On low for speed sets

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Oberwolfach Workshop “Computability Theory”

January 11, 2018
Lowness for speed

A recurring theme in computability theory:

\( \text{Low}(\mathcal{N}) \) = set of oracles \( X \) such that relativizing the notion \( \mathcal{N} \) to \( X \) leaves it unchanged.

- \( \mathcal{N} = \text{halting set} \rightarrow \text{Low}(\mathcal{N}) = \text{low} \)
- \( \mathcal{N} = \text{ML-random} \rightarrow \text{Low}(\mathcal{N}) = \text{K-trivials} \)
- \( \mathcal{N} = \text{weakly 1-generic (or Kurtz random)} \rightarrow \text{Low}(\mathcal{N}) = \text{non-dnr + hyperimmune-free} \)
Lowness for speed

Allender proposed to study **lowness for speed**: 

**Definition (Allender)**

$X$ is low for speed (l.f.s) if every *decidable* set/language $L$ that can be computed with oracle $X$ in time $f$ can be computed without oracle in time $\text{poly}(f)$.

(model of computation: Turing machine with a dedicated tape; the machine may write $n$ on this tape then query the oracle $X$ as to whether $n \in X$).
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(model of computation: Turing machine with a dedicated tape; the machine may write $n$ on this tape then query the oracle $X$ as to whether $n \in X$).

Does such an $A$ exist? Obviously yes: take $A$ to be in PTIME-computable! (note: $X$ computable but EXPTIME-complete would not work, so lowness for speed is **not** closed under $\equiv_T$).
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There exists $A$ non-computable and c.e. that is l.f.s.
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**Theorem (Bayer-Slaman)**

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Proof is a priority argument. One constructs $A$ to be sparse, so that at stage $t$ there are few candidates for $A | t$, thus for a functional $\Phi$ one can try to simulate all possible $\Phi^A$ in parallel (+ some very nice twist to handle Friedberg-Muchnik requirements).
Lowness for speed

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2. What is the situation outside c.e. sets? How big is the set LFS in terms of cardinality/category/measure? (category answered by Bayer and Slaman)
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2. What is the situation outside c.e. sets? How big is the set LFS in terms of cardinality/category/measure? (category answered by Bayer and Slaman)
3. Closing under $\equiv_T$: what are the $X$ such that equivalent to some low for speed? (note: every degree contains a non low for speed). Are such $X$ closed downwards? under join?
Within c.e. sets

Can we characterize the c.e. sets in LFS? Seems very hard, but one can get partial results.

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If a c.e. set $A$ is promptly simple, it is not l.f.s.
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**Theorem (Bayer-Slaman)**
There is a non-prompt c.e. set \( A \) such that \( A \) is not l.f.s., nor any \( B \equiv_T A \).
Within c.e. sets

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**Theorem (BD)**
If $A \geq_T \emptyset'$, then $A$ is not l.f.s. (does not require $A$ to be c.e.).
Within c.e. sets

How does lowness for speed fit in the high/low hierarchy?
Within c.e. sets

How does lowness for speed fit in the high/low hierarchy?

Theorem (BD)

- There is a high c.e. set that is low for speed.
- A non-computable c.e. low set $A$ cannot be low for speed(!)
- There is a non-computable low$_2$ c.e. set that is low for speed.
Outside the c.e. world

How common are low for speed sets? Can/should a generic be low for speed? How about randoms?
The strange case of generics

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So we might not know for a while whether LFS is meager or co-meager.
The strange case of generics

However, LFS contains an homeomorphic copy of the 1-generics. Consider a doubly-exponentially sparse set $S$ such as

$$S = \{2^{2^n} \mid n \in \mathbb{N}\}$$

and define

$$S_X = \{2^{2^n} \mid n \in X\}$$
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A fairly direct proof gives us:

**Theorem (BD)**

If $G$ is 1-generic, then $S_G$ is low for speed.
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Randomness vs lowness for speed

Like for generics, one could expect a conditional behaviour of randoms w.r.t. lowness for speed, for example a dependance on the answer to $P = BPP$. This is not the case:

Theorem (BD)
If $A$ is Schnorr random, it is not l.f.s.

A Schnorr random can however be equivalent to a l.f.s. (take a l.f.s. of high degree).

However, unlike for generics (assuming $P \neq NP$), the phenomenon disappears for Martin-Löf randomness. In fact:

Theorem (BD)
If $A$ has DNC degree, it is not low for speed.

Proof inspired by Blum's speedup theorem.
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Proof: Take a 2-generic $G_0$ and consider $G_1 = G_0 \Delta \emptyset'$, also 2-generic. Both $G_0$ and $G_1$ are Turing equivalent to a l.f.s. set, but $G_0 \oplus G_1 \geq_T \emptyset'$ is not.
Turing degrees and LFS

Some more results on the Turing degrees of l.f.s. sets.

Theorem (BD)
The Turing degrees of LFS are not closed downwards.

Proof: extend the earlier result to show that a low c.e. degree does not contain any l.f.s. set. Take a non-computable c.e. set $X$ which is l.f.s. and apply Sack's splitting theorem to get a low c.e. $Y$ with $0 < T_Y < T_X$.

How lowness for speed interacts with minimality is not fully solved, but we know at least:

Theorem (BD)
There exists a minimal Turing degree which does not contain any l.f.s. set.

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**Theorem (BD)**
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Proof: extend the earlier result to show that a low c.e. *degree* does not contain any l.f.s. set. Take a non-computable c.e. set $X$ which is l.f.s. and apply Sack’s splitting theorem to get a low c.e. $Y$ with $0 <_T Y <_T X$.

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Thank you!