From equivalence structures to topological groups

Alexander Melnikov

Dagstuhl, Feb 2017.
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What would be considered a “good” classification of structures in $\mathcal{K}$?

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Very few classes admit a Friedberg enumeration.

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- Structure and Anti-structure theorems (Goncharov and Knight)
- Effective classification of computable structures (MillerR., Lange, and Steiner)
- Effectively closed sets and enumerations (Brodhead and Cenzer)
- Theory of numberings (A book by Ershov)
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Question (Goncharov and Knight 2002)
Is there a Friedberg enumeration of the class of computable equivalence structures?

Goncharov and Knight conjectured that the answer is NO because the invariants are too complicated.

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Guessing isomorphism $E \cong F$ between eq. structures is a $\Pi^0_4$-complete problem.

Compare this to c.e. sets where $W_e = W_j$ is $\Pi^0_2$.

Earlier attempts by Goncharov and Knight, and by Miller R., Lange, and Steiner.

**Theorem (Downey, M., Ng)**

There **exists** a Friedberg enumeration of computable eq. structures.

This is a non-uniform $0'''$. 
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**Problem (Maltsev, in the 1960-s)**

Describe computably categorical abelian groups.

We have nice satisfactory classifications for:

- $p$-groups (Smith, indep. Goncharov)
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Missing cases:

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Case of study: Torsion abelian groups.

What would be considered a “good” classification of c.c. torsion abelian groups?

Theorem (M. and Ng)

There exists a $\mathcal{L}_{\omega_1 \omega}^\omega \Pi_4^c$-sentence $\Psi$ such that

$$A \models \Psi \iff A \text{ is a c.c. torsion abelian group.}$$

Furthermore, $\Pi_4^c$ is the optimal complexity. (The index set is $\Pi_4^0$-complete.)

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- $\Pi^c_4$-harness of the index set is the easy(er) part.
- $\Psi$ relies on several subtle algebraic reductions.
- $\Psi$ says that a certain diagonalization attempt on equivalence structures must fail.
- The analysis of computable equivalence structures is in the (scary) combinatorial core of the proof.
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Would my academic semi-grate grate grandfather be happy?
From computable groups to Polish groups
A **computable Polish group** is a computable Polish (metric) space equipped with computable group operations.

We consider Polish groups up to topological isomorphism.

Suppose $K$ is a natural class of Polish groups (e.g., connected compact groups).

**Can we classify members of $K$?**
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Theorem (M. and Khoussainov)

1. The index sets of **profinite** and of **connected compact** Polish groups are Arithmetical.

2. The topological isomorphism problems for **profinite abelian groups** and for **connected compact abelian** groups are \( \Sigma_1^1 \)-complete.

We can list all partial computable Polish groups: \( G_0, G_1, G_2, \ldots \)

- \( \{ i : G_i \text{ is a connected topological group} \} \) is Arithmetical.
- \( \{ (i, j) : G_i \cong G_j \text{ and } G_i, G_j \text{ are connected} \} \) is \( \Sigma_1^1 \)-complete.

The result is uniform. It follows connected and profinite (abelian) groups are **unclassifiable**.
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- Computable Polish space theory.
- **Computable (countable) abelian group theory** (e.g., the old result of Dobrica on bases, a result of Downey and Montalban, etc.).
- Pontryagin duality.
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(If there is time.)

**Definition**

Let $\mathbb{T}$ be the unit circle group. The **dual group** of a topological group $G$ is

$$\hat{G} = \{ \chi \mid \chi \text{ is a continuous group homomorphism from } G \text{ to } \mathbb{T} \}.$$ 

**Theorem (Pontryagin)**

Let $G$ be either discrete or compact abelian group. Then:

- $\hat{\hat{G}} \cong G$, and
- $G$ is compact iff $G$ is discrete.
- $G$ is **torsion** iff $\hat{G}$ is **profinite**.

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A profinite group is *recursive* if it is the limit of a computable surjective inverse system of finite groups.

($\hat{G}$ stands for the Pontryagin dual of $G$.)

Theorem (Khoussainov and M.)

Let $G$ be a countable torsion abelian group. Then

- $G$ is computable iff $\hat{G}$ is a recursive profinite group;
- $G$ is computably categorical iff $\hat{G}$ is computably categorical (as a recursive profinite group).

Corollary (follows from M. and Ng)

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