Our Integration Toolbox So Far

**Power Rule**
\[ \frac{d}{dx} x^n = nx^{n-1} \]

**Reverse Power Rule**
\[ \int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad \text{for } n \neq -1 \]

**Slogan:** Shoes & Socks

**Chain Rule**
\[ (f \circ g)'(x) = f'(g(x))g'(x) \]

**U-Substitution (Reverse Chain Rule)**
\[ \int f'(g(x))g'(x) \, dx = (f \circ g)(x) \]

**Slogan:** the derivative of \( u \) must appear multiplied by \( dx \)

**Product Rule**
\[ (fg)'(x) = f(x)g'(x) + f'(x)g(x) \]

**Integration By Parts**
\[ \int u \, dv = uv - \int v \, du \]

**Slogan:** - \( u \)'s derivative is simpler than \( u \)
- the integral of \( dv \) does not get more complicated

Some clever ways to use this method
- Invisible \( dv \)
- Use it multiple times in a row
- Use it multiple times to express your original integral in terms of itself e.g. \( \int e^x \cos(x) \, dx \)
Q: What about "Reverse Quotient Rule?"

A: Well, the quotient rule for derivatives is actually just the product rule in disguise.

$$\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

If we set $h(x) = \frac{1}{x}$ then $\frac{1}{g(x)} = h(g(x)) = (hog)(x)$

So \(\frac{d}{dx} \frac{1}{g(x)} = \frac{d}{dx} (hog)(x) = h'(g(x))g'(x)\)

\(h'(x) = \frac{d}{dx} h(x) = \frac{d}{dx} \frac{1}{x} = \frac{d}{dx} x^{-1} = -x^{-2} = -\frac{1}{x^2}\)
Putting everything together now

\[
\frac{d}{dx} \frac{f(x)}{g(x)} = f(x) \frac{d}{dx} \frac{1}{g(x)} + f'(x) \frac{1}{g(x)}
\]

Chain Rule

\[
= -f(x) \frac{g'(x)}{[g(x)]^2} + f'(x) \frac{1}{g(x)}
\]

Quotient Rule

\[
\frac{f(x)g(x) - f(x)g'(x)}{[g(x)]^2} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}
\]

So we do not need a separate rule for reversing derivatives that require the quotient rule. Integration By Parts + U-substitution should be enough

Today:  Trigonometric Integrals

\[
\int \sin^m(x) \cos^n(x) \, dx \quad \text{where } m \text{ & } n \text{ can be any exponents}
\]

Tomorrow:

\[
\int \tan^m(x) \sec^n(x) \, dx \quad \text{where again } m \text{ & } n \text{ can be any exponents}
\]
Products of Powers of Sine & Cosine

\[ \int \sin^m(x) \cos^n(x) \, dx \]

Some easy examples:

\[ \begin{align*}
    & \text{calc 1} \\
    & \begin{cases} 
        m = 1 \\
        n = 0
    \end{cases} \\
    & \int \sin(x) \, dx = -\cos(x) + C \\
    & \begin{cases} 
        m = 0 \\
        n = 1
    \end{cases} \\
    & \int \cos(x) \, dx = \sin(x) + C
\end{align*} \]

Either choice of u-sub:

\[ \begin{align*}
    & \begin{cases} 
        m = 1 \\
        n = 1
    \end{cases} \\
    & \int \sin(x) \cos(x) \, dx \\
    & \text{take } u = \sin(x) \quad \text{or} \quad u = \cos(x) \\
    & \begin{align*}
        & da = \cos(x) \, dx \\
        & du = -\sin(x) \, dx
    \end{align*} \\
    & \int u \, du = \frac{u^2}{2} + C \\
    & \int u \, du = -\frac{u^2}{2} + C
\end{align*} \]

\[ \begin{align*}
    & = \frac{\sin^2(x)}{2} + C \\
    & = -\frac{\cos^2(x)}{2} + C
\end{align*} \]

It might not be obvious that answers obtained from different solutions are in fact equal. The best way to check your work is by taking the derivative.
The First Place we Get Stuck Happens when

\[ m=2 \quad n=0 \quad \int \sin^2(x) \, dx \quad \text{OR} \quad m=0 \quad n=2 \quad \int \cos^2(x) \, dx \]

We need a new idea: In this case, the idea comes from storing endlessly at a list of trig identities.

Pythagorean Theorem
\[ \sin^2 \theta + \cos^2 \theta = 1 \]

Double Angle Formula (2x)
\[ \cos^2(\theta) = \frac{\cos(2\theta) + 1}{2} \]

From these we can deduce:

\[ 1 - \sin^2(\theta) = \cos^2(\theta) = \frac{\cos(2\theta) + 1}{2} \]

\[ \sin^2(\theta) = 1 - \frac{\cos(2\theta) + 1}{2} \]

\[ \sin^2(\theta) = \frac{1 - \cos(2\theta)}{2} \]
\[\sin^2(\theta) + \cos^2(\theta) = 1\]

\[
\frac{\sin^2(\theta)}{\cos^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}
\]

\[\frac{1}{\cos^2(\theta)} = \left(\frac{1}{\cos(\theta)}\right)^2\]

\[
\left(\frac{\sin(\theta)}{\cos(\theta)}\right)^2 + 1 = \left(\frac{1}{\cos(\theta)}\right)^2
\]

\[
\tan^2(\theta) + 1 = \sec^2(\theta)
\]

\[\tan^2(\theta) + 1 = \sec^2(\theta)\]

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**The Main 4 Trig Identities**

(for success in this course)

(\text{PYT})

\[\text{Pythagorean Theorem 1}\]

\[\text{divide by } \cos^2(\theta)\]

\[\sin^2(\theta) + \cos^2(\theta) = 1\]

(\text{PYT2})

\[\tan^2(\theta) + 1 = \sec^2(\theta)\]

\[\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}\]

\[\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}\]
So far, you only need to memorize one thing

$$\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}$$

but we will learn how to come up with this identity in a way that’s easy to remember (later) so it doesn’t matter if you actually memorize it or not (in terms of the final).

When faced with \( \int \sin^m x \cos^n x \) reach for your

**PYT**

- to exchange even powers of \( \sin \) for even powers of \( \cos \) or the other way around

\[
\sin^2(x) = 1 - \cos^2(x) \\
\cos^2(x) = 1 - \sin^2(x) \\
\sin^{2k}(x) = (1 - \cos^2(x))^k \\
\cos^{2k}(x) = (1 - \sin^2(x))^k
\]

**C2X**

- to exchange decrease exponent on cosine by 1 (provided it starts as \( \geq 1 \) or larger)

\[
\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2} \\
\cos^n(\theta) = \cos^{n-2}(\theta) \left[ \frac{1 + \cos(2\theta)}{2} \right]
\]

**S2X**

- replacing \( \cos \) with \( \sin \)

\[
\text{If replacing } \cos \text{ with } \sin
\]
\[ m=0 \quad n=2 \quad \int \cos^2 x \, dx = \int \frac{1 + \cos(2x)}{2} \, dx = \int \frac{1}{2} \, dx + \int \frac{\cos(2x)}{2} \, dx \]

\[ \overset{CRX}{=} \frac{1}{2} x + \overbrace{\frac{\sin(2x)}{4}} + C \]

**Double check**

\[ \frac{d}{dx} \left( \frac{1}{2} x + \frac{\sin(2x)}{4} + C \right) = \frac{1}{2} \frac{dx}{dx} + \frac{1}{4} \frac{d}{dx} \sin(x) + \frac{d}{dx} C \]

\[ = \frac{1}{2} + \frac{1}{4} \cdot 2 \cdot \cos(2x) + 0 = \frac{1}{2} + \frac{\cos(2x)}{2} \overset{CRX}{=} \cos^2 x \]

\[ \frac{d}{dx} (\frac{1}{2} x + \frac{\sin(2x)}{4} + C) = \frac{1}{2} + \frac{\cos(2x)}{2} \overset{CRX}{=} \cos^2 x \]

**Your Turn**

\[ m=2 \quad n=0 \quad \int \sin^2(x) \, dx = \frac{1}{2} x - \frac{\sin(2x)}{4} + C \]

---

**What do we know so far?**

\[ \int \sin^m x \cos^n x \, dx \]

- **Green** = These integrals are easy or memorized (Calc I)
- **Blue** = U-Sub with either \( u = \cos(x) \) or \( u = \sin(x) \)
- **Purple** = Use the appropriate double angle formula
- **Red** = Have not figured out how to integrate these yet

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(These dots in this picture (\& ones like it) represent integrals we either do or don’t know how to solve. The color indicates its status.)

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(\( (0,0) \)

\( m \) values

\( n \) values
We will now start populating this grid with strategies much faster.

\[ m \neq 0 \]
\[ n = 1 \]

\[ \int \sin^m(x) \cos(x) \, dx \]
\[ u = \sin(x) \]
\[ du = \cos(x) \, dx \]

looks like a good candidate for \( du \)

\[ = \int u^m \, du \]
\[ = \frac{u^{m+1}}{m+1} + C \]
\[ = \frac{\sin^{m+1}(x)}{m+2} + C \]

convert this to the case above

\[ m \neq 0 \]
& odd \times \cosines
\[ n = 2k+1 \]

\[ \int \sin^m(x) \cos^{2k+1}(x) \, dx \]

\[ = \int \sin^m(x) \cos^{2k}(x) \cos(x) \, dx \]

use \( \text{PYT} \) to convert into sines

\[ = \int \sin^m(x) (1-\sin^2(x))^k \cos(x) \, dx \]

\[ u = \sin(x) \]

\[ (= \int u^m (1-u^2)^k \, du) \]

Polyomial

Each integral above is now of the previous type (some \( \star \) of sines vs only a single cosine)
Here is an explicit example to help you get the feel for this last step

\[
\int \sin^4(x) \cos^5(x) \, dx
\]

\[= \int \sin^4(x) \cos^4(x) \cos(x) \, dx \quad m=4 \neq 0 \quad n=5 = 2 \cdot 2 + 1 \quad \text{(so \, } k=2)\]

\[= \int \sin^4(x) [\cos^2(x)]^2 \cos(x) \, dx\]

\[\overset{PyT}{=} \int \sin^4(x) [1 - \sin^2(x)]^2 \cos(x) \, dx\]

\[= \int \sin^4(x) (1 - \sin^2(x))(1 - \sin^2(x)) \cos(x) \, dx\]

\[= \int \sin^4(x) \left[1 - 2\sin^2(x) + \sin^4(x) \right] \cos(x) \, dx\]

\[= \int \sin^4(x) \cos(x) \, dx - 2 \int \sin^6(x) \cos(x) \, dx + \int \sin^8(x) \cos(x) \, dx\]

Set \, \text{ } u=\sin(x) \quad \text{for all 3 integrals}\]

\[du = \cos(x) \, dx\]

\[= \int u^4 \, du - 2 \int u^6 \, du + \int u^8 \, du = \frac{u^5}{5} - 2 \frac{u^7}{7} + \frac{u^9}{9} + C\]

\[= \frac{\sin^5(x)}{5} - 2 \frac{\sin^7(x)}{7} + \frac{\sin^9(x)}{9} + C\]
\[= \text{easy or memorized}\]
\[= \text{double angle formulas}\]
\[= \text{let } u = \sin(x)\]
\[= \text{exchange all even powers of } \cos \text{ for even powers of } \sin \text{ then set } u = \sin(x)\]
\[= \text{you should be able to figure out how to integrate these now that we have seen similar examples}\]
\[= \text{the strategy for } \square \& \square \text{ will both work}\]

\[\int \sin^m(x) \cos^n(x) \, dx\]
\[\int \sin^m(x) \, dx, \quad m \geq 3\]
\[\int \cos^n(x) \, dx, \quad n \geq 3\]
$$\int \cos^n(x) \, dx$$  Assuming \( n \geq 3 \) we can factor out a copy of cosine & still have at least 2 leftover

\[
\int \cos^{n-1}(x) \cos(x) \, dx \quad n \geq 3 \Rightarrow n-1 \geq 2
\]

Factoring out as many copies of \( \cos^2(x) \) from \( \cos^{n-1}(x) \) as possible (\( n-1 \geq 2 \) means there is always at least 1) we are either left with

\[
\int \cos^{n-1}(x) \cos(x) \, dx
\]

even

\[n-1=2k\]

OR

\[
\int [\cos^2(x)]^k \cos(x) \, dx
\]

\[
\int \cos^2(x) \, dx
\]

\[
\int \left[ \frac{1+\cos(2x)}{2} \right]^k \, dx
\]

PYT

we do know how to solve these

\[\begin{align*}
   n=0 & \quad m=1 \\
   n \neq 0 & \quad m=1
\end{align*}\]

This is a special formula territory

\[
\frac{1}{2}
\]

the problem has gotten easier

terms have lots of different values of \( n \)

\[
\int \left[ 1-\sin^2(x) \right]^k \cos(x) \, dx
\]

\[
\int \frac{1+\cos(2x)}{2} \, dx
\]

\[
\int \cos^2(x) \, dx
\]
Now you should be able to reason similarly for powers of \( \sin x \geq 3 \)

Finally

\[
\int \sin^{2M} (x) \cos^{2N} (x) \, dx
\]

When both are even powers

Use double angle laws to convert all sines to cosines with doubled input & all cosines to other cosines w/ doubled input

\[
\begin{align*}
\text{e.g.} & \\
\int \sin^{2M} x \cos^{2N} x \, dx &= \int \left( \frac{1-\cos(2x)}{2} \right)^M \cos^{2N} x \, dx \\
&= \int \left( \frac{1-\cos(2x)}{2} \right)^M \left( \frac{1+\cos(2x)}{2} \right)^N \, dx
\end{align*}
\]

Simplifies to a polynomial in \( \cos(2x) \) so this is now reduced to the case \( m = \text{anything} \), \( n = 0 \)

Sine double angle formula

Cosine double angle formula
Cosine Reduction Formula

\[ \int \cos^n(x) \, dx = \frac{1}{n} \cos^{n-1}(x) \sin(x) + \frac{n-1}{n} \int \cos^{n-2}(x) \, dx \]

\[ \int \cos^4(x) \, dx = \int \cos^3(x) \cos(x) \, dx = uv - \int v \, du \]
\[ du = -3\cos^2(x) \sin(x) \, dx \]
\[ v = \sin(x) \]
\[ -\int v \, du = 3 \int \cos^2(x) \sin^2(x) \, dx = 3 \int \cos^2(x) \sin^2(x) \, dx \]
\[ \int \cos^4(x) \, dx = \sin(x) \cos^3(x) + 3 \int \cos^2(x) \, dx = 3 \int \cos^2(x) \, dx \]
\[ \Rightarrow 4 \int \cos^4(x) \, dx = \sin(x) \cos^3(x) + 3 \int \cos^2(x) \, dx \]

\[ \int \cos^3(x) \, dx = \int \cos^2(x) \cos(x) \, dx = uv - \int v \, du \]
\[ du = -2 \sin(x) \cos(x) \, dx \]
\[ v = \sin(x) \]
\[ n \neq 0, m = 1 \]

\[ \int \cos^2(x) \, dx = \int \left[ 1 + \cos(2x) \right] \, dx = \frac{1}{2} x + \frac{\sin(2x)}{4} + C \]

\[ \int \cos^1(x) \, dx = \int \cos(x) \, dx = -\sin(x) + C \]

\[ \int \cos^0(x) \, dx = \int dx = x + C \]
So, a reduction formula works by taking one instance of a problem

calculate \( \int \cos^n(x) \, dx \)

assuming \( n = 5 \)

& "solves" it by reducing the problem to an easier version of itself (presumably a version of it you already know how to solve)

\[
\int \cos^5(x) \, dx = \frac{1}{5} \cos^4(x) \sin(x) + \frac{4}{5} \int \cos^3(x) \, dx
\]

Oh, this one is easy...

\[
\int \cos^3(x) \, dx = \frac{1}{3} \cos^2(x) \sin(x) + \frac{2}{3} \int \cos^1(x) \, dx
\]

Oh. This one? Its easy...
Derivation of Cosine Reduction Formula

\[ \int \cos^n(x) \, dx = \int \cos^{n-1}(x) \cos(x) \, dx \]

\[ du = (1-n) \cos^{n-2}(x) \sin(x) \, dx \]

\[ v = \sin(x) \]

\[ \int \cos^n(x) \, dx = \sin(x) \cos^{n-1}(x) + (n-1) \int \cos^{n-2}(x) \sin^2(x) \, dx \]

**Integration by Parts Formula**

\[ = \sin(x) \cos^{n-1}(x) + (n-1) \int \cos^{n-2}(x) [1 - \cos^2(x)] \, dx \]

\[ \therefore \text{ this means } \]

\[ \int \cos^n(x) \, dx = \sin(x) \cos^{n-1}(x) + (n-1) \int \cos^{n-2}(x) \, dx - (n-1) \int \cos^n(x) \, dx \]

\[ \text{divide both sides by } n \text{ & then you're done.} \]
More Reduction Formulas You Might Try To Prove Yourself

\[
\int \sin^m(x) \, dx = -\frac{1}{m} \sin^{m-1}(x) \cos(x) + \frac{m-1}{m} \int \sin^{m-2}(x) \, dx
\]

\[
\int \sec^n(x) \, dx = \frac{1}{n-1} \sec^{n-1}(x) \sin(x) + \frac{n-2}{n-1} \int \sec^{n-2}(x) \, dx
\]

\[
\int \tan^m(x) \, dx = \frac{\tan^{m-1}(x)}{m-1} - \int \tan^{m-2}(x) \, dx
\]

\[
\int [\ln(x)]^N \, dx = x \left(\ln(x)\right)^N - N \int [\ln(x)]^{N-1} \, dx
\]

The pattern to keep in mind is to use integration by parts & look out for convenient times to apply trig identities (at least for the first 3 above)
Summary

When faced with an integral line $\int \sin^m(\theta) \cos^n(\theta) d\theta$

(i.e. $\sin$ to some power

times $\cos$ to a potentially different power)

The problem is begging for one of the two following substitutions

$u = \cos(\theta)$ OR $u = \sin(\theta)$

But the fact that these choices result in

$du = -\sin(\theta) d\theta$ \& $du = \cos(\theta) d\theta$ (respectively)

means such a substitution will only work if there is

an odd \& of sines

(\(m\) is an odd number) OR an odd \& of cosines

(\(n\) is an odd number) (resp.)

When this does happen to be the case, we proceed as follows

convert the leftover \(n-1\)

cosines into even \&
sines with PVT

\[\sin^2(\theta) = (1-\cos^2(\theta))\]

convert the leftover \(m-1\)

sines into even \&
cosines with PVT

\[\cos(\theta) = (1-\sin^2(\theta))\]
This leaves you with an integral of the form

\[ \int (1 - \cos^2(\theta)) \cos^n(\theta) \sin^1(\theta) \, d\theta \]

\[ \div \] \[ \int \sin^m(\theta)(1 - \sin^2(\theta))^{\frac{n-1}{2}} \cos^{\frac{1}{2}}(\theta) \, d\theta \]

\[ \text{Remember: Here } u = \cos(\theta) \]
\[ \text{so } du = -\sin(\theta) \, d\theta \]

\[ \iff \int (1 - u^2)^{\frac{m-1}{2}} \cdot u \, du \]

\[ \iff \int u^m(1 - u^2)^{\frac{n-1}{2}} \, du \]

\[ \text{Here } u = \sin(\theta) \]
\[ \text{so } du = \cos(\theta) \, d\theta \]

In either case we could expand out the integrand & realize that all that's left to do now is integrate a Polynomial in the variable \( u \).

**Linearity of \( \int \) & Reverse Power Rule \( \Rightarrow \) can integrate all polynomials.

\[ \int (\text{polynomial in } u) \, du = \int [a_n u^n + a_{n-1} u^{n-1} + \ldots + a_2 u^2 + a_1 u + a_0] \, du \]

Fixed but unknown ree & constants
\[ a_i = \text{"coefficient of } u^i \text{"} \]

Linearity of \( \int \)

\[ = a_n \int u^n \, du + a_{n-1} \int u^{n-1} \, du + \ldots + a_2 \int u^2 \, du + a_1 \int u \, du + a_0 \int du \]

\[ \text{monomial of degree } n \]
\[ \text{monomial of degree } n-1 \]
\[ \text{monomial of degree 2} \]
\[ \text{monomial of degree 0} \]

\( U^0 = 1 \) is a monomial of degree zero.
Reverse Power Rule

\[ \int u^n \, du = \frac{u^{n+1}}{n+1} + C \]

Hence, \( \int \left( \text{polynomial in } u \text{ of degree } n \right) \, du \)

- Sum of Constants times monomials of various degrees
- Highest power of \( u \) that appears in the polynomial with non-zero coefficient

\[ = \frac{a_n x^{n+1}}{n+1} + \frac{a_{n-1} x^n}{n} + \cdots + \frac{a_3 x^3}{3} + \frac{a_2 x^2}{2} + a_1 x + C \]

- First integral
- Second integral
- \( (n-1) \text{st integral} \)
- \( n \text{th integral} \)
- \( (n+1) \text{st integral} \)

\[ \int \text{(polynomial)} = \text{polynomial of 1 degree higher} \]

- Linearity of the integral
  \[ \int [af + bg] \, dx = a \int f \, dx + b \int g \, dx \]

Together with Reverse Power Rule

\[ \Rightarrow \text{we can integrate ALL POLYNOMIALS} \]
The following strategy works when $n$ is odd or $m$ is odd.

\[
\int \sin^m(\theta) \cos^n(\theta) \, d\theta \quad \text{OR} \quad \int (\text{polynomial in } \sin u) \, \text{polynomial in } u
\]

$u = \sin(\theta)$ or $u = \cos(\theta)$

+ Pythagorean theorem to exchange even powers of $\sin/\cos$ for "even powers of the other."

When neither $m$ nor $n$ is odd

The double angle formulas decrease the values of $m$ or $n$

\[
\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}
\]

\[
\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}
\]

Squared! No squared trig functions

2 \rightarrow 1

So, you can use this idea repeatedly to get a bunch of integrals only involving 1 $\sin \theta$ or 1 $\cos \theta$ (odd # of sines/cosines)

\[m = 0 \implies \text{the above line is a recipe for the cosine reduction formula} \]

\[n = 0 \implies \text{the above line is a recipe for the sine reduction formula} \]

"Reduction formulas" always come from repeated integration by parts.