to show that $J \cup \bar{J} = [0, k^k]$, i.e., that there is no "gap" left. Suppose $m$ belongs to the gap. Then write $m$ as an $k$-ary number. We can then read out $g(i)$ from $m$ for all $i \in I$, contradicting the fact that $g$ is not coded on $I \times [0, k)$.

This concludes the proof of Lemma 2.3. $\qquad\square$

Lemmas 2.2 and 2.3 now immediately establish Theorem 1.2 as desired.

## 3. The proof of Theorem 1.4

Using Theorem 1.2, it suffices to prove PART from $D_2^2$. So suppose that $(M, \prec)$ is a linear order in $\mathcal{M}$ such that for any $x \in M$, exactly one of $\{y \in M : y \prec x\}$ and $\{y \in M : x \prec y\}$ is $\mathcal{M}$-finite. Let $A$ be the set of all $x \in M$ such that $\{y \in M : y \prec x\}$ is $\mathcal{M}$-finite, or equivalently, such that $\{y \in M : x \prec y\}$ is $\mathcal{M}$-infinite. Thus $A$ is a $\Delta_2^0$-definable subset of $M$. Applying $D_2^2$ (and by symmetry), let $B$ be an infinite subset of $A$ which exists in the second-order model $\mathcal{M}$. Then the $\prec$-downward closure $C$ of $B$ is a $\Sigma_1^0$-definable subset of $M$; and by our assumption on $(M, \prec)$, $C = A$. Now fix any $\mathcal{M}$-finite partition $\{a_i : i \leq k\}$ of $\langle \mathcal{M}, \prec \rangle$ (where $a_0$ and $a_k$ are the least and greatest element), and assume that for each $i < k$, the interval $[a_i, a_{i+1}]$ is $\mathcal{M}$-finite. By $\Sigma_1^0$-induction, we then have that for each $i \leq k$, the set $\{a_0, a_1, \ldots, a_i\}$ is a subset of $C$, and thus $[a_0, a_i]$ is $\mathcal{M}$-finite. But clearly $a_k \notin C$, giving the desired contradiction.

As a final remark, we note that Jockusch later observed a shorter but less direct proof, using Hirschfeldt and Shore's result [7, Proposition 4.6] that SADS implies $B\Sigma_2^0$ and thus requiring only a proof of SADS from $D_2^2$ as in the first half of the previous paragraph: Once the infinite set $B$ is obtained, one can argue immediately that it has order-type $\mathcal{M}$ or $\mathcal{M}^*$ and has thus established SADS.

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