Mathias generic sets

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Mathias generics

Definition.

1. A (computable Mathias) pre-condition is a pair (D, E) such that D is a finite set, E is a computable set, and $\max D < \min E$.

2. A (computable Mathias) condition is a pre-condition (D, E) such that E is infinite.

3. A pre-condition (D', E') extends (D, E), written $(D', E') \leq (D, E)$, if $D \subseteq D' \subseteq D \cup E$ and $E' \subseteq E$.

4. A set S satisfies a pre-condition (D, E) if $D \subseteq S \subseteq D \cup E$.

Mathias generics

A set S meets a set $\mathcal{C}$ of conditions if it satisfies some condition in $\mathcal{C}$.

A set S avoids a set $\mathcal{C}$ of conditions if it meets the set of conditions having no extension in $\mathcal{C}$.

Definition.

1. A Σ^0_n set of conditions is a Σ^0_n-definable set of pre-conditions, each of which is a condition.
2. A set G is Mathias n-generic if it meets or avoids every Σ^0_n set of conditions.
3. A set G is weakly Mathias n-generic if it meets every dense Σ^0_n set of conditions.

Indices of pre-conditions

An index for a pre-condition (D, E) is a pair $(d, e) \in \omega^2$ such that d is the canonical index of D and $E = \{x \in \omega : \varphi_e(x) \downarrow = 1\}$.

The set of all (indices for) pre-conditions is Π^0_1-definable.

Remark. There exists a computable set of (indices for) pre-conditions containing an index for every pre-condition.

We work entirely over this set from now on.

Indices of conditions

The set of all (indices for) conditions is Π^0_2-definable.

Definition. A set G is Mathias strongly n-generic if it meets or avoids every Σ^0_n-definable set of pre-conditions.

Proposition (Cholak, Dzhafarov, Hirst). For $n \geq 3$, a set is strongly n-generic if and only if it is n-generic. For $n < 3$, a set is strongly n-generic if and only if it is 3-generic.

Without further comment, n below will always be a number ≥ 3.

Familiar properties

Every n-generic set is weakly n-generic, is $(n-1)$-generic.

There exist n-generics $G \leq_T \emptyset^{(n)}$.

Every weakly n-generic set is hyperimmune relative to $\emptyset^{(n-1)}$.

Corollary. Not every n-generic set is weakly $(n+1)$-generic.

Corollary. The n-generic sets form a null class.

The degree of any n-generic forms a minimal pair with $\mathbf{0}^{(n-1)}$.

Less familiar properties

If G is weakly n-generic then G is cohesive, i.e., for every c.e. set W, either $G \subseteq^* W$ or $G \subseteq^* \overline{W}$.

Corollary. If $G = G_0 \oplus G_1$ is n-generic, then either $G_0 =^* \emptyset$ or $G_1 =^* \emptyset$. (In particular, van Lambalgen's theorem fails.)

Corollary. No Mathias n-generic can be even Cohen 1-generic.

Corollary. The class of Mathias generic sets is not comeager.

If G is weakly n-generic then $G' \geq_T \emptyset''$.

Corollary. No Cohen 2-generic can compute a Mathias 3-generic.

The forcing relation

Let $\varphi(X)$ be a Σ^0_0 formula of second-order arithmetic in one free set variable X, written in disjunctive normal form.

Let $P_{\varphi,i}$ be the set of all $n \in \omega$ such that $\underline{n} \in X$ appears as a conjunct of the ith disjunct.

Let $N_{\varphi,i}$ be the set of all $n \in \omega$ such that $\underline{n} \notin X$ appears as a conjunct of the ith disjunct.

Definition.

A condition (D, E) forces $\varphi(G)$, written $(D, E) \Vdash \varphi(G)$, if there is an i such that $P_{\varphi,i} \subseteq D$ and $N_{\varphi,i} \subseteq \overline{D \cup E}$.

For general $\varphi(X)$, define $(D, E) \Vdash \varphi(G)$ inductively according to the standard definition of (strong) forcing.

The forcing relation

Lemma (Cholak, Dzhafarov, Hirst). Let $\varphi(X)$ be a formula of second-order arithmetic in one free set variable X.

If φ is Σ^0_0, then the relation $(D, E) \Vdash \varphi(G)$ is computable.

If φ is Π^0_1, Σ^0_1, or Σ^0_2, then so is the relation $(D, E) \Vdash \varphi(G)$.

If φ is Π^0_n for some $n \geq 2$, then the relation $(D, E) \Vdash \varphi(G)$ is Π^0_{n+1}.

If φ is Σ^0_n for some $n \geq 3$, then the relation $(D, E) \Vdash \varphi(G)$ is Σ^0_{n+1}.

Proposition (Cholak, Dzhafarov, Hirst). Let G be n-generic, and let $\varphi(X)$ be a Σ^0_m formula for some $m \leq n$, or the negation of such a formula. If G satisfies a condition that forces $\varphi(G)$, then $\varphi(G)$ holds.

Jump properties

Theorem (Cholak, Dzhafarov, Hirst). If G is n-generic, then $G^{(n-1)} \equiv_T G' \oplus \emptyset^{(n)}$.

Proof sketch. One direction follows immediately from $G' \geq \emptyset''$.

For the other, fix any Σ^0_{n-1} formula $\varphi(X)$ and consider

$$\mathcal{C} = \{(D, E) \text{ condition} : (D, E) \Vdash \varphi(G)\} \in \Sigma^0_n$$

$$\mathcal{D} = \{(D, E) \text{ condition} : (D, E) \Vdash \neg\varphi(G)\} \in \Pi^0_n.$$

G meets $\mathcal{C}$ iff it avoids $\mathcal{D}$, one must happen, and $G' \oplus \emptyset^{(n)}$ knows which.

Corollary. The Π^0_{n+1} and Σ^0_{n+1} bounds in the definition of $\Vdash$ cannot be lowered even to Δ^0_{n+1}. So an n-generic set only decides Σ^0_{n-1} formulas.

Jump properties

Theorem (Cholak, Dzhafarov, Hirst). If G is n-generic, then $\deg(G) \in \mathbf{GH}_1$, i.e., $G' \equiv_T (G \oplus \emptyset')'$.

Proof sketch. For $i \in \omega$, a condition (D, E) forcing the formula $i \in (G \oplus \emptyset')'$ is Σ^0_2. Forcing the negation requires universally quantifying over all extensions of (D, E), and so appears Π^0_3. But in fact it suffices to quantify over finite extensions, which makes this relation Π^0_2. Now since $G' \geq_T G' \oplus \emptyset''$, proceed as in the previous theorem.

Corollary. Every Mathias n-generic has $\overline{\mathbf{GL}}_1$ degree. Hence, no Mathias n-generic has Cohen 1-generic degree, but every Mathias n-generic computes a Cohen 1-generic.

Coding into Mathias generics

Theorem (Kurtz). If $A >_T \emptyset^{(n-1)}$ is hyperimmune relative to $\emptyset^{(n-1)}$ then $A \equiv_T B^{(n-1)}$ for some weakly Cohen n-generic set G.

Theorem (Cholak, Dzhafarov, Hirst). If $A >_T \emptyset^{(n-1)}$ is hyperimmune relative to $\emptyset^{(n-1)}$ then $A \equiv_T G^{(n-2)}$ for some weakly Mathias n-generic set G.

Corollary. Not every weakly n-generic set is n-generic.

Coding into Mathias generics

Fix $A >_T \emptyset^{(n-1)}$ is hyperimmune relative to $\emptyset^{(n-1)}$. We want to build a weakly Mathias n-generic G such that $A \equiv_T G^{(n-2)}$.

We A-computably build a series of conditions $(\emptyset, \omega) = (D_0, E_0) \geq (D_1, E_1) \geq \cdots$, and take $G = \bigcup_s D_s$.

Ensuring that G is weakly n-generic uses the hyperimmunity of A relative to $\emptyset^{(n-1)}$. This is just as in Kurtz's proof, but the escaping function must be chosen a bit more carefully.

We force the jump along the way, which is easy since $A \geq_T \emptyset^{(n)} \geq_T \emptyset'$. This ensures that $G^{(n-2)} \equiv_T G' \oplus \emptyset^{(n-1)} \leq_T A$.

Coding into Mathias generics

In Kurtz's proof, where one constructs a sequence of finite strings, the bit $A(n)$ is coded at a certain stages by appending a long block of 1s.

We cannot code the same way: if we are at, say, (D_s, E_s), the reservoir E_s may be very sparse.

Instead, fix a sequence of disjoint co-immune sets $B_0, B_1, \ldots \leq_T \emptyset^{(n-1)}$ ahead of time. These can serve as coding markers. Since each E_s is computable, it must intersect each B_i infinitely often. So, instead of appending e many 1s, we append the least element of $B_e \cap E_s$.

Since G will be weakly n-generic, $G^{(n-2)}$ will compute $\emptyset^{(n-1)}$ and hence also the sequence of B_i, so $G^{(n-2)}$ will be able to do the decoding.

Coding into Mathias generics

Theorem (Cholak, Dzhafarov, Hirst). If G is Mathias n-generic and $B \leq_T \emptyset^{(n-1)}$ is bi-immune, then $G \oplus B$ computes a Cohen n-generic.

Proof. Given a c.e. set of strings V, let $\mathcal{C}$ be set of all conditions (D, E) such that $D \cap B$, regarded as an element of $2^{\min E}$, belongs to V.

Then $\mathcal{C}$ is Σ^0_3. If G meets $\mathcal{C}$, then $G \cap B$ meets V.

If G avoids $\mathcal{C}$, then $G \cap B$ must avoid V. Indeed, suppose G avoids $\mathcal{C}$ via (D, E). Since B and $\overline{B}$ are co-immune, they intersect E infinitely often, and so if $D \cap B$ had an extension τ in V, we could make a finite extension (D', E') of (D, E) so that $D' \cap B = \tau$.

Proposition (Cholak, Dzhafarov, Hirst). No Mathias n-generic m-computes a Cohen m-generic.

Questions

Does every Mathias n-generic compute a Cohen n-generic?

Is there a form of van Lambalgen's theorem for Mathias generics?

What is the reverse mathematical content of the principle asserting the existence, for every X, of an n-generic set for X-computable Mathias forcing? It is Π^1_1 conservative over RCA_0, how about over $\mathsf{B}\Sigma^0_2$?

Shore has asked whether there are any interesting degrees realizing properties of the form $\mathbf{d}^j = (\mathbf{d}^k \vee \mathbf{0}^l)^m$. The Cohen and Mathias generics realize two such properties. Do generics for other forcing notions realize other properties?

Thank you for your attention.