Weak König's Lemma in the absence of Σ^0_1 induction

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Background: reverse mathematics

Reverse mathematics studies the strength of axioms needed to prove mathematical theorems. This is done by deriving implications between the theorems and set existence principles expressed in the language of second-order arithmetic, which has:

- vbles $x, y, z, \ldots, i, j, k \ldots$ for natural numbers,
- vbles $X, Y, Z, \ldots$ for sets of naturals,
- non-logical symbols $+, \cdot, 2^x, \leq, 0, 1, \in$.

Often, the theorems studied are Π^1_2 of the form $\forall X \, \exists Y \, \psi$, and their strength is related to the difficulty of computing Y given X.

The implications are proved in a relatively weak base theory.

The usual base theory, and an important axiom

The usual base theory, RCA_0, has the following axioms:

- $+, \cdot, 2^x$ etc. have their usual basic properties.
- Δ^0_1 comprehension: if $\bar{X} = X_1, \ldots, X_k$ are sets and $\psi(x, \bar{X})$ is computable relative to $\bar{X}$, then $\{n : \psi(n, \bar{X})\}$ is a set.
- Σ^0_1 induction: if $\bar{X}$ are sets and $\psi(x, \bar{X})$ is c.e. relative to $\bar{X}$, then $\psi(0, \bar{X}) \wedge \forall n \left(\psi(n, \bar{X}) \Rightarrow \psi(n+1, \bar{X})\right) \Rightarrow \forall n\, \psi(n, \bar{X})$.

Possibly the most important theory in reverse mathematics, WKL_0, is axiomatized by RCA_0 and Weak König's Lemma WKL: "Every infinite tree in $\{0,1\}^{\mathbb{N}}$ has an infinite path".

This says essentially: "The interval $[0,1]$ is Heine-Borel compact". Or "For every set X there is a set Y of PA degree relative to X".

Properties of WKL

RCA_0 proves: $WKL_0 \equiv$ a plethora of theorems, from compactness of first-order logic to the Peano existence thm for ODE's.

WKL is not provable in RCA_0. On the other hand:

Theorem (Harrington 1977, independently Ratajczyk 1980's)
WKL_0 is Π^1_1-conservative over RCA_0, i.e. every Π^1_1 sentence provable in WKL_0 is also provable in RCA_0.

The proof is by adding a path to an infinite 0-1 tree T in a countable model of RCA_0, which is done via forcing with infinite subtrees of T.

A weaker base theory

In an alternative, weaker base theory RCA^*_0,
one replaces Σ^0_1 induction with Δ^0_1 induction:
if $\bar{X}$ are sets and $\psi(x, \bar{X})$ is computable relative to $\bar{X}$, then
$\psi(0, \bar{X}) \wedge \forall n \left(\psi(n, \bar{X}) \Rightarrow \psi(n+1, \bar{X})\right) \Rightarrow \forall n\, \psi(n, \bar{X})$.

- Used to identify some theorems that are equivalent to Σ^0_1 induction (e.g. "every non-zero poly has finitely many roots" [Simpson-Smith 1986]) and some that do not need it (e.g. Friedberg-Muchnik Thm [Chong-Mourad 1992]).
- Turns out to be useful in understanding some aspects of reverse mathematics over RCA_0 (e.g. [Belanger 20XX]).

Weak König's Lemma over RCA_0^*

The theory obtained by adding WKL to RCA_0^* is known as WKL_0^*.

Theorem (Simpson-Smith 1986)

WKL_0^* is Π^1_1-conservative over RCA_0^*.

The proof is a forcing argument similar to the one over RCA_0.

However: today's talk is about a property
that models of $\mathrm{WKL}_0^* + \neg \mathrm{I}\Sigma^0_1$ have but those of WKL_0 do not.

The isomorphism theorem

Theorem

Let $(M, \mathcal{X})$, $(M, \mathcal{Y})$ be countable models of WKL_0^*, and assume that $(M, \mathcal{X} \cap \mathcal{Y}) \models \neg \mathsf{I}\Sigma^0_1$. Then $(M, \mathcal{X}) \simeq (M, \mathcal{Y})$.

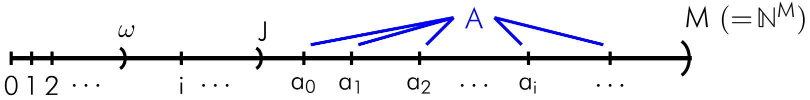
- This can be seen as a second-order generalization of results due to Kossak and Kaye saying that models of $\mathsf{I}\Delta_1 + \neg \mathsf{I}\Sigma_1$ have "many" automorphisms.
- There are many ω-models of WKL_0 ("Scott sets") that are neither isomorphic nor elementarily equivalent to one another.
- Any countable $(M, \mathcal{W}) \models \mathsf{RCA}_0$ will have extensions $(M, \mathcal{X}), (M, \mathcal{Y})$ satisfying WKL_0 with $(M, \mathcal{X}) \not\equiv (M, \mathcal{Y})$.

Plan for rest of talk

- Brief comment about the proof of the isomorphism theorem
- Consequences for WKL^*_0 in the absence of $\mathrm{I}\Sigma^0_1$.
- Consequences for reverse mathematics over RCA_0.

The isomorphism theorem: ideas behind proof

- Because $(M, \mathcal{X} \cap \mathcal{Y})$ satisfies $\neg I\Sigma^0_1$ there is a Σ^0_1-definable proper cut J closed under $x \mapsto 2^x$ and an infinite set $A \in \mathcal{X} \cap \mathcal{Y}$ s.t. $A = \{a_i : i \in J\}$ enumerated in increasing order.



- We use back-and-forth. At each step, we have finite tuples $\bar{r}, \bar{R}$ in the domain, $\bar{s}, \bar{S}$ in the range of the partial iso. The invariant is roughly: for each Δ_0 formula δ, each $i, k \in J$,

$$(M, \mathcal{X}) \models \delta(a_i, k, \bar{r}, \bar{R}) \text{ iff } (M, \mathcal{Y}) \models \delta(a_i, k, \bar{s}, \bar{S}).$$

- WKL needed to find "globally good" second-order element to add in the inductive step, based on "locally good" ones that are easier to find directly from inductive assumption. □

Consequences for WKL^*_0: the analytic hierarchy

For a set W, let $W_k = \{n : \langle k, n\rangle \in W\}$.

If $(M, \mathcal{X}) \models \mathrm{WKL}^*_0$ and $A \in \mathcal{X}$, then there exists $W \in \mathcal{X}$ such that $W_0 = A$ and $(M, \{W_k : k \in M\}) \models \mathrm{WKL}^*_0$.
We say that W codes a model of WKL^*_0. If it satisfies $\neg\mathrm{I}\Sigma^0_1$, then by the isomorphism theorem it is elementarily equivalent to $(M, \mathcal{X})$!

Corollary

For any formula $\psi(\bar{x}, \bar{X})$, TFAE provably in $\mathrm{WKL}^*_0 + \neg\mathrm{I}\Sigma^0_1$:

(i) $\psi(\bar{x}, \bar{X})$,

(ii) "there exists a coded model of $\mathrm{WKL}^*_0 + \neg\mathrm{I}\Sigma^0_1 + \psi(\bar{x}, \bar{X})$",

(iii) "there is no coded model of $\mathrm{WKL}^*_0 + \neg\mathrm{I}\Sigma^0_1 + \neg\psi(\bar{x}, \bar{X})$".

Thus, in $\mathrm{WKL}^*_0 + \neg\mathrm{I}\Sigma^0_1$ the analytic hierarchy collapses to Δ^1_1.

Consequences for WKL^*_0: conservativity

Corollary

Let ψ be a Π^1_2 statement. Then:

(i) ψ is Π^1_1-conservative over $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1$ iff $\mathsf{WKL}^*_0 + \neg\mathsf{I}\Sigma^0_1 \vdash \psi$.

(ii) if ψ is Π^1_1-conservative over RCA^*_0, then $\mathsf{WKL}^*_0 + \neg\mathsf{I}\Sigma^0_1 \vdash \psi$.

In contrast, the set of Π^1_2 sentences ψ that are Π^1_1-conservative over RCA_0 is Π_2-complete. [Towsner 2015]

It also contains some combinatorially natural principles that do not follow from WKL_0, such as the cohesive set principle COH:

"for every family $\{R_x : x \in \mathbb{N}\}$ of subsets of $\mathbb{N}$, there exists infinite $C \subseteq \mathbb{N}$ s.t. for each x, either $\forall^\infty z \in C\,(z \in R_x)$ or $\forall^\infty z \in C\,(z \notin R_x)$".

Consequences for WKL^*_0: failure of low basis

Corollary

If $(M, \mathcal{X}) \models \mathsf{RCA}^*_0$, and $A \in \mathcal{X}$ is such that $\neg I\Sigma^A_1$ holds, then there is a computable in A infinite 0-1 tree T such that no model $(M, \mathcal{Y}) \models \mathsf{RCA}^*_0$ contains any infinite path through T that is arithmetically definable in A.

- This is a failure of the low basis theorem: T is $\Delta_1(A)$, but has not just no low $\Delta_2(A)$ path, but even no arithmetically-in-A definable one, at least one contained in a model of RCA^*_0.
- In contrast, the low basis theorem is provable in RCA_0. [Hájek-Kučera 1989].

Back over RCA_0

A major problem in reverse mathematics:
describe the Π^1_1 consequences of $\mathsf{RCA}_0 + \mathsf{RT}^2_2$.
(Here RT^2_2 is Ramsey's Thm for pairs and two colours.)

$\mathsf{RCA}_0 + \mathsf{RT}^2_2$ is Π^1_1-conservative over $\mathsf{I}\Sigma^0_2$ and proves $\mathsf{I}\Delta^0_2$.
So, it remains to characterize its behaviour over $\mathsf{I}\Delta^0_2 + \neg\mathsf{I}\Sigma^0_2$.

But if $(M, \mathcal{X}) \models \mathsf{RCA}_0 + \mathsf{I}\Delta^0_2 + \neg\mathsf{I}\Sigma^0_2$,
then $(M, \Delta^0_2\text{-Def}(M, \mathcal{X})) \models \mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1$!

Is there a neat statement ensuring that Δ^0_2-Def satisfies WKL?

A closer look at COH

Provably in $\mathsf{RCA}_0 + \mathsf{I}\Delta^0_2$, the statement

"The Δ^0_2-definable sets satisfy WKL"

or

"For every set X, there is Y such that Y′ has PA degree relative to X′ "

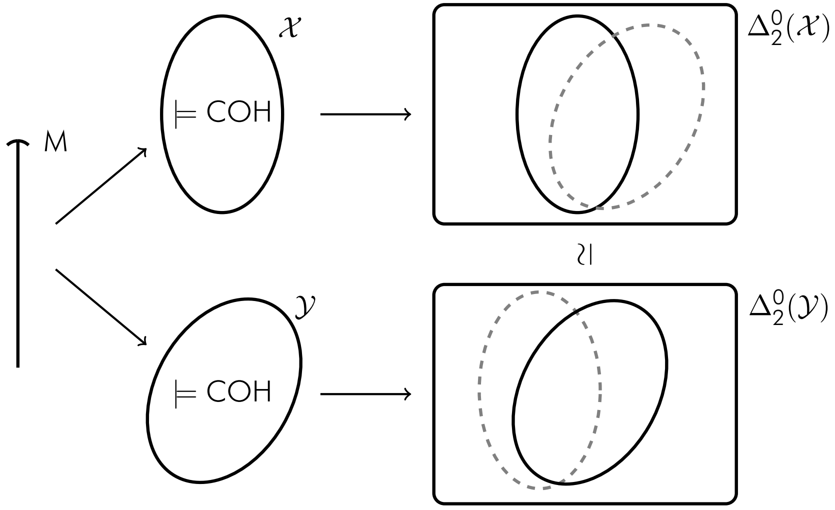
is equivalent to COH! [Belanger 20XX]

So, the isomorphism theorem for WKL^*_0 gives:

Corollary

Let $(M, \mathcal{X}), (M, \mathcal{Y})$ be countable models of $\mathsf{RCA}_0 + \mathsf{I}\Delta^0_2 + \mathsf{COH}$. If $(M, \mathcal{X} \cap \mathcal{Y}) \models \neg \mathsf{I}\Sigma^0_2$, then $(M, \Delta^0_2\text{-Def}(M, \mathcal{X})) \simeq (M, \Delta^0_2\text{-Def}(M, \mathcal{Y}))$.

The isomorphism theorem for COH, pictured



A consequence of RT^2_2

RT^2_2 says: "For every $f\colon [\mathbb{N}]^2 \to 2$ there is a homogeneous set H".

Consider the following sentence γ:

"For every Z, if $\neg\mathsf{I}\Sigma^Z_2$, then for every $f\colon [\mathbb{N}]^2 \to 2$ with $f \leq_T Z$
and every set Y such that Y′ has PA degree relative to Z′,
there is a Δ^0_2-set $\widetilde{H}$ homogeneous for f s.t. $(\widetilde{H} \oplus Z)' \leq_T Y'$."

(For those who care: γ says that if $\mathsf{I}\Sigma^0_2$ fails then the first-jump control argument of [CJS 2001] has to work for adding homogeneous sets for 2-colourings of pairs.)

- γ is Π^1_1, in fact $\forall\Pi^0_5$.
- $\mathsf{RCA}_0 + \mathsf{RT}^2_2 \vdash \gamma$. (Clear over $\mathsf{I}\Sigma^0_2$. Over $\mathsf{I}\Delta^0_2 + \neg\mathsf{I}\Sigma^0_2$, argue using $\mathsf{RCA}_0 + \mathsf{RT}^2_2 \vdash \mathsf{COH}$ and the iso thm for COH.)
- If $\mathsf{RCA}_0 + \mathsf{I}\Delta^0_2 \vdash \gamma$, then $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ is Π^1_1-conservative over $\mathsf{I}\Delta^0_2$.

Characterizing conservativity of RT^2_2

Corollary

$\mathsf{RCA}_0 + \mathsf{RT}^2_2$ is Π^1_1-conservative over $\mathsf{RCA}_0 + \mathsf{I}\Delta^0_2$ iff $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ is $\forall\Pi^0_5$-conservative over $\mathsf{RCA}_0 + \mathsf{I}\Delta^0_2$.

Note:

- $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ is $\forall\Pi^0_3$-conservative over $\mathsf{RCA}_0 + \mathsf{I}\Delta^0_2$. [PY 2018]
- $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ is $\forall\Pi^0_4$-conservative over $\mathsf{RCA}_0 + \mathsf{I}\Delta^0_2 + \{\mathsf{WO}(\omega), \mathsf{WO}(\omega^\omega), \ldots\}$. (Essentially [CSY 2017].)
- $\mathsf{RCA}^*_0 + \mathsf{RT}^2_2$ is $\forall\Pi^0_3$- but not $\forall\Pi^0_4$-conservative over RCA^*_0. [KKY 20XX]

Another result on conservativity

Theorem (Towsner 2015)

For each n, the set of Π^1_2 sentences ψ that are Π^1_1-conservative over $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_n$ is Π_2-complete.

Towsner asked whether this also works for $\mathsf{I}\Delta^0_n$ in place of $\mathsf{I}\Sigma^0_n$. Much of our analysis of RT^2_2 carries over to any Π^1_2 sentence, giving:

Corollary (of the isomorphism thm)

For each n, the set of Π^1_2 sentences ψ that are Π^1_1-conservative over $\mathsf{RCA}^*_0 + \mathsf{I}\Delta^0_n + \neg\mathsf{I}\Sigma^0_n$ is c.e.

However, a completely different argument shows:

Theorem

For each n, the set of Π^1_2 sentences ψ that are Π^1_1-conservative over $\mathsf{RCA}^*_0 + \mathsf{I}\Delta^0_n$ is Π_2-complete.