Theories of Classes of Structures

Asher M. Kach
University of Chicago

(Joint Work with Antonio Montalbán)

Computability Theory Meeting
Oberwolfach, Germany
February 2012

Outline

1. Context and Questions

2. Some Answers and More Questions
   - Cardinals
   - Linear Orders
   - Boolean Algebras
   - Groups

3. Open Questions

A Fixed Question...

Question

Is there an element x such that $x \neq x + x$ and $x = x + x + x$?

A Fixed Question...

Question

Is there an element x such that $x \neq x + x$ and $x = x + x + x$?

- Is there a cardinal x such that $x \neq x + x$ and $x = x + x + x$?
- Is there a linear order x such that $x \ncong x + x$ and $x \cong x + x + x$?
- Is there a Boolean algebra x such that $x \ncong x \oplus x$ and $x \cong x \oplus x \oplus x$?
- Is there a group x such that $x \ncong x \times x$ and $x \cong x \times x \times x$?

Tarski's Cube Problem...

Remark

The last (i.e., for Boolean algebras) is known as Tarski's Cube Problem. It remained open for decades, with Ketonen giving a positive answer in 1978.

Tarski's Cube Problem...

Remark

The last (i.e., for Boolean algebras) is known as Tarski's Cube Problem. It remained open for decades, with Ketonen giving a positive answer in 1978.

Indeed, Ketonen showed any commutative semigroup embeds into $\mathbb{BA}_{\aleph_0}^{\oplus} := (\mathrm{BA}_{\aleph_0}; \oplus)$, the commutative monoid of countable Boolean algebras under direct sum. Consequently, the Σ_1-theory of $\mathbb{BA}_{\aleph_0}^{\oplus}$ is decidable.

Ketonen asked whether the first-order theory of $\mathbb{BA}_{\aleph_0}^{\oplus}$ is decidable.

The General Question...

Question

Let S be a set of isomorphism types of structures. Let $\mathbb{S}$ be the structure with universe S with (natural) relations and functions. How complicated is the first-order theory of $\mathbb{S}$?

The General Question...

Question

Let S be a set of isomorphism types of structures. Let $\mathbb{S}$ be the structure with universe S with (natural) relations and functions. How complicated is the first-order theory of $\mathbb{S}$?

- How complicated is the first-order theory of $\mathbb{CARD}_\kappa^+$?
- How complicated is the first-order theory of $\mathbb{LO}_\kappa^+$?
- How complicated is the first-order theory of $\mathbb{BA}_\kappa^\oplus$?
- How complicated is the first-order theory of $\mathbb{GR}_\kappa^\times$?

The General Question...

Question

Let S be a set of isomorphism types of structures. Let $\mathbb{S}$ be the structure with universe S with (natural) relations and functions. How complicated is the first-order theory of $\mathbb{S}$?

- How complicated is the first-order theory of $\mathbb{CARD}_\kappa^+$?
- How complicated is the first-order theory of $\mathbb{LO}_\kappa^+$?
- How complicated is the first-order theory of $\mathbb{BA}_\kappa^\oplus$?
- How complicated is the first-order theory of $\mathbb{GR}_\kappa^\times$?

Do any of these questions depend on κ (provided $\kappa \geq \aleph_0$)?

Outline

1 Context and Questions

2 Some Answers and More Questions
- Cardinals
- Linear Orders
- Boolean Algebras
- Groups

3 Open Questions

Answers for $\mathbb{CARD}_\kappa^+$...

Theorem (Feferman and Vaught)

[ZFC] The first-order theory of $\mathbb{CARD}_\kappa^+$ is decidable. Moreover, it depends only on the remainder of dividing α by $\omega^\omega + \omega^\omega$, where $\kappa = \aleph_\alpha$.

Proof.

For decidability, recursively transform any sentence φ into a sentence ψ_φ such that

$$\mathbb{CARD}_\kappa^+ \models \varphi \text{ if and only if } \mathbb{CARD}_\kappa^+ \models \psi_\varphi$$

and any (in)equality of ψ_φ is explicitly of finite cardinals or of infinite cardinals. It then suffices to show that $(\mathbb{N}; +)$ and $(\alpha + 1; \max)$ are decidable.

For the characterization, exploit that $\aleph_{\delta+\omega^k \cdot n_k + \cdots + \omega \cdot n_1 + n_0}$ is a definable singleton of CARD_κ (provided it exists) using $\aleph_\delta$ as a parameter. □

More Questions for $\mathbb{CARD}^+_\kappa$...

Question

Is there a model of *ZF* with a set C of cardinals such that $\mathbb{CARD}^+_C$ is not decidable? At the very least, require C to be *nice*, for example downward closed and closed under addition, if not an initial segment.

Remark

An obvious method would be to construct a model of *ZF* having a maximal antichain of cardinals of size $n \in \mathbb{N}$ if and only if n is in some predescribed set $T \subseteq \mathbb{N}$.

Unfortunately, my understanding is that set theorists do not know if this possible.

Answers for $\mathbb{LO}_\kappa^+$...

Theorem

The first-order theory of $\mathbb{LO}_\kappa^+$, for $\kappa \geq \aleph_0$, computes true second-order arithmetic. Moreover, the structure $\mathbb{LO}_{\aleph_0}^+$ is bi-interpretable with second-order arithmetic.

Theorem

The first-order theory of $\mathbb{LO}_\kappa^+$, for $\kappa \geq \aleph_0$, computes true second-order arithmetic. Moreover, the structure $\mathbb{LO}_{\aleph_0}^+$ is bi-interpretable with second-order arithmetic.

Proof.

Repeatedly exploit the relation $u \trianglelefteq v$ that holds exactly if

$$(\exists w_1)(\exists w_2)\left[v = w_1 + u + w_2\right].$$

Using it, establish the definability of various order types: ω, ζ, ζ^2, and so on. □

Answers for $\mathbb{LO}_\kappa^+$...

Proof.

Encode the integer $n \in \mathbb{N}$ by the order type $\mathbf{n}$. Then the set of natural numbers is a definable subset of LO_κ, namely the set of all x with $x \lhd \omega$.

The less than relation $\leq$ is definable as $m \leq n$ if and only if $\mathbf{m} \unlhd \mathbf{n}$.

Addition is definable as $m + n = p$ if and only if $\mathbf{m} + \mathbf{n} = \mathbf{p}$.

Code an ℓ-tuple $\overline{n} = (n_1, \ldots, n_\ell) \in \mathbb{N}^\ell$ by the order type

$$t_\ell(\overline{n}) := \zeta^2 + \mathbf{n_1} + \zeta + \cdots + \mathbf{n}_\ell + \zeta + \zeta^2.$$

Any order type $x \in \mathrm{LO}_\kappa$ codes a set of ℓ-tuples, namely the set of all $\overline{n} \in \mathbb{N}^\ell$ such that $t_\ell(\overline{n}) \unlhd x$. Conversely, if $S \subseteq \mathbb{N}^\ell$, then the order type $\sum_{\overline{n} \in S} t_\ell(\overline{n})$ codes S. □

Answers for $\mathbb{LO}^+_\kappa$...

Corollary

The structure $\mathbb{LO}^+_{\aleph_0}$ *is rigid.*

Let $K \subset LO^k_{\aleph_0}$ *be a definable subset in second-order arithmetic. Then* K *is definable in* $\mathbb{LO}^+_{\aleph_0}$.

Answers for $\mathbb{LO}_\kappa^+$...

Corollary

The structure $\mathbb{LO}_{\aleph_0}^+$ *is rigid.*

Let $K \subset LO_{\aleph_0}^k$ *be a definable subset in second-order arithmetic. Then* K *is definable in* $\mathbb{LO}_{\aleph_0}^+$*.*

Remark

The last implies the definability of some subsets that might not seem otherwise definable: the scattered order types, the set of triples (x, y, z) of order types such that $x \cdot y = z$, the set of order types with condensation rank α, and so on.

Answers for $\mathbb{LO}_c^+$...

Theorem

The first-order theory of $\mathbb{LO}_c^+$ is 1-equivalent to the ωth jump of Kleene's $\mathcal{O}$.

Proof.

In order to show $Th(\mathbb{LO}_c^+) \leq_1 \mathcal{O}^{(\omega)}$, note that $\mathcal{O}$ suffices to determine if two computable order types are isomorphic. Thus, Kleene's $\mathcal{O}$ suffices to compute the universe of $\mathbb{LO}_c^+$ together with the additive operation. Its theory is then computable from $\mathcal{O}^{(\omega)}$.

In order to show $\mathcal{O}^{(\omega)} \leq Th(\mathbb{LO}_c^+)$, note that the parameter that codes multiplication [as before] is (can be taken to be) computable. Alter the earlier encoding to code pairs $(\mathcal{L}, a) \in \mathrm{LO}_c \times \mathbb{N}$. Define a predicate for $\mathcal{O}$ by exploiting that ω^α is an infinite, computable, (right) additively indecomposable linear order whenever α is a nonzero computable ordinal. □

Answers for $\mathbb{BA}^{\oplus}_{\aleph_0}$...

Theorem

The first-order theory of $\mathbb{BA}^{\oplus}_{\aleph_0}$ computes true second-order arithmetic.

Proof.

Encode an integer $n \in \mathbb{N}$ by the interval algebra of $\omega^n \cdot (1 + \eta)$.

Code an ℓ-tuple $\overline{n} = (n_1, \ldots, n_\ell) \in \mathbb{N}^\ell$ by

$$t_\ell(\overline{n}) := \text{IntAlg}\left(\sum_{i \in 1+\eta} (\omega^{n_1} \cdot (1 + \eta) + \cdots + \omega^{n_\ell} \cdot (1 + \eta))\right).$$

Any Boolean algebra $x \in \mathrm{BA}_\kappa$ codes a set of ℓ-tuples, namely the set of all $\overline{n} \in \mathbb{N}^\ell$ such that $t_\ell(\overline{n})$ is a relative algebra of x. Conversely, if $S \subseteq \mathbb{N}^\ell$, then the interval algebra of $\bigoplus_{\overline{n} \in S} t_\ell(\overline{n})$ codes S. □

More Questions About $\mathbb{BA}_{\aleph_0}^{\oplus}$...

Conjecture

The first-order theory of $\mathbb{BA}_{\kappa}^{\oplus}$, for $\kappa > \aleph_0$, computes true second-order arithmetic.

More Questions About $\mathbb{BA}_{\aleph_0}^{\oplus}$...

Conjecture

The first-order theory of $\mathbb{BA}_{\kappa}^{\oplus}$, for $\kappa > \aleph_0$, computes true second-order arithmetic.

Remark

The first-order theories of $\mathbb{BA}_{\aleph_0}^{\oplus}$ and $\mathbb{BA}_{\kappa}^{\oplus}$ differ for $\kappa > \aleph_0$: The former has exactly two [nontrivial] minimal elements, namely the atom and the atomless algebra; the latter has more.

Our proof is not known to work for $\kappa > \aleph_0$ because there are "more" elements whose set of relative algebras is linearly ordered by $\trianglelefteq$.

More Questions About $\mathbb{BA}_{\aleph_0}^{\oplus}$...

Conjecture

The first-order theory of $\mathbb{BA}_{\kappa}^{\oplus}$, for $\kappa > \aleph_0$, computes true second-order arithmetic.

Remark

The first-order theories of $\mathbb{BA}_{\aleph_0}^{\oplus}$ and $\mathbb{BA}_{\kappa}^{\oplus}$ differ for $\kappa > \aleph_0$: The former has exactly two [nontrivial] minimal elements, namely the atom and the atomless algebra; the latter has more.

Our proof is not known to work for $\kappa > \aleph_0$ because there are "more" elements whose set of relative algebras is linearly ordered by $\trianglelefteq$.

Question

Is the structure $\mathbb{BA}_{\aleph_0}^{\oplus}$ bi-interpretable with second-order arithmetic? In particular, is it under the previous encoding?

Answers for $\mathbb{GR}_\kappa^{\times,\leq}$...

Theorem

The first-order theory of $\mathbb{GR}_\kappa^{\times,\leq}$, for $\kappa \geq \aleph_0$, computes true second-order arithmetic.

Proof.

Encode the integer $n \in \mathbb{N}$ by the group $\mathbb{Z}^n$. Then less than is definable as $m \leq n$ if and only if $\mathbb{Z}^m \leq \mathbb{Z}^n$. Addition is definable as $m + n = p$ if and only if $\mathbb{Z}^m \times \mathbb{Z}^n = \mathbb{Z}^p$.

Encode a tuple $\overline{n} = (n_1, \dots, n_\ell) \in \mathbb{N}^\ell$ by $t_\ell(\overline{n}) := \mathbb{Z}^{n_1} \times \cdots \times \mathbb{Z}^{n_\ell}$. Any group $x \in \mathrm{GR}_\kappa$ codes a set of ℓ-tuples, namely the set of all $\overline{n} \in \mathbb{N}^\ell$ such that $t_\ell(\overline{n}) \leq x$. Conversely, if $S \subseteq \mathbb{N}^\ell$, then the group

$$\bigoplus_{\overline{n} \in S} (\mathbb{Z}^{n_1} \times \cdots \times \mathbb{Z}^{n_\ell})$$

codes S.

Answers for $\mathbb{GR}_\kappa^{\times,\leq}$...

Theorem

The first-order theory of $\mathbb{GR}_\kappa^{\times,\leq}$, for $\kappa \geq \aleph_0$, computes true second-order arithmetic.

Proof.

Encode the integer $n \in \mathbb{N}$ by the group $\mathbb{Z}^n$. Then less than is definable as $m \leq n$ if and only if $\mathbb{Z}^m \leq \mathbb{Z}^n$. Addition is definable as $m + n = p$ if and only if $\mathbb{Z}^m \times \mathbb{Z}^n = \mathbb{Z}^p$.

Encode a tuple $\overline{n} = (n_1, \ldots, n_\ell) \in \mathbb{N}^\ell$ by $t_\ell(\overline{n}) := \mathbb{Z}_{q_1}^{n_1} \times \cdots \times \mathbb{Z}_{q_\ell}^{n_\ell}$. Any group $x \in \mathrm{GR}_\kappa$ codes a set of ℓ-tuples, namely the set of all $\overline{n} \in \mathbb{N}^\ell$ such that $t_\ell(\overline{n}) \leq x$. Conversely, if $S \subseteq \mathbb{N}^\ell$, then the group

$$\prod^{\star}_{\overline{n} \in S} \left(\mathbb{Z}_{q_{\overline{n}}} \times \mathbb{Z}_{q_1}^{n_1} \times \cdots \times \mathbb{Z}_{q_\ell}^{n_\ell}\right)$$

codes S. Decode using Kurosch's Theorem. □

Answers for $\mathbb{GR}_{\kappa}^{\times,\leq}$...

Theorem (Kurosch's Theorem)

A subgroup H of a free product $\prod_j^\star A_j$ is itself a free product of the form

$$F \star \prod_k^\star x_k^{-1} U_k x_k$$

where F is a free group and each $x_k^{-1} U_k x_k$ is the conjugate of a subgroup U_k of one of the factors A_j by an element of the free group $\prod_j^\star A_j$.

Answers for $\mathbb{GR}_\kappa^{\times,\leq}$...

Theorem (Kurosch's Theorem)

A subgroup H of a free product $\prod_j^\star A_j$ is itself a free product of the form

$$F \star \prod_k^\star x_k^{-1} U_k x_k$$

where F is a free group and each $x_k^{-1} U_k x_k$ is the conjugate of a subgroup U_k of one of the factors A_j by an element of the free group $\prod_j^\star A_j$.

Proof (Continued...)

Unfortunately, none of $\mathbb{Z}$ or $\mathbb{Z}_q$ is (seemingly) definable in $\mathbb{GR}_\kappa^{\times,\leq}$. Instead, fix $\leq$-minimal elements $w_0, w_1, \ldots, w_\ell$. Show that, for any i and j, the set of pairs (w_i^k, w_j^k) is definable in $\mathbb{GR}_\kappa^{\times,\leq}$ with w_i and w_j as parameters. □

Outline

1 Context and Questions

2 Some Answers and More Questions
- Cardinals
- Linear Orders
- Boolean Algebras
- Groups

3 Open Questions

Open Questions...

Question

How complicated is the first-order theory of $\mathbb{GR}_{\kappa}^{\times}$?

How complicated is the first-order theory of $\mathbb{GR}_{\kappa}^{\leq}$?

Open Questions...

Question

How complicated is the first-order theory of $\mathbb{GR}_\kappa^\times$?

How complicated is the first-order theory of $\mathbb{GR}_\kappa^\leq$?

Theorem (Tamvana Makuluni)

The first-order theory of $\mathbb{F}_\kappa^\leq$ (fields with the subfield relation) computes true second-order arithmetic.

Open Questions...

Question

Does the Σ_1 theory of $\mathbb{LO}_\kappa^+$ admit a *nice* characterization?

Open Questions...

Question

Does the Σ_1 theory of $\mathbb{LO}_\kappa^+$ admit a *nice* characterization?

Remark

Any finite preorder embeds into $\mathbb{LO}_\kappa^\trianglelefteq$, so the Σ_1-theory of $\mathbb{LO}_\kappa^\trianglelefteq$ admits a nice characterization.

Open Questions...

Question

Does the Σ_1 theory of $\mathbb{LO}_\kappa^+$ admit a *nice* characterization?

Remark

Any finite preorder embeds into $\mathbb{LO}_\kappa^\trianglelefteq$, so the Σ_1-theory of $\mathbb{LO}_\kappa^\trianglelefteq$ admits a nice characterization.

Question

How complicated is the first-order theory of $\mathbb{LO}_\kappa^\preccurlyeq$ (order types with embeddability)?

Fixed Answers...

Question

Is there an element x such that $x \neq x + x$ and $x = x + x + x$?

Fixed Answers...

Question

Is there an element x such that $x \neq x + x$ and $x = x + x + x$?

- Is there a cardinal x such that $x \neq x + x$ and $x = x + x + x$? NO.
- Is there a linear order x such that $x \ncong x + x$ and $x \cong x + x + x$? NO.
- Is there a Boolean algebra x such that $x \ncong x \oplus x$ and $x \cong x \oplus x \oplus x$? YES.
- Is there a group x such that $x \ncong x \times x$ and $x \cong x \times x \times x$? YES.

References

S. Feferman and R. L. Vaught.
The first order properties of products of algebraic systems.
Fund. Math., 47:57–103, 1959.

Asher M. Kach and Antonio Montalbán.
Decidability and undecidability of the theories of classes of structures.
submitted.

Jussi Ketonen.
On isomorphism types of countable Boolean algebras.
unpublished.

Jussi Ketonen.
The structure of countable Boolean algebras.
Ann. of Math. (2), 108(1):41–89, 1978.

Alfred Tarski.
Cardinal Algebras. With an Appendix: Cardinal Products of Isomorphism Types, by Bjarni Jónsson and Alfred Tarski.
Oxford University Press, New York, N. Y., 1949.