Amenable Groups and Randomness

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The Setting

- A – a countable alphabet.
- G – a computable group.
- A^G – the set of functions from G to A.

Place the product topology on A^G.

Question:
Can we transfer the theory of algorithmic randomness, particularly prefix-free complexity to A^G?

The Uninteresting Part

- Fix an isomorphism between G and $\mathbb{N}$.
- This gives a homeomorphism between A^G and $A^{\mathbb{N}}$.
- Transfer the theory via this homeomorphism.
- Use this to define 1-randomness.

Uninteresting because this ignores the group structure of G. But where exactly does the groups structure come in?

Let's postpone the answer to this question until after we look at initial segment complexity.

Approximation Sequences

Call $(F_i)_{i\in\mathbb{N}}$ an *approximation sequence* to G if

- Each F_i is a finite subset of G.
- Each element of G is contained in all but finitely many F_i.

Example

- An approximation sequence to $\mathbb{Z}$ is given by $F_i = [-i, \ldots, i]$.
- If G is finitely generated, we can take

$$F_n = \{g \in G \colon g = s_1 \ldots s_n \text{ where each } s_i \text{ is a generator or its inverse or } 1_G\}.$$

If $x \in A^G$ and $(F_i)_{i\in\mathbb{N}}$ is an approximation sequence to G, then we will think of the "initial segments" of x as being $x\restriction_{F_0}, x\restriction_{F_1}, \ldots$.

Initial Segment Complexity for Elements of A^G

- If $x \restriction_{F_n}$ is an initial segment of x, then what is its prefix free complexity?
- If $\sigma \in A^{F_n}$, then we can regard σ as a finite subset of $G \times A$. $K(\sigma)$ can be defined to be the complexity of this finite subset.
- Note that given a description of σ we can uniformly compute the domain of σ as well as the values of $\sigma(g)$ for each g in the domain.

Dimension

We will use initial segment complexity to look at analogues of effective Hausdorff dimension and effective packing dimension for elements of A^G.

Definition

- $\dim(x) := \liminf_{n\to\infty} \frac{K(x \restriction_{F_n})}{|F_n|}$
- $\operatorname{Dim}(x) := \limsup_{n\to\infty} \frac{K(x \restriction_{F_n})}{|F_n|}$

Note that this definition is dependent on the approximation sequence picked, but we will see that in certain cases we can remove this dependence.

Group Actions as Dynamical Systems

- Let X be a set and let $T : X \to X$ be an automorphism of X.
- We can regard this as $\mathbb{Z}$ acting on X with the action defined by

$$a(n, x) = T^n(x).$$

We will use $\cdot$ to denote the left-shift action of G on A^G. If $g \in G$ and $x \in A^G$, then $g \cdot x$ is the element of A^G is defined by

$$(g \cdot x)(h) = x(g^{-1}h).$$

Invariance of Dimension

- Given a description of $x \restriction_{F_n}$ and g, how much more information do we need to determine $(g \cdot x) \restriction_{F_n}$?
- If $h \in gF_n$ then we already know $(g \cdot x)(h)$.

[Because $(g.x)(h) = x(g^{-1}h)$.]

Hence $\quad K((g \cdot x) \restriction_{F_n}) \leq K(x \restriction_{F_n}) + K(g) + \lceil \log(A) \rceil |F_n \setminus gF_n|$

$$\liminf_{n\to\infty} \frac{K((g \cdot x) \restriction_{F_n})}{|F_n|} \leq \liminf_{n\to\infty} \frac{K(x \restriction_{F_n})}{|F_n|} + \lceil \log(A) \rceil \liminf_{n\to\infty} \frac{|F_n \setminus gF_n|}{|F_n|}$$

If this last term tends to 0 then $\dim(g \cdot x) \leq \dim(x)$.

Følner Sequences

Definition

An approximation sequence $(F_i)_{i\in\mathbb{N}}$ to G is called a *Følner sequence* if for all $g \in G$,

$$\lim_n \frac{|gF_n \Delta F_n|}{|F_n|} = 0.$$

If we define dimension using Følner sequences, then for all $g \in G$ and all $x \in A^G$

- $\dim(g \cdot x) = \dim(x)$
- $\mathrm{Dim}(g \cdot x) = \mathrm{Dim}(x)$

When does a group have a Følner sequence?

Amenable Groups

Theorem

A countable group G has a Følner sequence if and only if it is amenable.

Definition

A group G is *amenable* if there exists a finitely additive measure μ on the powerset of G such that $\mu(G) = 1$ and for all $g \in G$ and $E \subseteq G$, $\mu(gE) = \mu(E)$.

Theorem (Tarski)

A group G is paradoxical if and only if it is not amenable.

Examples of Amenable Groups

- All abelian groups are amenable.
- All finitely generated groups of polynomial growth are amenable.
- Subgroups of amenable groups are amenable.
- If N is a normal subgroup of G and each of N, G/N are amenable then so is G

Topological Entropy

- Let X be a closed subset of A^G that is also closed under the left shift action i.e. $g \in G$ and $x \in X$ implies $g \cdot x \in X$.
- As the mappings $x \mapsto g \cdot x$ are continuous, we can consider X and the left shift as a topological dynamical system.
- Let $X \restriction_{F_n} = \{x \restriction_{F_n} : x \in X\}$

Definition

The *topological entropy* of X is denoted $\mathrm{ent}_T(X)$ and defined to be

$$\lim_{n \to \infty} \frac{\log |X \restriction_{F_n}|}{|F_n|}.$$

Topological Entropy and Dimension

Theorem

Let G be a computable amenable group and let X be a computable subshift of A^G. If for all $x \in X$, $\dim(x) \leq s$, then $\mathrm{ent}_T(X) \leq s$.

This implies that

$$\mathrm{ent}_T(X) = \inf_{Z \in 2^{\mathbb{N}}} \sup\{\dim^Z(x) \colon x \in X\}.$$

Hence $\mathrm{ent}_T(X)$ is equal to the Hausdorff dimension of X (by Lutz, Mayordomo and Hitchcock).

- Case G is $\mathbb{N}$ is due to Furstenberg.
- Case G is $\mathbb{N}^d$ or $\mathbb{Z}^d$ is due to Simpson (2014).
- Case G is an amenable group is new. (Dimension must be defined using an appropriate Følner sequence.)

Ornstein and Weiss's Work

- Used to prove Shannon-McMillan-Brieman theorem for a subclass of amenable groups.
- After Lindenstrauss adapted their techniques to give a new proof for all countable amenable groups.
- Need to restrict to bi-invariant Følner sequences that are tempered

$$\left| \bigcup_{i \leq n} F_i^{-1} F_{n+1} \right| \leq b \left| F_{n+1} \right|$$

Ergodic Group Actions

Let $(X, \mathcal{X})$ be a measurable space and μ a probability measure on this space. A group action $a : G \times X \to X$ is *measure preserving* if

- For each $g \in G$, $x \mapsto a(g, x)$ is measurable.
- For each $g \in G$ and $E \in \mathcal{X}$, $\mu(a(g, E)) = \mu E$.

A measure preserving group action is *ergodic* if for all $E \in \mathcal{X}$ and all $g \in G$ $a(g, E) \subseteq E$ implies that $\mu E = 0$ or $\mu E = 1$.

Question

If $a : G \times X$ is an ergodic action for $(X, \mathcal{X})$, and $E \in \mathcal{X}$ is it true that for μ almost all $x \in X$,

$$\lim_{n\to\infty} \frac{|\{g \in F_n \colon a(g, x) \in E\}|}{|F_n|} = \mu E?$$

i.e. Does Birkhoff's ergodic theorem hold.

Lindenstrass Theorem

- (Lindenstrass 1999) Birkhoff's ergodic theorem holds if G is an amenable group.
- Provided we use tempered bi-invariant Følner sequences.
- (Moriakov 2017) Has effectivised this proof and shown that it holds for all 1-random points.
- Lindenstrass also generalised the Shannon-McMillan-Breiman Theorem to amenable groups.

Entropy

Definition

Let P be a discrete probability measure on the countable set $\{c_1, c_2, \ldots\}$. The *Shannon entropy* of P is defined by

$$H(P) = \sum_{i=1}^{\infty} -P(c_i) \log P(c_i).$$

"The expected length of an optimal prefix-free code."

Kolmogorov-Sinai Entropy

Let's return to the space A^G and the left shift action of G on A^G.

Let μ be a measure on A^G such that left shift action is ergodic.

- If $\sigma \in A^{F_n}$, denote by $[\![\sigma]\!]$ the set $\{x \in A^G : x \restriction_{F_n} = \sigma\}$.
- Define $H_n = \sum_{\sigma \in A^{F_n}} \mu[\![\sigma]\!] . \log \mu[\![\sigma]\!]$.
- The Kolmogorov-Sinai entropy of μ is defined to be

$$h(\mu) = \lim_{n \to \infty} \frac{H_n}{|F_n|}.$$

Shannon-McMillan-Breiman

Theorem (Lindenstrass (1999))

Let μ be an ergodic measure for the left-shift action on A^G. Let h be the Kolmogorov-Sinai entropy of $(A^G, \cdot, \mu)$. Let (F_n) be a tempered Følner sequence for G. Then for μ-almost all $x \in A^G$.

$$\lim_n \frac{-\log \mu[\![x \restriction_{F_n}]\!]}{|F_n|} = h.$$

This is a simplified version of Lindenstrass's result.

Effective Version

By ergodicity, there are h_u and h_l such that for μ almost all $x \in A^G$,

$$\dim(x) = h_l \quad \text{and} \quad \mathrm{Dim}(x) = h_u.$$

Theorem (Shannon-McMillan-Breiman effective version)

Let G be a computable group and let μ be a computable ergodic measure for the left-shift action on A^G. Let h be the Kolmogorov-Sinai entropy of $(A^G, \cdot, \mu)$. If dimension is defined using a tempered Følner sequence, then

If x is μ 1-random, $\dim(x) = \mathrm{Dim}(x) = h$.

The case that G is $\mathbb{N}$ was proved by V'yugin (1998), Hoyrup (2013).

Entropy as an Isomorphism Invariant

- Let A be an alphabet of size n. Call the system (A^G, μ) where μ is the product of uniform measures on A, the *full n shift over G*.
- Kolmogorov-Sinai entropy originates in the proof that the full 2 shift over $\mathbb{Z}$ is not isomorphic to the full 3 shift over $\mathbb{Z}$.
- In fact there is no factor map from the full 2 shift over $\mathbb{Z}$ to the full 3 shift over $\mathbb{Z}$.
- Reason: factor maps must be decreasing in entropy.

Theorem (Ornstein-Weiss)

If G is infinite and amenable then the Kolmogorov-Sinai entropy classifies Bernoulli shifts over G up to isomorphism

Entropy for Non-Amenable Groups

- Bowen - Entropy for Free groups and then generalised to Sofic groups.
- Seward - Rokhlin entropy.

Future directions analyse these entropies from the perspective of algorithmic randomness.