Existence of fixed points for monotone operators

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Joint work with Sean Walsh.

Out of paradox.

There are numerous approaches for getting around the various famous paradoxes of self-reference:

- hierarchies/types (Russell, Gödel);
- truth predicates (Hilbert and Bernays, Tarski).

A formal theory of truth is a first-order theory that treats truth as a primitive unary predicate (of Gödel numbers of sentences).

Our project has two different but related goals:

- Analyze the complexity of constructions that show up in the study of formal theories of truth.
- Given a formal theory of truth, determine a subsystem of second-order arithmetic with the same first-order consequences.

Kripke's strong logic.

Let $L[\mathbb{T}]$ be the language of first-order arithmetic augmented by one new unary predicate symbol $\mathbb{T}$.

The strong Kleene valuation scheme gives a 3-valued interpretation for $L[\mathbb{T}]$ that is constructed iteratively through the ordinals.

- Interpret all arithmetical sentences (those not involving $\mathbb{T}$) as usual.

- Now suppose S is a consistent set S of $L[\mathbb{T}]$-sentences. We interpret $\mathbb{T}(\sigma)$ to be true if $\sigma \in S$, and false if $\neg\sigma \in S$.

(So if $\sigma \notin S$ and $\neg\sigma \notin S$ for some σ then $\mathbb{T}(\ulcorner\sigma\urcorner)$ has no truth value.)

Let S' be the set of sentences that come out true under this interpretation. Then $S \subseteq S'$ and S' is consistent.

Kripke fixed points.

Identify $L[\mathbb{T}]$-sentences with their Gödel numbers.

- Let $S_0 = \emptyset$.

- Given S_α, let $S_{\alpha+1} = S'_\alpha$

- For λ limit, let $S_\lambda = \bigcup_{\alpha<\lambda} S_\alpha$.

Thus, $S_\alpha \subseteq S_\beta \subseteq \omega$ for all $\alpha \leq \beta$.

By construction, for all σ in $L[\mathbb{T}]$ we have $\sigma \in S_\alpha \leftrightarrow \mathbb{T}(\ulcorner\sigma\urcorner) \in S_{\alpha+1}$.

Fix a γ such that $S_\gamma = S_{\gamma+1}$. Then we have $\sigma \in S_\gamma \leftrightarrow \mathbb{T}(\ulcorner\sigma\urcorner) \in S_\gamma$.

S_γ is the Kripke fixed point.

Monotone operators.

A monotone operator is a function $\theta : 2^\omega \to 2^\omega$ such that if $X \subseteq Y$ then $\theta(X) \subseteq \theta(Y)$.

A monotone operator θ is computable or arithmetical if the relation $m \in \theta(X)$ is uniformly Δ^0_1 in X or arithmetical in X, respectively.

The complexity of *least* fixed points of various kinds of definable monotone operators was investigated by Cenzer and Remmel (2005).

The least fixed point can always be obtained simply by iterating the operator on $\emptyset$. (This is what we saw in the Kripke construction.)

Other fixed points may exist, and in general their complexities will be very different from the complexity of the least fixed point.

Computable monotone operators.

A computable monotone operator is a a total Turing functional Φ satisfying $X \subseteq Y \implies \Phi(X) \subseteq \Phi(Y)$ for all oracles X, Y.

The (non-empty) collection of fixed points of such a Φ forms a Π^0_1 class. The least fixed point is thus always computable from $\emptyset'$.

Cenzer and Remmel (2005) note that there exists a Φ as above such that $\emptyset'$ is 1-1 reducible to its least fixed point.

Formalizing this in RCA_0 yields that the existence of a least fixed point for every computable monotone operator is equivalent to ACA_0.

Theorem (D. and Walsh)**.** Over $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$, the existence of a fixed point for every computable monotone operators is equivalent to WKL_0.

The extra induction seems necessary in both directions of this proof.

Arithmetical monotone operators.

Proposition (D. and Walsh)**.** Over RCA_0, the existence of a fixed point for every arithmetical monotone operator implies ACA_0.

For the implication to ACA_0, we want an arithmetical monotone operator θ such that if X is any fixed point of θ then $\emptyset' \leq_T X$.

Here is a perfectly good one: $\theta(X) = \emptyset'$ for all X.

Alas, RCA_0 cannot prove that θ is total.

Question. Is the existence of a fixed point for every arithmetical monotone operator provable in ACA_0?

Theorem (D. and Walsh)**.** Over RCA_0, the existence of a *least* fixed point for every arithmetical monotone operator is equivalent to Π^1_1-CA_0.

Reverse mathematics and KFP.

Kripke's construction of a fixed point can be formalized in $\Pi^1_1\text{-}\mathsf{CA}_0$. Combining with our our result that the least fixed point for every arithmetical monotone operator is equivalent to $\Pi^1_1\text{-}\mathsf{CA}_0$ yields:

Theorem (essentially Burgess, 1989)**.** Over RCA_0, the existence of a least Kripke fixed point is equivalent to $\Pi^1_1\text{-}\mathsf{CA}_0$.

This gives an upper bound on the complexity of the existence of Kripke fixed points in general.

Theorem (D. and Walsh)**.** Over RCA_0, the existence of Kripke fixed points is provable in $\Pi^1_1\text{-}\mathsf{CA}_0$ and implies ACA_0.

Question. Does the existence of a Kripke fixed point imply $\Pi^1_1\text{-}\mathsf{CA}_0$?

Kripke-Feferman Axioms (KF).

KF consists of PA[$\mathbb{T}$] plus 13 additional "truth schemes", e.g.:

- $\mathbb{T}(\ulcorner\varphi\urcorner) \leftrightarrow \varphi$ is satisfiable, for $\varphi \in \Sigma^0_0$;

- $\mathbb{T}(\ulcorner\neg\varphi\urcorner) \leftrightarrow \neg\varphi$ is satisfiable, for $\varphi \in \Sigma^0_0$;

- $\mathbb{T}(\ulcorner\neg\neg\varphi\urcorner) \leftrightarrow \mathbb{T}(\ulcorner\varphi\urcorner)$;

- $\mathbb{T}(\ulcorner\varphi \wedge \psi\urcorner) \leftrightarrow \mathbb{T}(\ulcorner\varphi\urcorner) \wedge \mathbb{T}(\ulcorner\psi\urcorner)$;

- $\mathbb{T}(\ulcorner\mathbb{T}(\ulcorner\varphi\urcorner)\urcorner) \leftrightarrow \mathbb{T}(\ulcorner\varphi\urcorner)$;

- $\mathbb{T}(\ulcorner\neg\mathbb{T}(\ulcorner\varphi\urcorner)\urcorner) \leftrightarrow \mathbb{T}(\ulcorner\neg\varphi\urcorner)$.

Notably absent from KF is the $\mathbb{T}$-schema, $\mathbb{T}(\ulcorner\sigma\urcorner) \leftrightarrow \sigma$.

Feferman's conservation results.

The first-order part of KF is somewhat understood.

Theorem (Feferman, 1991)**.**

Let φ be an L-sentence.

- If KF proves φ, then ACA proves φ.

- If ACA proves φ, then $\mathsf{KF} + \mathsf{Consistency}$ proves φ.

Here, $\mathsf{Consistency}$ is $\forall\varphi\ \neg(\mathbb{T}(\ulcorner\varphi\urcorner) \wedge \mathbb{T}(\ulcorner\neg\varphi\urcorner))$.

Feferman's proof proceeds via ordinal analysis and a fine analysis of ramified hierarchies.

We give more direct proofs using methods familiar from, e.g., the study of subsystems of second order arithmetic.

New proof via model expansions.

Theorem. If If ACA proves φ, then KF + Consistency proves φ.

KF + Consistency proves that if $\mathbb{T}(\ulcorner\varphi\urcorner) \vee \mathbb{T}(\ulcorner\neg\varphi\urcorner)$ holds for some $L[\mathbb{T}]$-formula φ then $\mathbb{T}(\ulcorner\varphi)\urcorner \leftrightarrow \varphi$.

Given a model N of KF + Consistency we define a model M of ACA:

- Let M and N have the same first-order part, $\mathbb{N}$.

- Let the second-order part of M consist of all D-sets: a set X is a D-set if there is a an $L[\mathbb{T}]$-formula φ such that
  - $N \models \mathbb{T}(\ulcorner\varphi\urcorner) \vee \mathbb{T}(\ulcorner\neg\varphi\urcorner)$,
  - $X = \{x \in \mathbb{N} : \varphi(x)\}$.

M satisfies full second-order induction, and Σ^0_1-comprehension.

Further conservation results.

We can use these and similar methods to obtain extensions of Feferman's conservation theorem.

Theorem (D. and Walsh)**.**

Let φ be an L-sentence.

If ACA proves φ, then $\mathsf{KF} + \mathsf{Completeness}$ proves φ.

Here, $\mathsf{Completeness}$ is $\forall\varphi\ (\mathbb{T}(\ulcorner\varphi\urcorner) \vee \mathbb{T}(\ulcorner\neg\varphi\urcorner))$.

Question.

Given an an φ be an L-sentence, is it the case that if ACA proves φ then necessarily KF proves φ?

Thank you for your attention!