Complements up to Homotopy

The goal of this document is to provide some techniques that may help you identify more useful descriptions of spaces obtained as complements of things in $\mathbb{R}^n$ or S^n (especially for $n=3$).

First, recall that S^n is the one point compactification of $\mathbb{R}^n$. Therefore, whenever you remove one or more points from S^n it can be helpful to imagine one of the points as being the one used to construct S^n from $\mathbb{R}^n$. In this way, you should be able to convince yourself that.

a) $S^n \setminus \{m \text{ points}\} \simeq \mathbb{R}^n \setminus \{m-1 \text{ points}\}$

b) $S^n \setminus S^1 \simeq \mathbb{R}^n \setminus \{\text{coordinate axis}\}$

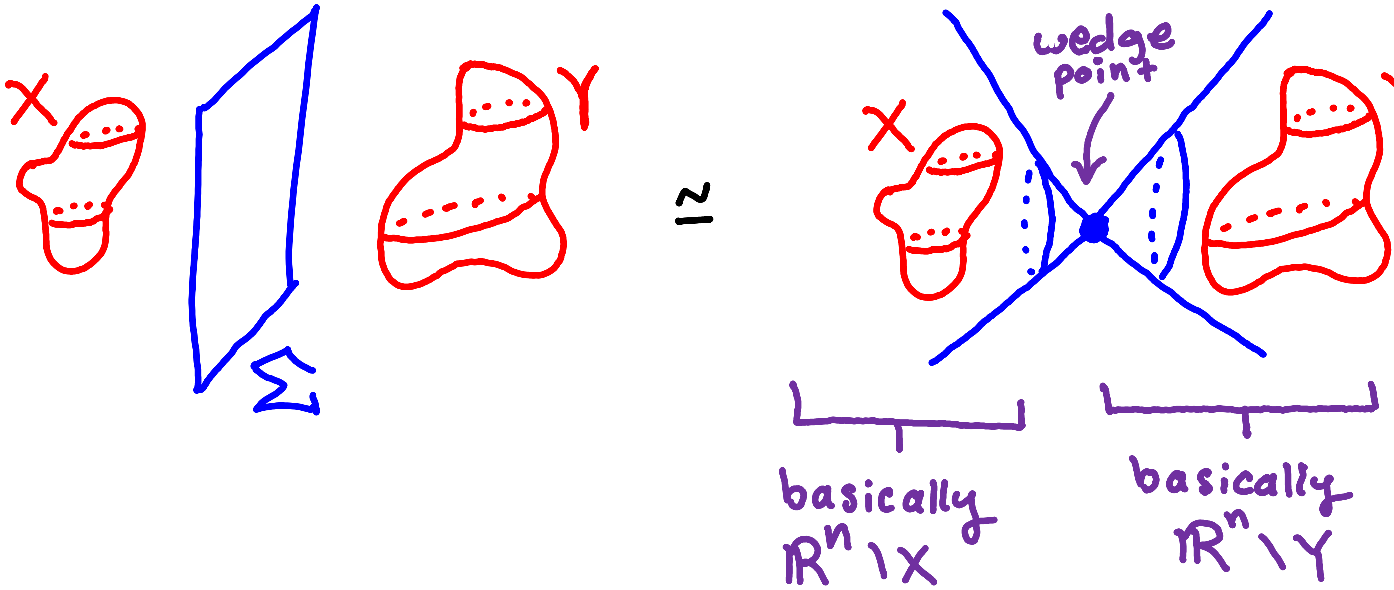
c) $S^3 \setminus \{\text{pair of linked circles}\} \simeq \mathbb{R}^3 \setminus (\{z\text{-axis}\} \cup \{x^2+y^2=1\})$

d) $S^3 \setminus \{n \text{ component unlink}\} \simeq \mathbb{R}^3 \setminus (\{z\text{-axis}\} \cup \{n-1 \text{ copies of } S^1\})$

Next, suppose $X, Y \subseteq \mathbb{R}^n$ s.t. $\exists$ a hyperplane Σ for which X and Y are in distinct connected components of $\mathbb{R}^n \setminus \Sigma$.

Then it follows that $\mathbb{R}^n \setminus (X \cup Y) \simeq (\mathbb{R}^n \setminus X) \vee (\mathbb{R}^n \setminus Y)$.

In other words, the complements of X and Y in $\mathbb{R}^n$ may be treated separately & the results wedged together to compute $\mathbb{R}^n \setminus (X \cup Y)$. To justify this, simply form the quotient $\mathbb{R}^n / \Sigma$ which collapses Σ down to a point - the wedge point.



The two resulting 'halves' of $\mathbb{R}^n$ that remain in $\mathbb{R}^n / \Sigma$ are technically homotopy equivalent to halfspaces $\mathbb{H}^n := \{(x_i) \mid x_1 > 0\}$

So a complete argument would mention $\mathbb{H}^n \simeq \mathbb{R}^n$.

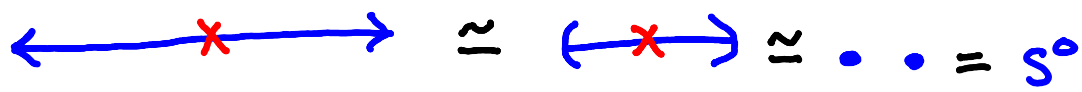
We can now focus on complements of sets in $\mathbb{R}^n$ that have no 'separating hyperplanes' in the sense above.

1) $\mathbb{R}^n \setminus \{\text{point}\} \simeq S^{n-1}$

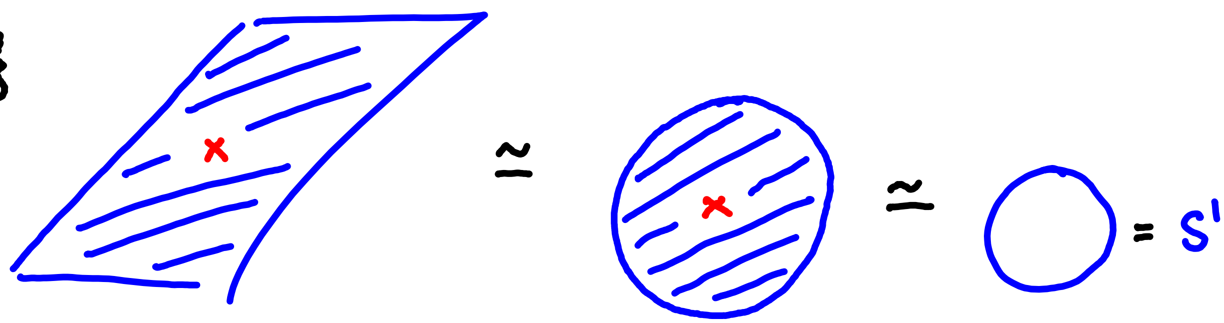
An explicit continuous map $\mathbb{R}^n \setminus \{\text{origin}\} \longrightarrow S^n$ which can be used to construct a homotopy is $\vec{x} \mapsto \dfrac{\vec{x}}{\|\vec{x}\|}$.

in the examples below we perform the homotopy in two stages first collapsing $\mathbb{R}^n\setminus\{\text{point}\}$ to $D^n\setminus\{\text{point}\}$ then to S^n.

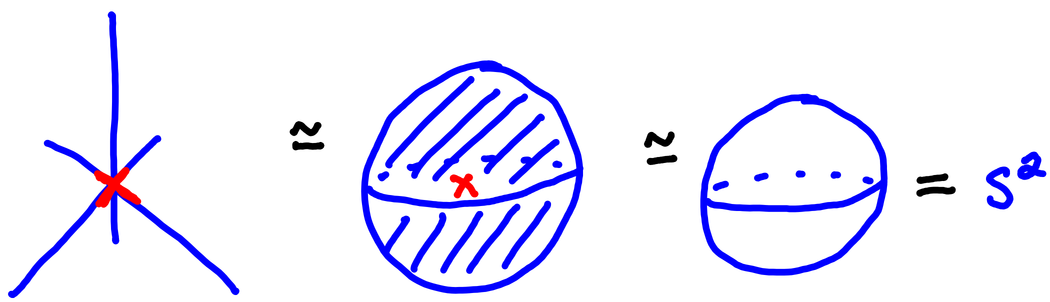
e.g. $\mathbb{R}\setminus\{0\}$



$\mathbb{R}^2\setminus\{(0,0)\}$



$\mathbb{R}^3\setminus\{(0,0,0)\}$



Steriographic projection

⚠ Even though $\mathbb{R}^n\setminus\{\text{point}\} \simeq S^{n-1}$ and $S^k\setminus\{\text{point}\} \simeq \mathbb{R}^k$ it is not the case that $\mathbb{R}^n\setminus\{2 \text{ points}\} \simeq \mathbb{R}^{n-1}$. Since any pair of points in $\mathbb{R}^n$ can be separated by a hyperplane the argument above can be modified to show $\mathbb{R}^n\setminus\{m \text{ points}\} \simeq \bigvee_{i=1}^{m} S^{n-1}$.

The conclusion one can draw from this observation is that the operation of taking complements of sets is not well-defined on homotopy classes of spaces.

2) Let $n>1$. $\mathbb{R}^n \setminus \{\text{a coordinate axis}\} \simeq S^{n-2}$

This problem can be reduced to the previous problem of showing $\mathbb{R}^K \setminus \{\text{point}\} \simeq S^{K-1}$ by projecting $\mathbb{R}^n$ onto the (n-1)-dimensional subspace orthogonal to the 'missing' coordinate axis. In this way, one can construct a homotopy from $\mathbb{R}^n \setminus \{\text{coordinate axis}\}$ to $\mathbb{R}^{n-1} \setminus \{\text{origin}\}$.

To handle the complement of multiple disjoint lines one can

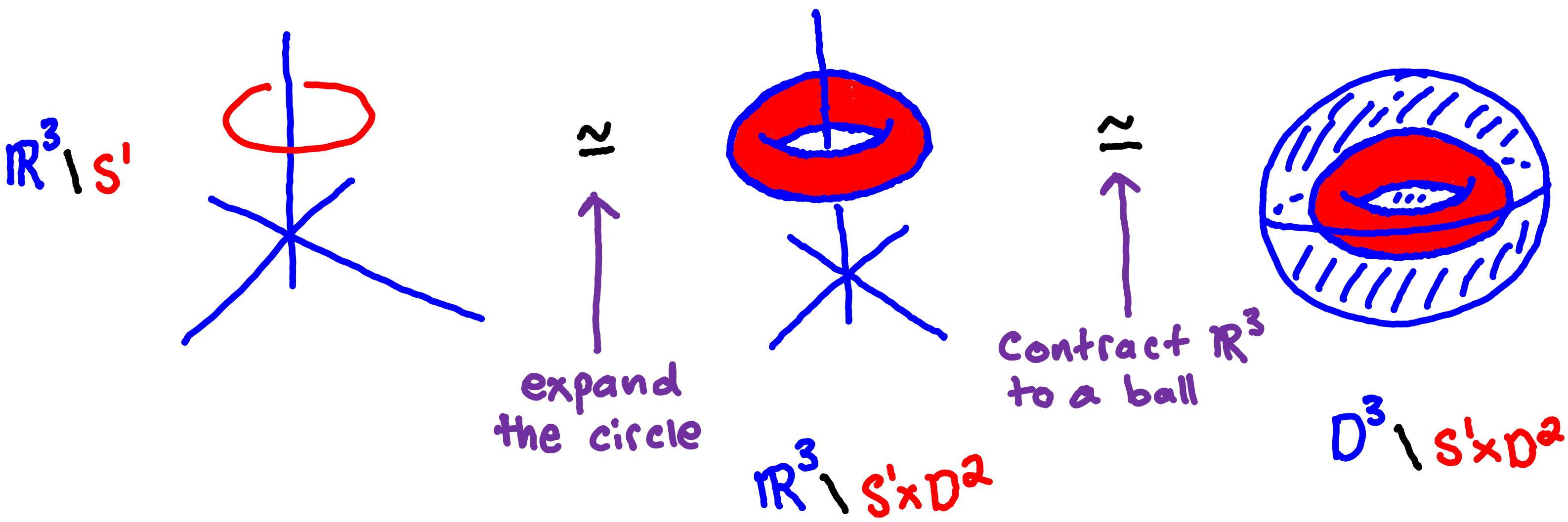
i) separate them with hyperplanes then quotient ; or

ii) homotope the space so that all of the complements of the lines are parallel and then project.

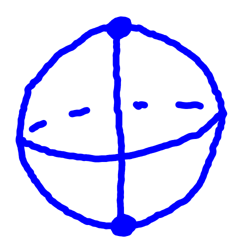
Thus $\mathbb{R}^n \setminus \{m \text{ disjoint lines}\} \simeq \mathbb{R}^{n-1} \setminus \{m \text{ points}\} \simeq \bigvee_{i=1}^{m} S^{n-2}$.

3) $\mathbb{R}^3 \setminus S^1$

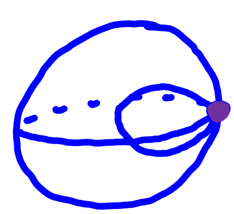
The complement of a circle in $\mathbb{R}^3$ can be "expanded" to the complement of a solid torus.



Next, the 'solid torus complement' can gradually be expanded. this has the effect of hollowing out the ball. Eventually all that's left is the boundary of the ball $\partial D^3 \simeq S^2$ and a diameter of the sphere which is due to the 'donut hole' of the solid torus. Therefore we get $\mathbb{R}^3 \setminus S^1 \simeq$



a 2-sphere with a diameter. By contracting an injective path on the surface of the sphere which connects the endpoints of the diameter we get

 $=$  $= S^2 \vee S^1$

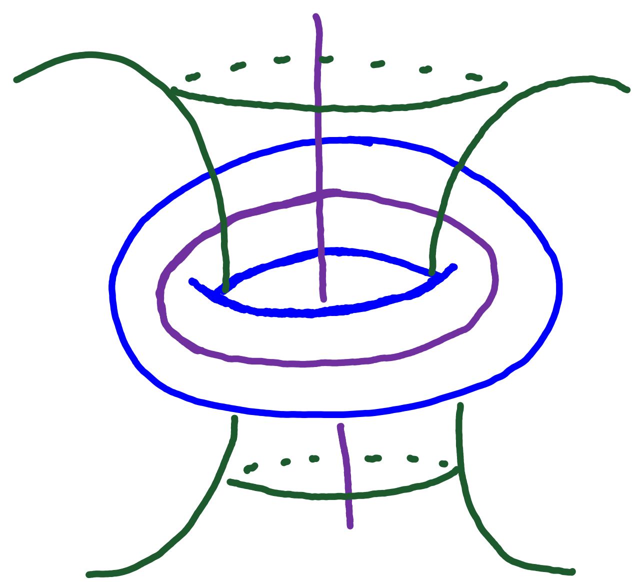
Since we have quotiented by a contractible subspace, the result is homotopy equivalent to the space we started with.

this proves that $\mathbb{R}^3 \setminus S^1 \simeq S^2 \vee S^1$.

4) $S^3 \setminus \{\text{linked circles}\} \simeq \mathbb{R}^3 \setminus (\{z\text{-axis}\} \cup \{x^2+y^2=1\}) \simeq \mathbb{T}^2$

First, recall that there is a Heegaard splitting of S^3 into two solid tori glued along their boundaries. The core circles of these tori are therefore linked with one another.

Here is a picture of the image of the two tori (and their core circles) in $\mathbb{R}^3$ under steriographic projection of S^3 from a point on one of the core circles. Although this is not



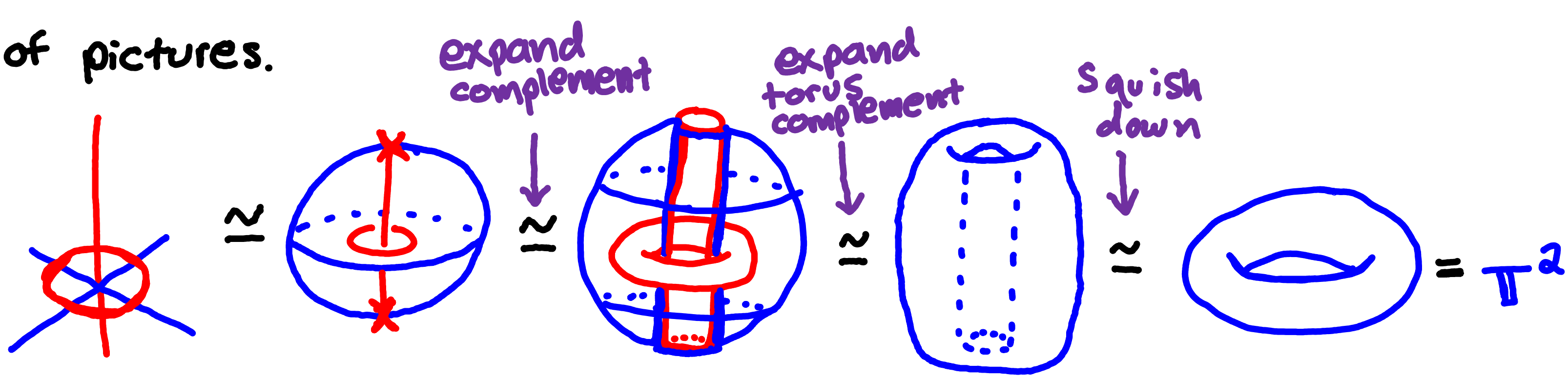
how donuts work, one might think of these core circles as the 'jelly' part of the donut. More precisely, these core circles may be identified with the unit circles in each copy of $\mathbb{C}$ if S^3 is viewed as $\{(z,w) \in \mathbb{C}^2 \cdot |z|^2 + |w|^2 = 1\}$.

Then steriographic projection from $(0,1)$ is a surjective map $S^3 \setminus \{(0,1)\} \longrightarrow \mathbb{R}^3$ that sends $\{(e^{i\theta}, 0) \mid \theta \in [0, 2\pi)\}$ to $\{(x,y,0) \mid x^2+y^2=1\} \subseteq \mathbb{R}^3$ which is the core circle of the blue torus in the picture above. Similarly, $\{(0, e^{i\theta}) \mid \theta \in (0, 2\pi)\}$ is mapped to the z-axis which is the core circle of the green 'ambient' torus which can be seen in the picture above 'fountaining' out in both directions from the other torus' donut hole.

Therefore, after removing these core circles the empty space left behind can be expanded to thicker and thicker complements of solid tori until all that is left is the common boundary torus along which the two original solid tori are glued to create S^3 in the first place.

This proves $S^3 \setminus \{\text{linked circles}\} \simeq \mathbb{T}^2$.

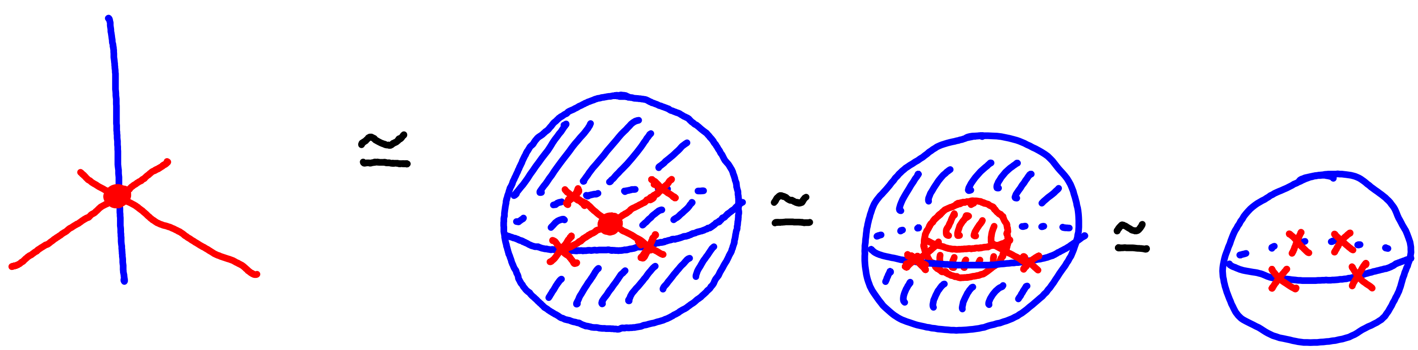
On the other hand, we can argue the fact that $\mathbb{R}^3 \setminus (\{z\text{-axis}\} \cup \{x^2+y^2=1\}) \simeq \mathbb{T}^2$ directly in a sequence of pictures.



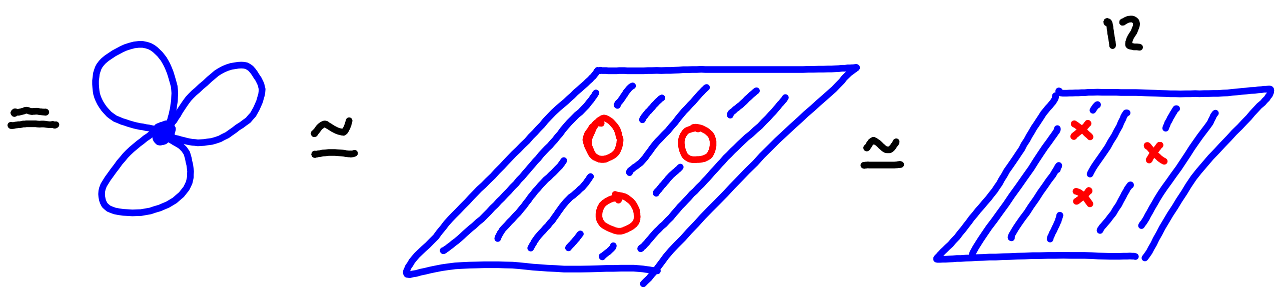
5) $\mathbb{R}^n \setminus \{m \text{ lines intersecting at a point}\} \simeq \bigvee_{i=1}^{2m-1} S^{n-2}$

Contract $\mathbb{R}^n$ to a ball centered at the intersection point of the lines that have been removed. Expand the complement of the intersection point to the boundary of the ball. This leaves S^{n-1} with $2m$ points removed. We know that $S^{n-1} \setminus \{2m \text{ points}\} \simeq \mathbb{R}^{n-1} \setminus \{2m-1 \text{ points}\}$ and that $\mathbb{R}^{n-1} \setminus \{2m-1 \text{ points}\} \simeq \bigvee_{i=1}^{2m-1} S^{n-2}$ which finishes the proof.

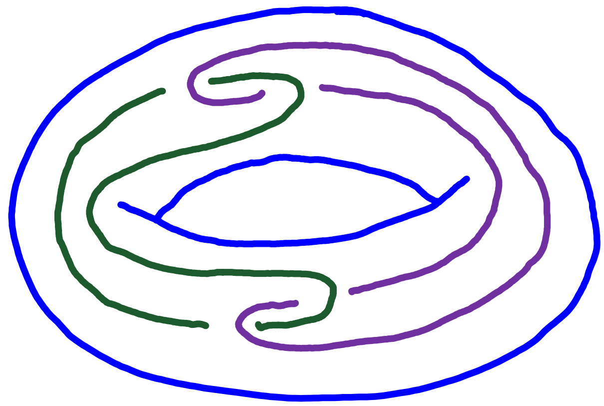
e.g. $\mathbb{R}^3 \setminus \{x \text{ \& } y\text{-axes}\}$



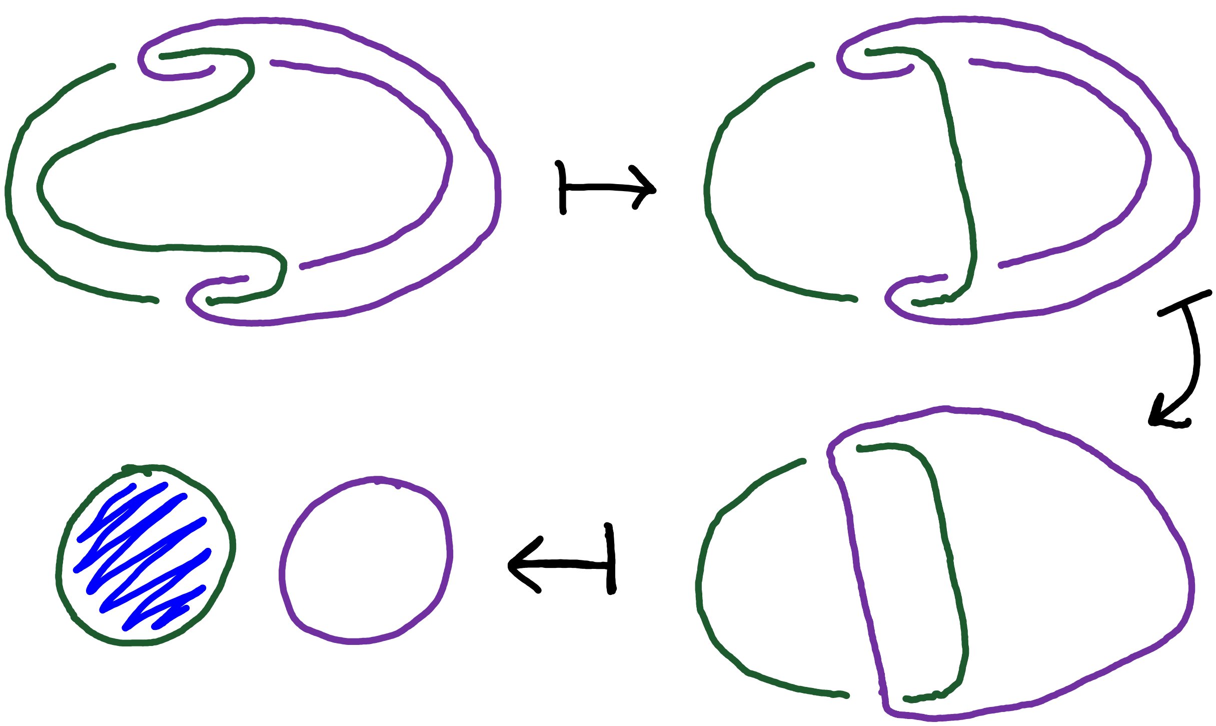
$\bigvee_{i=1}^{3} S^1 =$

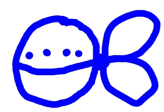


Here is an application of these ideas: Consider two curves A and B in the solid torus $T = D^2 \times S^1$ in $\mathbb{R}^3$ as in the figure below.



First, notice that the pattern of crossings (over v.s. under) is not that of a linked pair of circles as viewed inside of the ambient $\mathbb{R}^3$. Here is an explicit homotopy which shows that A bounds a disk in $\mathbb{R}^3 \setminus B$.

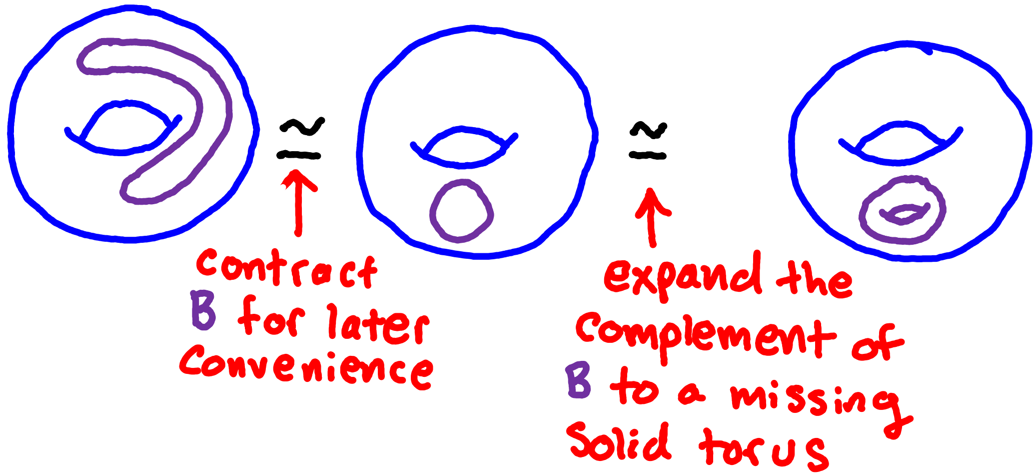


However, we can prove that A is not null-homotopic to a constant in $T \setminus B$. Abstractly, we know that $T \setminus B$ is homotopic to $S^3 \setminus \{2 \text{ component unlink}\}$ which is homotopic to $\mathbb{R}^3 \setminus (\{z\text{-axis}\} \cup \{\text{circle}\})$ where the circle is separated from the z-axis by a plane, and its not difficult to show, using the ideas above, that this is space is homotopic to $S^1 \vee S^1 \vee S^2 =$ . However, to understand the map induced on fundamental groups by the inclusion $\iota : S^1 \longrightarrow T \setminus B$, which is of type

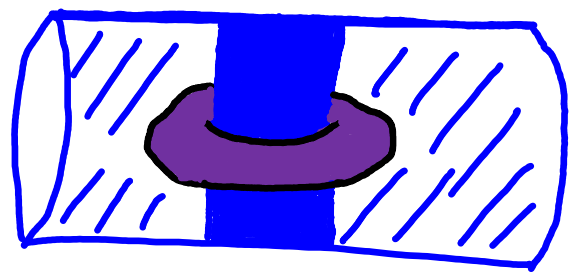
$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Z} * \mathbb{Z}, \\ \wr\wr & & \wr\wr \\ \pi_1(S^1) & & \pi_1(T \setminus B) \end{array}$$

we must understand the homotopy $T \setminus B \simeq S^1 \vee S^1 \vee S^2$ more concretely & keep track of the image of the loop A.

let's begin with a series of pictures beginning with $T \setminus B$ & ending with $S^1 \vee S^1 \vee S^2$.

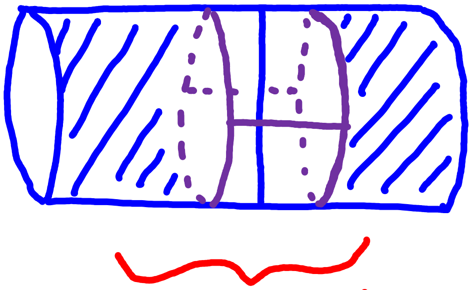


Zooming in on the section containing the inflated complement of B, we can see that the inside of the hole is filled in by $D^2 \times I$ which can be shrunk to an interval $\{x\} \times I$ by expanding the complement of B so that the donut hole becomes as small as possible. Another way to think about what's happening in this step is to realize that $D^2 \simeq \{x\}$. In any case, we end up with



$$= D^2 \times I \setminus D^2 \times S^1$$

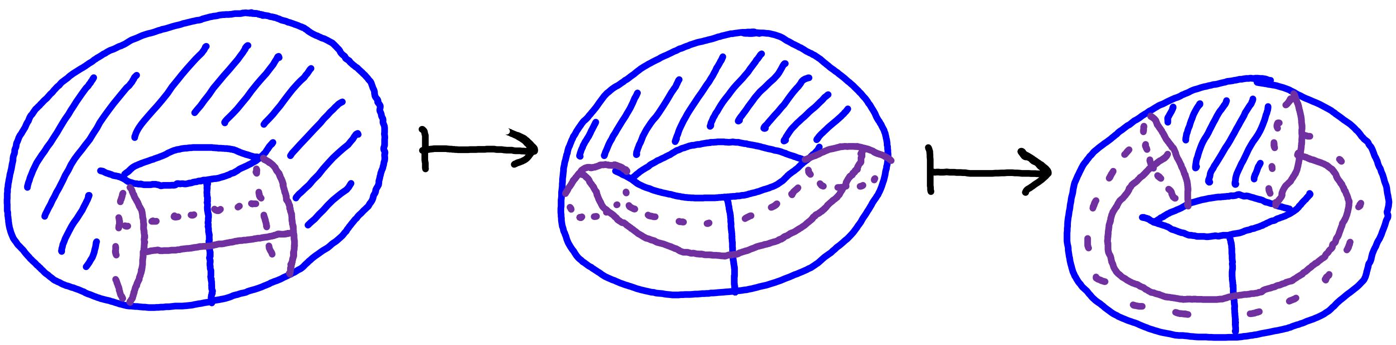
After elongating the (purple) part that's being removed the picture looks like



now the missing part basically looks like an annulus cross an interval

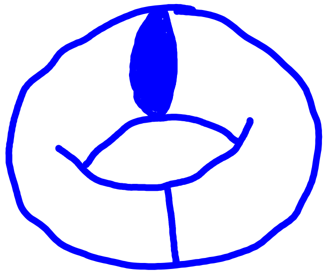
and zooming back out we see

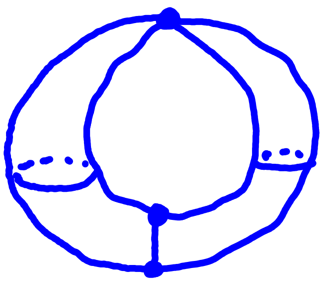
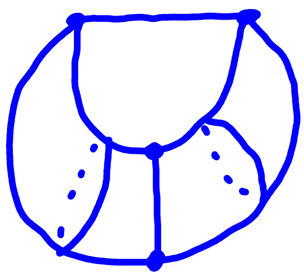
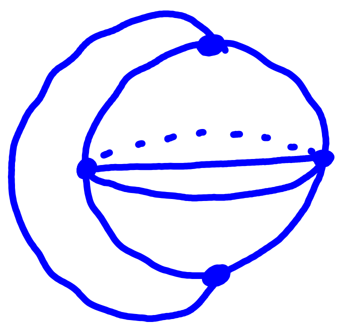
then the remaining inner part of T can be slowly pushed back around the torus as illustrated below



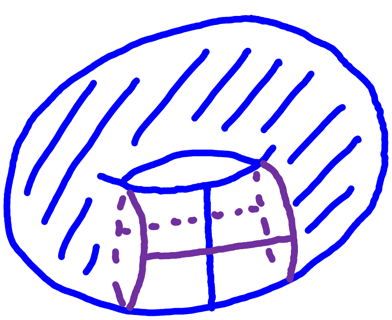
the point is that the filled in part of this solid torus is like $D^2 \times I$ if the interval inside the annulus $\times I$ is ignored & the two moves pictured above are essentially just contracting the interval direction down. Since $I \simeq \{y\}$

we can contract this part down to a disk which leaves

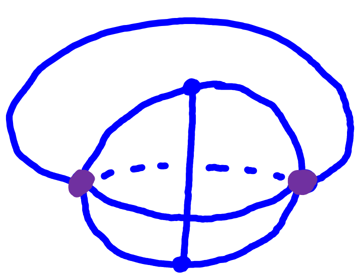
$T^2 \cup D^2 \cup I =$  Since disks are contractible, we can collapse the disk and the resulting space will be homotopy equivalent to the one we started with.

 $\simeq$  $\simeq$ 

If you don't like that perspective, you can think about



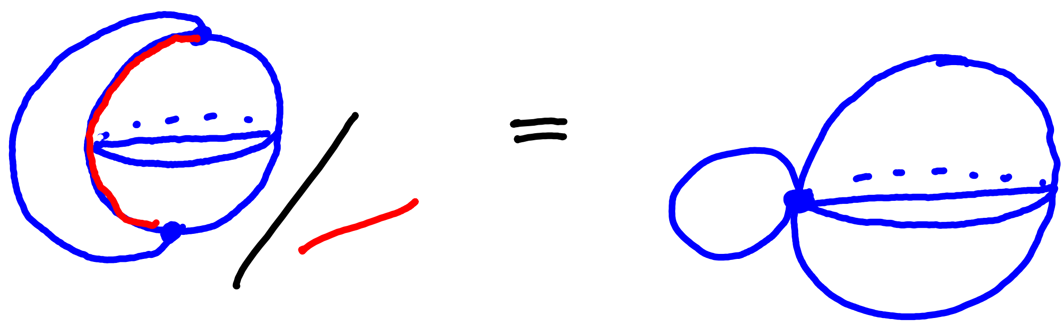
collapsing the disk direction of the solid cylinder outside the tall annulus which jumps immediately from the picture above to



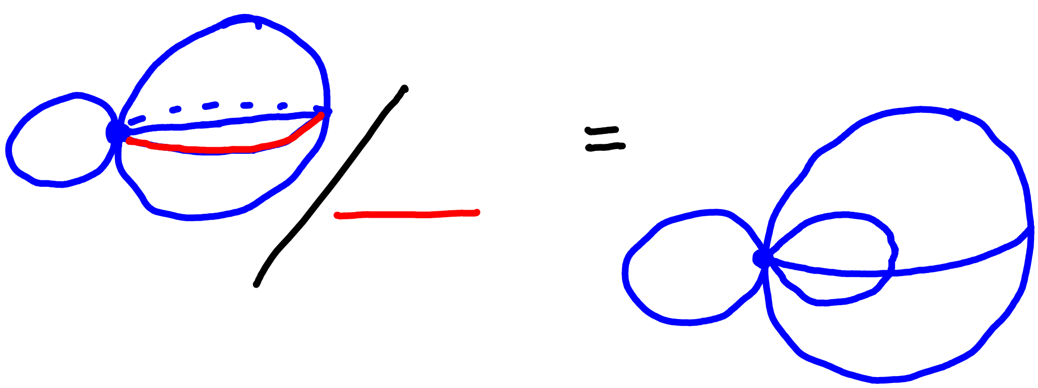
but either way

we end up with a copy of S^2 that has two diameters one on the inside and one on the outside. By connecting the endpoints of a diameter to one another by a path on the surface of the sphere, we get a contractible subspace that identifies the ends of the diameter once this subspace is quotiented out.

e.g.



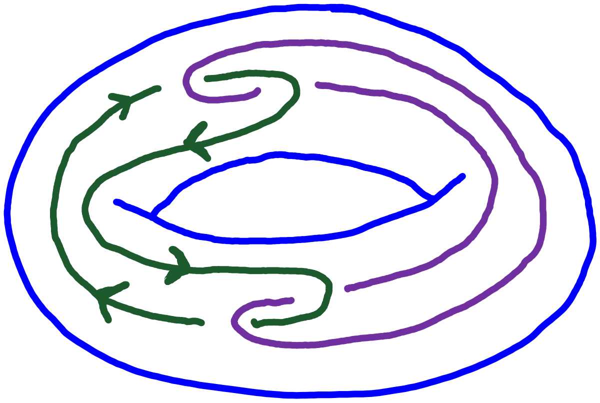
and



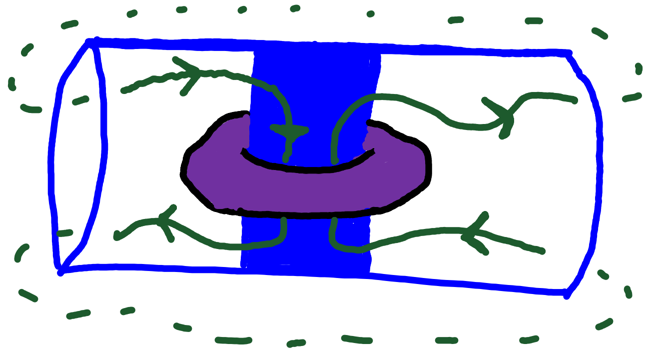
$= S^1 \vee S^1 \vee S^2$

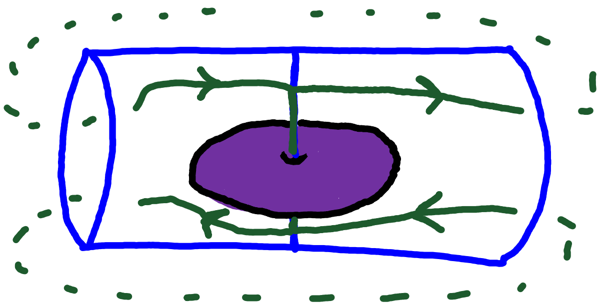
finally, we can trace back through these homotopies keeping track of the loop A. When A passes through the loop B inside T, this is like traversing the interval that came from shrinking the hole in the

Solid torus complement. So, after orienting A



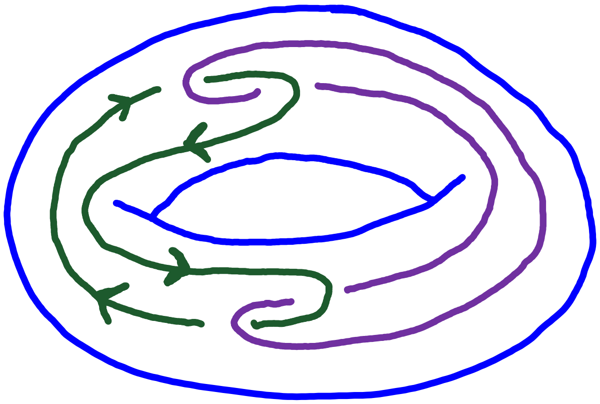
looks like



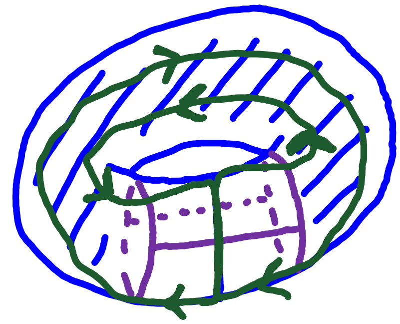


and its clear that this interval is traversed twice in two different directions

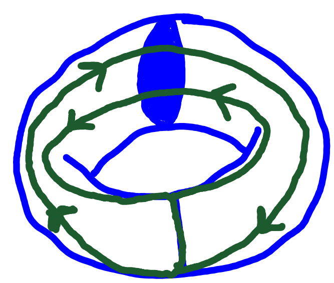
the other copy of S^1 in the result comes from the Solid part of the torus that was either squished down to a disk and then quotiented out or turned into the outter diameter of S^1 immediately. In the case of the former A gets pushed to the boundary of the torus as more of the interior of the Solid torus is removed, and its clear what happens to A in the other case. Since it is not as clear what happens with A in the first case, we finish the argument for that version of solving the problem.



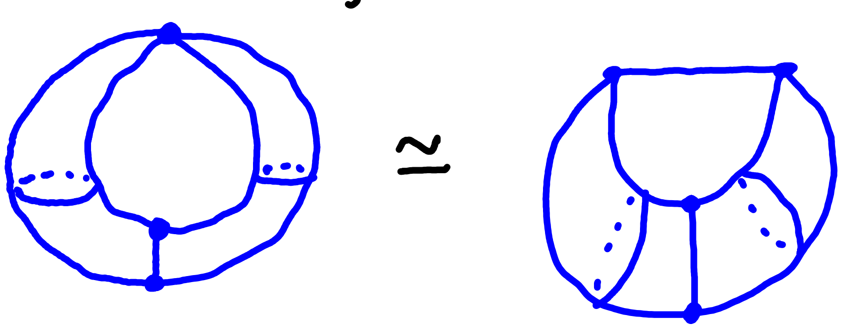
translates to



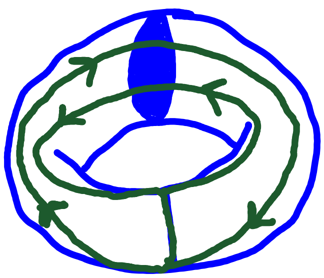
A can be in the interior of the part shaded blue, but must be on the boundary of the torus in the purple part unless it is passing through the interval.



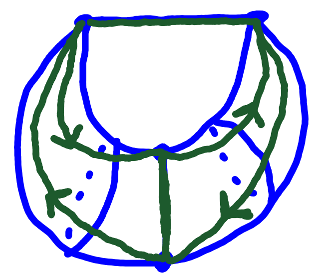
So once all that remains is $T^2 \cup D^2 \cup I$ the curve A must mainly be travelling on the boundary of the torus Then at the stage where the disk is collapsed



the point that the disk collapses to is what is stretched out to an interval that eventually becomes the second copy of S^1. Therefore, when A passes through this disk in either direction, this corresponds to a non-trivial loop in S^1.



becomes



All and all we see that $i : A \hookrightarrow T \setminus B$ induces $i_* : \mathbb{Z} \longrightarrow \mathbb{Z} * \mathbb{Z}$ where the image of the generator of the domain is $aba^{-1}b^{-1}$ where a & b are the generators of the two factors of the codomain. Since $\mathbb{Z} * \mathbb{Z}$ is not commutative we don't expect $aba^{-1}b^{-1}$ to be trivial, and indeed it isn't. We conclude that A is not null-homotopic in $T \setminus B$.