TD implies $CC_{\mathbb{R}}$

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AD

Definition

Given a set $A \subseteq \omega^\omega$,

1. A game G_A has two players, say I and II, so that each player plays a natural number.
2. I wins if the final outcome belongs to A; Otherwise, II wins.
3. A strategy is a function $\hat{\sigma} : \omega^{<\omega} \to \omega$.
4. $\hat{\sigma}$ is a winning strategy for I if the final outcome always belong to A as long as I plays according to $\hat{\sigma}$; similarly for II.
5. Axiom of Determinacy, AD, says that for any set A, either I or II has a winning strategy.

TD

Definition

Turing determinacy (TD) says that for every set A of *Turing degrees*, either A or the complement of A contains an upper cone.

Consequences of AD

Theorem (Martin)

Over ZF, $\mathrm{AD} \to \mathrm{sTD} \to \mathrm{TD}$.

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TD *is more natural than* AD.

Axiom of Choice

Definition

Given a nonempty set A,

1. CC_A, the countable choice for subsets of A, says that for any countable sequence $\{A_n\}_{n\in\omega}$ of nonempty subsets of A, there is a function $f : \omega \to A$ so that $\forall n(f(n) \in A_n)$.
2. DC_A, the dependent choice for subsets of A, says that for any binary relation $R \subseteq A \times A$, if $\forall x \in A \exists y \in A R(x, y)$, there is a countable sequence elements $\{x_n\}_{n\in\omega}$ so that $\forall n R(x_n, x_{n+1})$.

Determinacy v.s. Choice (1)

Clearly AD implies ¬AC.

Theorem (Mycielski)

ZF + AD *implies* $\mathrm{CC}_{\mathbb{R}}$.

Proof.

Given a sequence nonempty sets $\{A_n\}_{n\in\omega}$ of reals, set
$A = \{n^\frown(x \oplus y) \mid n \in \omega \wedge x \notin A_n \wedge y \in \omega^\omega\}$.
I does not have a winning strategy for G_A. By AD, II does. The winning strategy $\hat{\tau}$ codes a choice function. □

Determinacy v.s. Choice (2)

Theorem (Kechris)

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Question

Does $\mathrm{ZF} + \mathrm{AD}$ *imply* $\mathrm{DC}_{\mathbb{R}}$*?*

TD v.s. Choice

Theorem (Peng and Y.)

$\mathrm{ZF} + \mathrm{TD}$ *implies* $\mathrm{CC}_{\mathbb{R}}$.

TD v.s. Choice

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Question

1. *Does* $\mathrm{ZF} + \mathrm{TD}$ *imply* $\mathrm{DC}_{\mathbb{R}}$?
2. *Does* $\mathrm{ZF} + V = L(\mathbb{R}) + \mathrm{TD}$ *imply* $\mathrm{DC}_{\mathbb{R}}$?

The double jumps of minimal covers

Theorem (Spector-Sacks)

Within ZF, for any real x, there is a perfect tree $T \leq_T x''$ so that

- *For any different reals $z_0, z_1 \in [T]$, $z_0 \not\equiv_T z_1$;*
- *For each $z \in [T]$, z is a minimal cover of x.*

Note that for any different reals $z_0, z_1 \in [T]$, if y has the property that $y \leq_T z_0$ and $y \leq_T z_1$, then $y \leq_T x$. Moreover the double jumps of the members in $[T]$ range over an upper cone.

A weaker version of $\mathrm{CC}_{\mathbb{R}}$

Lemma

If $\{A_n\}_{n\in\omega}$ is a sequence of countable nonempty sets of reals, then there is a choice function for the sequence.

Proof.

Suppose not. For any x, let

$$n_x = \min\{n \mid \forall y \in A_n(y \not\leq_T x)\}.$$

Then n_x is defined for every x. But by the Spector-Sacks theorem and the countability of A_n, there is some $y >_T x$ so that $n_y = n_x$ but $n_{y''} > n_x$.

By TD, $n_{y''} > n_y$ over an upper cone. Then $n_{y^{(\omega)}}$ is not defined. □

So every countable set of Turing degrees has an upper bound.

Constant function

Lemma

Suppose that $f : \mathbb{R} \to \mathbb{R}$ *is a degree invariant function, then* $f(x) = f(x')$ *over an upper cone.*

Proof.

Suppose not. Define

$$l_x = \min\{n \mid f(x)(n) \neq f(x')(n)\}.$$

By TD, $l_x \leq l_y$ for any $x \leq_T y$ over an upper cone.
For some $i \in \{0, 1\}$, $\{x \mid f(x)(l_x) = i\}$ contains an upper cone. So $l_x \neq l_{x'}$ and so $l_x < l_{x'}$ over an upper cone. □

Note that, in the lemma, the jump operator can be replaced with any degree increasing function.

Degree decreasing function

Lemma

If f is a degree invariant function so that $f(x) \leq_T x$ over an upper cone, then the range of f is at most countable over an upper cone.

Proof.

By the previous Lemma, $f(x') = f(x) \leq_T x <_T x'$ over an upper cone and so $f(x) <_T x$ over an upper cone.
Now by the Spector-Sacks theorem, given any x over the upper cone, there are two reals $y_0, y_1 >_T x$ so that $y_0'' \equiv_T y_1'' \geq_T x''$. Then $y_0 >_T f(y_0) = f(y_0'') = f(y_1'') = f(y_1) <_T y_1$ and so $f(y_0'') = f(y_0) \leq_T x$.
So every member in the range of f over the upper cone must be Turing below x.

□

Countability of degree invariant funciton

Lemma

Suppose that f is a degree invariant function, then the range of f must be at most countable.

Proof.

By the previous lemma, we may assume that $f(x) \not\leq_T x$ over an upper cone. Let

$$\Phi(x) = f(x) \oplus x.$$

Then $\Phi(x) >_T x$ over an upper cone and can be view as a "jump operator", By applying the previous lemma, $\Phi(x) \geq_T f(x) = f(\Phi(x))$ over an upper cone. So $f(x) \leq_T x$ over an upper cone, a contradiction. □

$\mathrm{ZF} + \mathrm{TD} \vdash \mathrm{CC}_{\mathbb{R}}$ (1)

This is where the set theory argument comes in.
Given a sequence $\{A_n\}_{n\in\omega}$ of nonempty sets of reals. We may assume that each one is Turing upward closed and the sequence is nonincreasing.
Let $B_n = A_n \setminus A_{n+1}$ and $f(x) = \{n \mid \exists y \in B_n(y \geq_T x)\}$.
Then the range of f is countable over an upper cone, enumerated as $\{a_i\}_{i\in\omega}$. Note that each a_i is infinite.
The idea is that the sets $\{d \mid \bigcup_{n\in d} B_n$ contains an upper cone $\}$ generates an ultrafilter. Then $\{a_i\}_{i\in\omega}$ can be viewed as a "countable decomposition" of the measure.

$\mathrm{ZF} + \mathrm{TD} \vdash \mathrm{CC}_{\mathbb{R}}$ (2)

Pick up a set $a \subseteq \omega$ so that $a \cap a_i \neq \emptyset$ and $a_i \setminus a \neq \emptyset$ for each i.
Set

$$C_0 = \bigcup_{n \in a} B_n; \text{ and } C_1 = \bigcup_{n \notin a} B_n.$$

There must be some k so that C_k ranges over an upper cone.
If $k = 0$, then C_1 is bounded and so $f(x) \subseteq a$ for an upper cone of degrees, a contradiction to $a_i \setminus a \neq \emptyset$;
If $k = 1$, then C_0 is bounded and so $f(x) \cap a = \emptyset$ for an upper cone of degrees, a contradiction to $a_i \cap a \neq \emptyset$. This is not possible.

An application

Theorem (Woodin)

Assume $\mathrm{ZF} + \mathrm{TD} + \mathrm{CC}_{\mathbb{R}}$*, every set of reals is Suslin.*

Now we may remove the assumption $\mathrm{CC}_{\mathbb{R}}$.

More applications

We have found a number applications of such methods, via point-to-set principle.

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