Effectiveness of the dual Ramsey theorem

Damir D. Dzhafarov
University of Connecticut

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Joint work with Stephen Flood, Reed Solomon, and Linda Brown Westrick.

Dual Ramsey's theorem.

The dual Ramsey's theorem is a variant of the well-known Ramsey's theorem.

For $k \leq \omega$, let $[\omega]^k$ denote the set of all subsets of ω of size k.

Ramsey's theorem for $k, \ell < \omega$ (RT^k_ℓ).
If $[\omega]^k = \bigcup_{i<\ell} C_i$, there is $H \in [\omega]^\omega$ such that $[H]^k \subseteq C_i$ for some i.

For $k \leq \omega$, let $(\omega)^k$ denote the set of all partitions of ω into k parts.

Given $x \in (\omega)^\omega$ and $k < \omega$, let $(x)^k$ be the set of all coarsenings $y \in (\omega)^k$ of x.

Dual Ramsey's theorem for $k, \ell < \omega$ (DRT^k_ℓ).
If $(\omega)^k = \bigcup_{i<\ell} C_i$ is Borel, there is $x \in (\omega)^\omega$ such that $(x)^k \subseteq C_i$ for some i.

A short history.

Introduced and proved by **Carlson and Simpson** (1984). Extended by **Prömel and Voigt** (1985) to colorings with the Baire property.

Miller and Solomon (2004) showed $\mathsf{RCA}_0 \vdash \text{Open-}\mathsf{DRT}^{k+1}_\ell \to \mathsf{RT}^k_\ell$.

Blass, Hirst, and Simpson (1987), **Miller and Solomon**, and **Erhard** (2013) all studied the Carlson-Simpson lemma (CSL), the combinatorial core of DRT.

Blass, Hirst, and Simpson (1987) showed $\Pi^1_2\text{-}\mathsf{CA}_0 \vdash \mathsf{CSL}$. **Slaman** (unpublished) improved this to $\Pi^1_1\text{-}\mathsf{CA}_0 \vdash \mathsf{CSL}$. It is unknown whether $\mathsf{RCA}_0 \vdash \mathsf{CSL}^2_2$.

Miller and Solomon (2004) showed $\mathsf{WKL}_0 \nvdash \mathsf{VW}^2_2$. **Erhard** (2013) showed that $\mathsf{COH} \nvdash \mathsf{VW}^2_2$ and $\mathsf{SRT}^2_2 \nvdash \mathsf{OVW}^2_2$.

All of this deals essentially only with open colorings.

Combinatorial principles.

Definition. A coloring $(\omega)^k = \bigcup_{i<\ell} C_i$ is reduced if the color of $x \in (\omega)^k$ depends only on the least element a of the kth block of x and which blocks the numbers $b < a$ belong to.

Carlson-Simpson lemma (CSL^k_ℓ).
If $(\omega)^k = \bigcup_{i<\ell} C_i$ is reduced, there is $x \in (\omega)^\omega$ and $i < \ell$ such if $y \in (x)^k$ keeps the first $k-1$ blocks of x distinct then $y \in C_i$.

Note that every reduced coloring is open.

The following natural variant the CSL is proved by $\omega \cdot k$ many iterations of CSL.

Combinatorial dual Ramsey's theorem (CDRT^k_ℓ).
If $(\omega)^k = \bigcup_{i<\ell} C_i$ is reduced, there is $x \in (\omega)^\omega$ and $i < \ell$ such that $(x)^k \subseteq C_i$.

The Baire DRT.

We code a Baire ℓ-coloring by open sets $O_0, \ldots, O_{\ell-1}$ and a sequence of dense open sets $\{D_n\}_{n\in\omega}$, representing that $O_i \cap \bigcap_n D_n \subseteq C_i$ for all i.

This allows us to formulate Baire-DRT^k_ℓ in second-order arithmetic.

Theorem (Dzhafarov, Flood, Solomon, and Westrick).
Over RCA_0, the following are equivalent for all $k, l < \omega$:

1. Open-DRT^k_ℓ;
2. Baire-DRT^k_ℓ;
3. CDRT^k_ℓ.

In particular, we obtain bounds for CDRT that we lack for CSL.

Relationships.

$$\begin{tikzcd}
\mathsf{ACA}_0^+ \arrow[d] \arrow[drr] & & \mathsf{CDRT}^4_\ell \arrow[d] \arrow[dll] \\
\mathsf{ACA}_0 \arrow[dr] & & \mathsf{CDRT}^3_\ell \arrow[dl] \\
& \mathsf{RT}^2_2 \arrow[d] & \\
& \mathsf{RCA}_0 \leftrightarrow \mathsf{CDRT}^2_\ell &
\end{tikzcd}$$

Folklore. $\mathsf{ACA}_0^+ \rightarrow$ Hindman's theorem $\rightarrow \mathsf{CDRT}^3_\ell$.

The implication $\mathsf{CDRT}^4_\ell \rightarrow \mathsf{ACA}_0$ follows by results of **Miller and Solomon**.

The Borel DRT.

Theorem (Dzhafarov, Flood, Solomon, and Westrick).
Over RCA_0, the following are equivalent for all $k, \ell < \omega$:

1. Borel-DRT^k_ℓ;
2. Baire-$\mathsf{DRT}^k_\ell + \mathsf{ATR}_0$.

The implication from Borel-DRT to Baire-DRT is a coding argument.

That Borel-DRT implies ATR_0 is not for any deep reason; the statement that for every Borel set, there a point in it or not in it, already implies ATR_0.

The implication from 2 to 1 uses ATR_0 to formalize that every Borel set is Baire.

Since Baire-DRT $\leftrightarrow$ CDRT, it follows that the strength of (Borel) DRT can be understood entirely in combinatorial terms.

Effective analysis, $k \geq 3$.

Throughout, fix $k \geq 3$.

Recall that a modulus is a function f such that if $f \leq g$ then $f \leq_T g$.

Lemma. Let f be a modulus. There is an f-computable clopen coloring $(\omega)^k = C_0 \cup C_1$ such that $f \leq_T x$ for each homogeneous $x \in (\omega)^\omega$.

Lemma. If $a \in \mathcal{O}$ and $(\omega)^k = C_0 \cup C_1$ where the C_i are H_a-computable and clopen, then the C_i have computable codes as topologically Δ^0_a sets.

Theorem (Dzhafarov, Flood, Solomon, and Westrick).
For each computable ordinal α there is a computable, topologically $\Delta^0_{\alpha+1}$ coloring $(\omega)^k = C_0 \cup C_1$ such that $\emptyset^{(\alpha)} \leq_T x$ for each homogeneous $x \in (\omega)^\omega$.

Every hyperarithmetic set A has a self-modulus, i.e., a modulus $f \equiv_T A$.

Effective analysis, $k = 2$.

Though Borel-$\mathrm{DRT}^2_\ell \to \mathrm{ATR}_0$, we lack the $\emptyset^{(\alpha)}$-coding that we had for $k \geq 3$.

For sufficiently nice colorings, there are more effective solutions.

Proposition (Dzhafarov, Flood, Solomon, and Westrick).
If $(\omega)^2 = C_0 \cup C_1$ where C_0 is effectively open, then there is a computable $x \in (\omega)^\omega$ such that $(x)^2 \subseteq C_i$ for some i.

This extends a result of **Katz** (unpublished), who established the same for C_0 effectively clopen. Our proof is necessarily non-uniform.

Theorem (Dzhafarov, Flood, Solomon, and Westrick).
If $(\omega)^2 = C_0 \cup C_1$ where C_0 is effectively Σ^0_2, then there is either a computable $x \in (\omega)^\omega$ with $(x)^2 \subseteq C_1$, or a $\emptyset'$-computable $x \in (\omega)^\omega$ with $(x)^2 \subseteq C_0$.

Simple colorings.

A coloring $\omega^2 = C_0 \cup C_1$ is simple if the color of $x \in (\omega)^2$ depends only on the least element of the non-zero block.

Let Δ^0_n-D^2_2 be the statement that given $c : [\omega]^n \to 2$ such that $\lim_{s_{n-2}} \cdots \lim_{s_0} c(k, s_0, \ldots, s_{n-2})$ exists for all k, there is an i and an infinite set L such that $\lim_{s_{n-2}} \cdots \lim_{s_0} c(k, s_0, \ldots, s_{n-2}) = i$ for all $k \in L$.

So Δ^0_2-D^2_2 is the well-known D^2_2, which is equivalent to SRT^2_2 over RCA_0.

Proposition (Dzhafarov, Flood, Solomon, and Westrick).
Over RCA_0, the following are equivalent:

1. Simple-DRT^2_2 for effectively Σ^0_n colorings;
2. Δ^0_n-SRT^2_2.

Summary.

$$
\begin{tikzcd}
& \Pi^1_1\text{-CA}_0 \arrow[d] & \\
& \text{Borel-DRT}^k_\ell \arrow[dl] \arrow[ddr] & \\
\text{ATR}_0 \arrow[d] & & \\
\text{ACA}_0^+ \arrow[d] \arrow[drr] & & \text{CDRT}^4_\ell \arrow[d] \arrow[dll] \\
\text{ACA}_0 \arrow[dr] & & \text{CDRT}^3_\ell \arrow[dl] \\
& \text{RT}^2_2 \arrow[d] & \\
& \text{SRT}^2_2 \leftrightarrow \Delta^0_2\text{-Simple-DRT}^2_2 \arrow[d] & \\
& \text{RCA}_0 \leftrightarrow \text{CDRT}^2_\ell &
\end{tikzcd}
$$

Questions.

1. What is the strength of CDRT^k_ℓ?

2. Is CDRT strictly stronger than CSL?

Thank you for your attention.