

Effectiveness of the dual Ramsey theorem

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October 4, 2015

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Dual Ramsey's theorem.

The dual Ramsey's theorem is a variant of the well-known Ramsey's theorem.

For $k \leq \omega$, let $[\omega]^k$ denote the set of all subsets of ω of size k .

Ramsey's theorem for $k, \ell < \omega$ (RT_ℓ^k).

If $[\omega]^k = \bigcup_{i < \ell} C_i$, there is $H \in [\omega]^\omega$ such that $[H]^k \subseteq C_i$ for some i .

For $k \leq \omega$, let $(\omega)^k$ denote the set of all partitions of ω into k parts.

Given $x \in (\omega)^\omega$ and $k < \omega$, let $(x)^k$ be the set of all coarsenings $y \in (\omega)^k$ of x .

Dual Ramsey's theorem for $k, \ell < \omega$ (DRT_ℓ^k).

If $(\omega)^k = \bigcup_{i < \ell} C_i$ is Borel, there is $x \in (\omega)^\omega$ such that $(x)^k \subseteq C_i$ for some i .

A short history.

Introduced and proved by **Carlson and Simpson** (1984). Extended by **Prömel and Voigt** (1985) to colorings with the Baire property.

Miller and Solomon (2004) showed $\text{RCA}_0 \vdash \text{Open-DRT}_\ell^{k+1} \rightarrow \text{RT}_\ell^k$.

Blass, Hirst, and Simpson (1987), **Miller and Solomon**, and **Erhard** (2013) all studied the Carlson-Simpson lemma (CSL), the combinatorial core of DRT.

Blass, Hirst, and Simpson (1987) showed $\Pi_2^1\text{-CA}_0 \vdash \text{CSL}$. **Slaman** (unpublished) improved this to $\Pi_1^1\text{-CA}_0 \vdash \text{CSL}$. It is unknown whether $\text{RCA}_0 \vdash \text{CSL}_2^2$.

Miller and Solomon (2004) showed $\text{WKL}_0 \not\vdash \text{VW}_2^2$. **Erhard** (2013) showed that $\text{COH} \not\vdash \text{VW}_2^2$ and $\text{SRT}_2^2 \not\vdash \text{OVW}_2^2$.

All of this deals essentially only with open colorings.

Combinatorial principles.

Definition. A coloring $(\omega)^k = \bigcup_{i < \ell} C_i$ is **reduced** if the color of $x \in (\omega)^k$ depends only on the least element a of the k th block of x and which blocks the numbers $b < a$ belong to.

Carlson-Simpson lemma (CSL $_{\ell}^k$).

If $(\omega)^k = \bigcup_{i < \ell} C_i$ is reduced, there is $x \in (\omega)^\omega$ and $i < \ell$ such if $y \in (x)^k$ keeps the first $k - 1$ blocks of x distinct then $y \in C_i$.

Note that every reduced coloring is open.

The following natural variant the CSL is proved by $\omega \cdot k$ many iterations of CSL.

Combinatorial dual Ramsey's theorem (CDRT $_{\ell}^k$).

If $(\omega)^k = \bigcup_{i < \ell} C_i$ is reduced, there is $x \in (\omega)^\omega$ and $i < \ell$ such that $(x)^k \subseteq C_i$.

The Baire DRT.

We code a Baire ℓ -coloring by open sets $O_0, \dots, O_{\ell-1}$ and a sequence of dense open sets $\{D_n\}_{n \in \omega}$, representing that $O_i \cap \bigcap_n D_n \subseteq C_i$ for all i .

This allows us to formulate Baire-DRT $_{\ell}^k$ in second-order arithmetic.

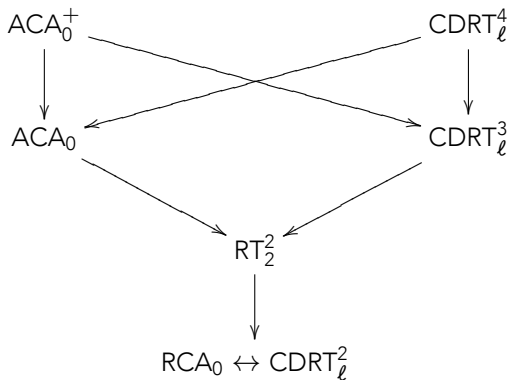
Theorem (Dzhafarov, Flood, Solomon, and Westrick).

Over RCA_0 , the following are equivalent for all $k, l < \omega$:

1. Open-DRT $_{\ell}^k$.
2. Baire-DRT $_{\ell}^k$.
3. CDRT $_{\ell}^k$.

In particular, we obtain bounds for CDRT that we lack for CSL.

Relationships.



Folklore. $ACA_0^+ \rightarrow$ Hindman's theorem $\rightarrow CDRT_\ell^3$.

The implication $CDRT_\ell^4 \rightarrow ACA_0$ follows by results of **Miller and Solomon**.

The Borel DRT.

Theorem (Dzhafarov, Flood, Solomon, and Westrick).

Over RCA_0 , the following are equivalent for all $k, \ell < \omega$:

1. $\text{Borel-DRT}_{\ell}^k$;
2. $\text{Baire-DRT}_{\ell}^k + \text{ATR}_0$.

The implication from Borel-DRT to Baire-DRT is a coding argument.

That Borel-DRT implies ATR_0 is not for any deep reason; the statement that for every Borel set, there a point in it or not in it, already implies ATR_0 .

The implication from 2 to 1 uses ATR_0 to formalize that every Borel set is Baire.

Since $\text{Baire-DRT} \leftrightarrow \text{CDRT}$, it follows that the strength of (Borel) DRT can be understood entirely in combinatorial terms.

Effective analysis, $k \geq 3$.

Throughout, fix $k \geq 3$.

Recall that a **modulus** is a function f such that if $f \leq g$ then $f \leq_T g$.

Lemma. Let f be a modulus. There is an f -computable clopen coloring $(\omega)^k = C_0 \cup C_1$ such that $f \leq_T x$ for each homogeneous $x \in (\omega)^\omega$.

Lemma. If $a \in \mathcal{O}$ and $(\omega)^k = C_0 \cup C_1$ where the C_i are H_a -computable and clopen, then the C_i have computable codes as topologically Δ_a^0 sets.

Theorem (Dzhafarov, Flood, Solomon, and Westrick).

For each computable ordinal α there is a computable, topologically $\Delta_{\alpha+1}^0$ coloring $(\omega)^k = C_0 \cup C_1$ such that $\emptyset^{(\alpha)} \leq_T x$ for each homogeneous $x \in (\omega)^\omega$.

Every hyperarithmetic set A has a self-modulus, i.e., a modulus $f \equiv_T A$.

Effective analysis, $k = 2$.

Though $\text{Borel-DRT}_\ell^2 \rightarrow \text{ATR}_0$, we lack the $\emptyset^{(\alpha)}$ -coding that we had for $k \geq 3$.

For sufficiently nice colorings, there are more effective solutions.

Proposition (Dzhafarov, Flood, Solomon, and Westrick).

If $(\omega)^2 = C_0 \cup C_1$ where C_0 is effectively open, then there is a computable $x \in (\omega)^\omega$ such that $(x)^2 \subseteq C_i$ for some i .

This extends a result of Katz (unpublished), who established the same for C_0 effectively clopen. Our proof is necessarily non-uniform.

Theorem (Dzhafarov, Flood, Solomon, and Westrick).

If $(\omega)^2 = C_0 \cup C_1$ where C_0 is effectively Σ_2^0 , then there is either a computable $x \in (\omega)^\omega$ with $(x)^2 \subseteq C_1$, or a \emptyset' -computable $x \in (\omega)^\omega$ with $(x)^2 \subseteq C_0$.

Simple colorings.

A coloring $\omega^2 = C_0 \cup C_1$ is **simple** if the color of $x \in (\omega)^2$ depends only on the least element of the non-zero block.

Let $\Delta_n^0\text{-D}_2^2$ be the statement that given $c : [\omega]^n \rightarrow 2$ such that $\lim_{s_{n-2}} \cdots \lim_{s_0} c(k, s_0, \dots, s_{n-2})$ exists for all k , there is an i and an infinite set L such that $\lim_{s_{n-2}} \cdots \lim_{s_0} c(k, s_0, \dots, s_{n-2}) = i$ for all $k \in L$.

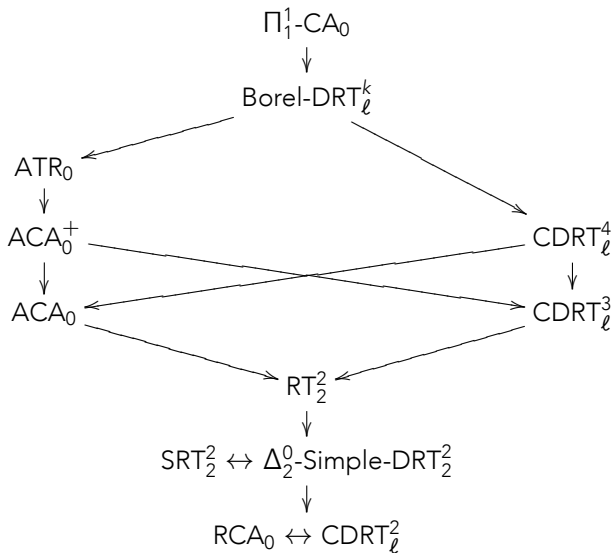
So $\Delta_2^0\text{-D}_2^2$ is the well-known D_2^2 , which is equivalent to SRT_2^2 over RCA_0 .

Proposition (Dzhafarov, Flood, Solomon, and Westrick).

Over RCA_0 , the following are equivalent:

1. Simple-DRT $_2^2$ for effectively Σ_n^0 colorings;
2. $\Delta_n^0\text{-SRT}_2^2$.

Summary.



Questions.

1. What is the strength of CDRT_{ℓ}^k ?
2. Is CDRT strictly stronger than CSL?

Thank you for your attention.