Π^0_1 equivalence structures and their isomorphisms

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- An equivalence structure $\mathcal{A} = (\omega, E^{\mathcal{A}})$ is *computable* if its relation $E^{\mathcal{A}}$ is computable.

- $\mathcal{A} = (\omega, E^{\mathcal{A}})$ is *c.e.* (or $\boldsymbol{\Sigma}^0_1$) if $E^{\mathcal{A}}$ is a c.e. set.
  $\mathcal{A}$ is *co-c.e.* (or $\boldsymbol{\Pi}^0_1$) if $E^{\mathcal{A}}$ is a co-c.e. set.

- Equivalence class of a: $[a]^{\mathcal{A}} = \{x \in A : xE^{\mathcal{A}}a\}$
  Character:
  $\chi(\mathcal{A}) = \{\langle k, n\rangle : n, k > 0$ and $\mathcal{A}$ has $\geq n$ equivalence classes of size $k\}$
  Bounded character: k is bounded

- For any c.e. equivalence structure $\mathcal{A}$:

  (a) $\{\langle k, a\rangle : card([a]^{\mathcal{A}}) \geq k\}$ is a c.e. set;

  (b) $Inf^{\mathcal{A}} = \{a : [a]^{\mathcal{A}}$ is infinite$\}$ is a Π^0_2 set;

  (c) $\chi(\mathcal{A})$ is a Σ^0_2 set.

- $K \subseteq \langle(\omega - \{0\}) \times (\omega - \{0\})\rangle$ is a *character* if for all $n > 0$ and k:

$$\langle k, n+1\rangle \in K \Rightarrow \langle k, n\rangle \in K$$

- (Calvert-Cenzer-Harizanov-Morozov 2006) For any $\mathbf{\Sigma}_2^0$ character K, there exists a computable equivalence structure $\mathcal{A}$ with infinitely many infinite equivalence classes and character K.

- (Corollary) If $\mathcal{A}$ is a c.e. equivalence structure with infinitely many infinite equivalence classes, then $\mathcal{A}$ is isomorphic to a computable equivalence structure.

- (Cenzer-Harizanov-Remmel 2011)
  For any $\boldsymbol{\Sigma}^0_2$ character K and any finite r,
  there is a c.e. equivalence structure with character K and
  with exactly r infinite equivalence classes.

- (Corollary)
  There exists a c.e. equivalence structure
  (with finitely many infinite equivalence classes),
  which is not isomorphic to any computable equivalence structure.

- A function $f : \omega^2 \rightarrow \omega$ is a (Khisamiev's) s-*function* if for every i and s:
  $f(i,s) \leq f(i,s+1)$, and the limit $m_i = lim_s f(i,s)$ exists.

- f is called an s_1-*function* if, in addition:
  $m_0 < m_1 < \cdots < m_i < m_{i+1} < \cdots$

  $\{m_i : i \in \omega\}$ is a Δ^0_2 set.

- Let $\mathcal{A}$ be a computable equivalence structure with finitely many infinite equivalence classes and infinite character $\chi(\mathcal{A})$.

- There exists a computable s-function f with limits m_i such that:

$$\langle k, n \rangle \in \chi(\mathcal{A}) \Leftrightarrow card(\{i : k = m_i\}) \geq n$$

- If $\chi(\mathcal{A})$ is unbounded, then there is a computable s_1-function f such that $\mathcal{A}$ contains an equivalence class of size m_i for each i.

- Let K be a $\mathbf{\Sigma}^0_2$ character, and $r \in \omega$.

- If f is a computable s-function with the limits m_i such that

$$\langle k, n \rangle \in K \Leftrightarrow card(\{i : k = m_i\}) \geq n,$$

then there is a computable equivalence structure $\mathcal{A}$ with $\chi(\mathcal{A}) = K$ and with exactly r infinite equivalence classes.

- If f is a computable s_1-function such that $\langle m_i, 1 \rangle \in K$ for all i, then there is a computable equivalence structure $\mathcal{A}$ with $\chi(\mathcal{A}) = K$ and exactly r infinite equivalence classes.

- There is an infinite Δ^0_2 set D such that for any computable equivalence structure $\mathcal{A}$ with unbounded character K and no infinite equivalence classes, $\{k : \langle k, 1\rangle \in K\}$ is not a subset of D.

  Hence, for any computable s_1-function f with $m_i = lim_s f(i, s)$
  $m_0 < m_1 < \cdots$,
  there exists i_0 such that $m_{i_0} \notin D$.

- (Corollary)
  A c.e. equivalence structure with character $\{\langle k, 1\rangle : k \in D\}$ and no infinite equivalence classes is not isomorphic to any computable equivalence structure.

Let $\mathcal{C}$ be a *computable* structure.

- $\mathcal{C}$ is Δ^0_n *categorical* if for all computable $\mathcal{B} \cong \mathcal{C}$, there is a Δ^0_n isomorphism from $\mathcal{C}$ onto $\mathcal{B}$.

- $\mathcal{C}$ is *relatively* Δ^0_n *categorical* if for all $\mathcal{B} \cong \mathcal{C}$, there is an isomorphism from $\mathcal{C}$ onto $\mathcal{B}$, which is Δ^0_n relative to the atomic diagram of $\mathcal{B}$.

(Calvert-Cenzer-Harizanov-Morozov 2006)

- A computable equivalence structure $\mathcal{A}$ is *computably categorical iff*:
(1) $\mathcal{A}$ has finitely many finite equivalence classes, or
(2) $\mathcal{A}$ has finitely many infinite classes, bounded character, and
at most one finite $k > 0$ with infinitely many classes of size k.

- Every computable equivalence structure is Δ^0_3 *categorical*.

- Let $\mathcal{A}$ be a computable equivalence structure with
infinitely many infinite equivalence classes, and with
unbounded character that has a computable s_1-function.
Then $\mathcal{A}$ is *not* Δ^0_2 *categorical*.

- A *Scott family* for a *countable* structure $\mathcal{C}$ is a countable set Φ of $L_{\omega_1\omega}$ formulas, with a fixed finite tuple of parameters in C, such that:

  (i) each tuple in C satisfies some $\psi \in \Phi$;

  (ii) if $\overline{a}$, $\overline{b}$ are tuples in C satisfying the *same* formula $\psi \in \Phi$, then there is an automorphism of $\mathcal{C}$ taking $\overline{a}$ to $\overline{b}$.

- (Ash-Knight-Manasse-Slaman 1989, Chisholm 1990)
  A computable structure $\mathcal{C}$ is *relatively* Δ^0_n categorical *iff*
  $\mathcal{C}$ has a c.e. Scott family consisting of computable Σ_n formulas.

(Calvert-Cenzer-Harizanov-Morozov 2006)

- Every computable *computably categorical* equivalence structure is relatively computably categorical.

- Every computable equivalence structure is *relatively* $\mathbf{\Delta}^0_3$ *categorical*.

- A computable equivalence structure $\mathcal{A}$ is *relatively* $\mathbf{\Delta}^0_2$ *categorical iff*:
  (i) $\mathcal{A}$ has finitely many infinite equivalence classes, or
  (ii) $\mathcal{A}$ has bounded character.

- If $\mathcal{A}$ is a computable equivalence structure with bounded character, then $\mathcal{A}$ is relatively $\mathbf{\Delta}_2^0$ categorical.

  Let k be the maximum size of any finite equivalence class.
  $[a]^{\mathcal{A}}$ is infinite *iff* $[a]^{\mathcal{A}}$ contains at least $k+1$ elements ($\mathbf{\Sigma}_1^0$ condition).

- If $\mathcal{A}$ is a computable equivalence structure with finitely many infinite equivalence classes, then $\mathcal{A}$ is relatively $\mathbf{\Delta}_2^0$ categorical.

  Choose representatives $c_1, \ldots, c_l$ for the finitely many infinite equivalence classes.

- (Goncharov 1980)
  There is a rigid computable graph that is
  computably categorical, but *not relatively* computably categorical.

- (Goncharov-Harizanov-Knight-McCoy-Miller-Solomon 2005)
  For every computable successor ordinal $\alpha > 1$, there is a computable structure that is Δ^0_α *categorical*, but *not relatively* Δ^0_α categorical.

- (Kach-Turetsky 2009) There is a computable Δ^0_2 *categorical* equivalence structure that is not *relatively* Δ^0_2 categorical.

(Cenzer-Harizanov-Remmel 2011)

- Let $\mathcal{A}$ be a c.e. equivalence structure, and let $\mathcal{B}$ be a computable structure isomorphic to $\mathcal{A}$ such that $\mathcal{B}$ is relatively Δ_2^0 categorical. Then $\mathcal{A}$ and $\mathcal{B}$ are Δ_2^0 *isomorphic*.

- (Corollary) Let $\mathcal{A}$ and $\mathcal{B}$ be isomorphic c.e. equivalence structures such that:
  (i) $\mathcal{A}$ has finitely many infinite equivalence classes, or
  (ii) $\mathcal{A}$ has bounded character.
  Then $\mathcal{A}$ and $\mathcal{B}$ are Δ_2^0 *isomorphic*.

(Cenzer-Harizanov-Remmel 2011)

- Let $\mathcal{A}$ and $\mathcal{B}$ be isomorphic Π^0_1 equivalence structures such that:

  (i) *either* $\mathcal{A}$ has only finitely many finite equivalence classes, *or*

  (ii) $\mathcal{A}$ has finitely many infinite equivalence classes and bounded character, and there is exactly one finite k such that
  $\mathcal{A}$ has infinitely many equivalence classes of size k.

  Then $\mathcal{A}$ and $\mathcal{B}$ are Δ^0_2 *isomorphic*.

- Proof. If $\mathcal{B}$ is a Π^0_1 equivalence structure, and $\mathcal{C}$ is an isomorphic computable structure that is computably categorical, then, since $\mathcal{C}$ is also relatively computably categorical, $\mathcal{C}$ and $\mathcal{B}$ are Δ^0_2 isomorphic.

- Suppose that $\mathcal{B}$ is a computable equivalence structure with bounded character, for which there exist $k_1 < k_2 \leq \omega$ such that $\mathcal{B}$ has infinitely many equivalence classes of size k_1 and infinitely many equivalence classes of size k_2.

  Then there exists a Π^0_1 structure $\mathcal{A}$ isomorphic to $\mathcal{B}$ such that $\mathcal{A}$ is not Δ^0_2 isomorphic to $\mathcal{B}$.
  Moreover, $\mathcal{A}$ is not Δ^0_2 isomorphic to any c.e. structure.

- Proof. We first suppose that $\mathcal{B}$ has no other equivalence classes.

  It suffices to build a Π^0_1 equivalence structure $\mathcal{A}$ such that $\{a : card([a]^{\mathcal{A}}) = k_2\}$ is not a Δ^0_2 set.

  That is, for any Σ^0_1 structure, the set of elements that belong to an equivalence class of (finite) size k is a Δ^0_2 set. So if $\mathcal{A}$ were Δ^0_2 isomorphic to a Σ^0_1 structure, then $\mathcal{A}$ would also have this property.

- For simplicity, let $\mathcal{A}$ have universe $\omega \setminus \{0\}$.

  Let $\phi : \omega^3 \to \{0, 1\}$ be a computable function such that
  for every $\mathbf{\Delta}_2^0$ set D, there is some e for which for all $n \in \omega$, the limit
  $\delta_e(n) =_{def} \lim_{t \to \infty} \phi(t, e, n)$ exists and
  δ_e is the characteristic function of D.

  The function ϕ exists by the Limit Lemma.

  If $\delta_e(n)$ is defined for all n, we let $D_e = \{n : \delta_e(n) = 1\}$.

  We will construct the equivalence relation $E = E^{\mathcal{A}}$ so that
  for each e, if D_e exists, then
  $card([2^e]^{\mathcal{A}}) = k_2$ if and only if $2^e \notin D_e$.

- We construct $E^{\mathcal{A}}$ in stages.

  At each stage s, we define a computable equivalence relation E_s so that $E_{s+1} \subseteq E_s$ for all s, and $E^{\mathcal{A}} = \bigcap_s E_s$.

  Let $[a]_s$ denote the equivalence class of a in E_s.

  At each stage s, we also define an *intended* equivalence class $I_s[2^e]$, either of size k_1 or of size k_2.

  We will ensure that for each e, there is some stage s_e such that for all $s \geq s_e$, we have $[2^e] = I_s[2^e]$.
  Furthermore, for all s, $[2^e]_{s+1} \subseteq [2^e]_s$, and $\bigcap_s [2^e]_s = [2^e]$.

  We also define a number of *permanent* classes $[a]$ of size k_1 at each s.

Construction

- *Stage* 0.
  We start with the equivalence classes $\{2^e(2k+1) : k \in \omega\}$ for $e \geq 0$.

  For each $e \geq 0$, let $I_0[2^e] = \{2^e, 3 \cdot 2^e, 5 \cdot 2^e, \ldots, (2k_1 - 1) \cdot 2^e\}$.

- *Stage* $s+1$.
  At the end of stage s, assume that for each e, we have defined
  the intended equivalence class $I_s[2^e]$,
  so that $I_s[2^e]$ is an initial subset of $[2^e]_s$,
  with cardinality either k_1 or k_2.

  Moreover, assume that if $\phi(s, e, 2^e) = 1$, then $I_s[2^e]$ has cardinality k_1,
  and if $\phi(s, e, 2^e) = 0$, then $I_s[2^e]$ has cardinality k_2.

- For each e, we say that the element 2^e *requires attention* at stage $s+1$ if $\phi(s+1, e, 2^e) \neq \phi(s, e, 2^e)$.
  We can assume this occurs for exactly one e.

  Let $[2^e]_s = \{2^e, a_1, a_2, \ldots\}$.

- If 2^e **requires attention at stage** $s+1$, we take the following action according to whether $I_s[2^e]$ has cardinality k_1 or k_2.

- *Case* (i): $card(I_s[2^e]) = k_2$
  Let $I_{s+1}[2^e] = \{2^e, a_1, \ldots, a_{k_1-1}\}$,
  let $[2^e]_{s+1} = \{2^e, a_1, \ldots, a_{k_1-1}, a_{2k_1}, a_{2k_1+1}, \ldots\}$, and create
  a permanent equivalence class $\{a_{k_1}, a_{k_1+1}, \ldots, a_{2k_1-1}\}$ of size k_1.

- *Case* (ii): $card(I_s[2^e]) = k_1$

- Assume that k_2 is finite.

  Let $I_{s+1}[2^e] = \{2^e, a_1, \ldots, a_{k_2-1}\}$,
  let $[2^e]_{s+1} = \{2^e, a_1, \ldots, a_{k_2-1}, a_{k_2+k_1}, a_{k_2+k_1+1}, \ldots\}$, and create a permanent equivalence class $\{a_{k_2}, a_{k_2+1}, \ldots, a_{k_2+k_1-1}\}$ of size k_1.

- Assume $k_2 = \omega$.

  Let $I_{s+1}[2^e] = [2^e]_{s+1} = [2^e]_s$.

- If 2^e **does not require attention**, there are two cases.

- If $k_2 = \omega$, $I_s[2^e] = [2^e]_s$ is infinite, then let $I_{s+1}[2^e] = [2^e]_{s+1} = [2^e]_s$.

- If $card([I_s[2^e]) = k_m$ is finite ($m \in \{1, 2\}$), then let $I_{s+1}[2^e] = \{2^e, a_1, \ldots, a_{k_m-1}\}$, let $[2^e]_{s+1} = \{2^e, a_1, \ldots, a_{k_m-1}, a_{k_m+k_1}, a_{k_m+k_1+1}, \ldots\}$, and create a permanent equivalence class $\{a_{k_m}, a_{k_m+1}, \ldots, a_{k_m+k_1-1}\}$ of size k_1.

- Clearly, the equivalence relation E_s is uniformly computable, and $E_{s+1} \subseteq E_s$ for every s.

  Thus, $E = \bigcap_s E_s$ is a Π^0_1 equivalence relation.

- Every equivalence class in E has either k_1 or k_2 elements. $A = \{n : card([2^n]) = k_2\}$ is not a Δ^0_2 set.

Corollary

- Suppose that $\mathcal{B}$ is a computable equivalence structure with bounded character, which is not computably categorical.

  Then there exists a Π^0_1 structure $\mathcal{A}$ isomorphic to $\mathcal{B}$, which is not Δ^0_2 isomorphic to $\mathcal{B}$.
  Moreover, $\mathcal{A}$ is not Δ^0_2 isomorphic to any c.e. structure.

- Suppose that $\mathcal{B}$ is a computable equivalence structure, which is relatively Δ^0_2 categorical and has unbounded character, hence has only finitely many infinite equivalence classes.

  Then there exists a Π^0_1 structure $\mathcal{A}$ that is isomorphic to $\mathcal{B}$, but not Δ^0_2 isomorphic to $\mathcal{B}$.

  Moreover, $\mathcal{A}$ is not Δ^0_2 isomorphic to any c.e. structure.

- Proof. There is a computable s_1-function f such that for each i, there exists finite $lim_s f(i,s) = m_i$ and $\mathcal{B}$ has an equivalence class of size m_i.

  $M = \{m_i : i \in \omega\}$ is a Δ^0_2 set.

  Thus, there exists a computable equivalence structure which consists of exactly one equivalence class of size m_i for each i.

- First, assume that $\mathcal{B}$ consists of exactly one equivalence class of size m_i for each i.

- It suffices to build an isomorphic Π^0_1 equivalence structure $\mathcal{A}$ such that $\{a : card([a]^{\mathcal{A}}) = m_{2i} \text{ for some } i\}$ is not a Δ^0_2 set.

- That is, we observe that the functions f_E and f_O, defined by $f_E(i, s) = f(2i, s)$ and $f_O(i, s) = f(2i + 1, s)$ are also s_1-functions.

- Hence the sets $M_0 = \{m_{2i} : i \in \omega\}$ and $M_1 = \{m_{2i+1} : i \in \omega\}$ are both Δ^0_2.

- There exist computable structures $\mathcal{B}_0$ and $\mathcal{B}_1$, which consist of precisely one class of size m_{2i} for $\mathcal{B}_0$ and of size m_{2i+1} for $\mathcal{B}_1$.

- In the structure $\mathcal{B}_0 \oplus \mathcal{B}_1$, the set $\{x : card([x]) \in M_0\}$ is computable.

- Since we have assumed that $\mathcal{B}$ is relatively $\mathbf{\Delta}^0_2$ categorical, it follows that for any $\mathbf{\Sigma}^0_1$ equivalence structure with character $\{(m, 1) : m \in M_0 \cup M_1\}$, the set $\{x : card([x]) \in M_0\}$ is $\mathbf{\Delta}^0_2$.

- Suppose that $\mathcal{B}$ is a computable equivalence structure, which is relatively Δ^0_2 categorical, but not computably categorical.

  Then there exists a Π^0_1 structure $\mathcal{A}$ isomorphic to $\mathcal{B}$, which is not Δ^0_2 isomorphic to $\mathcal{B}$. Moreover, $\mathcal{A}$ is not Δ^0_2 isomorphic to any c.e. structure.

- Previous theorem does not cover all computable Δ^0_2 categorical equivalence structures.

  Kach and Turetsky showed that there exists a computable Δ^0_2 categorical equivalence structure $\mathcal{B}$, which has infinitely many infinite equivalence classes and unbounded character, but has no computable s_1-function, and has only finitely many equivalence classes of size k for any finite k.

- Let $\mathcal{B}$ be a computable equivalence structure with infinitely many infinite equivalence classes and with unbounded character such that for each finite k, there are only finitely many equivalence classes of size k.

  Then there is a Π^0_1 structure $\mathcal{A}$, which is isomorphic to $\mathcal{B}$, such that $Inf^{\mathcal{A}}$ is Π^0_2 complete.

  Furthermore, if $\mathcal{B}$ is Δ^0_2 categorical, then $\mathcal{A}$ is not Δ^0_2 isomorphic to any computable structure.

- Suppose that $\mathcal{B}$ is a computable equivalence structure, which is not computably categorical.

  Then there is a Π^0_1 structure $\mathcal{A}$ that is isomorphic to $\mathcal{B}$ such that $\mathcal{A}$ is not Δ^0_2 isomorphic to $\mathcal{B}$.