long division

\[
\frac{108}{3} = \frac{10 \times 10 + 8 \times 1}{3} = \frac{10 \text{ tens} + 8 \text{ ones}}{3}
\]

= \frac{10 \text{ tens}}{3} + \frac{8 \text{ ones}}{3} = 3 \text{ tens} + \frac{1 \text{ ten}}{3} + \frac{8 \text{ ones}}{3}

= 3 \text{ tens} + \frac{18 \text{ ones}}{3} = 3 \times 10 + 6 \times 1 = \boxed{36} \leftarrow \text{quotient}

\[
3 \longdiv{108}
\]

Standard long division algorithm does the same thing with less writing.

108 is evenly divisible by 3 since there is no remainder.

\[\Rightarrow \text{no remainder}\]

All this shows is \((36) \cdot 3 = 108\)

When there is no remainder after division, the larger number can be written as a product of the quotient & divisor.
\[
\frac{123}{4} = \frac{12 \times 10 + 3 \times 1}{4} = \frac{12 \text{ tens} + 3 \text{ ones}}{4} = 3 \text{ tens} + \frac{3 \text{ ones}}{4} = 30 \text{ R } 3
\]

\[
3 \quad \text{leaves a remainder of } 3
\]

\[
\text{Quotient } = 30
\]

\[
30 \cdot 4 + 3 = 123
\]

Note: The remainder must always be smaller than the divisor. In this case, 
\(3 < 4\)

When there is a remainder after division, the larger number can be expressed as the quotient times the divisor plus the remainder.

Polynomial Long Division Works Exactly The Same Way

\[
\frac{a(x)}{b(x)} = q(x) \divides r(x)
\]

(We call these "Rational Functions")

\[
a(x) = q(x) \cdot b(x) + r(x)
\]
\[
\begin{array}{c}
x - 4) 7x^3 - 2x^2 - 9x - 32 \quad \text{\textarrow{quotient}} \\
\underline{-7x^4 + 4x^3} \\
7x^4 + 2x^3 - x^2 + 8 \\
\underline{-2x^3 - x^2 + 8} \\
\; -2x^3 + 8x^2 \\
\underline{\quad -9x^2 + 8} \\
\; -9x^2 + 32x \\
\underline{\quad -32x + 8} \\
\; -32x + 128 \\
\underline{\quad -128} \\
\text{\textarrow{remainder}}
\end{array}
\]

\text{Therefore}

\[(x - 4)(7x^3 - 2x^2 - 9x - 32) - 120 = 7x^4 + 2x^3 - x^2 + 8\]
Polynomial of degree $N$

Polynomial of degree $M$

$N \geq M \Rightarrow$ Non-zero quotient

maybe remainder

$N \lt M \Rightarrow$ No quotient

numerator is remainder

$\frac{6x - 4}{x^2 + 1}(6x^3 - 4x^2 + x + 5)$

$\frac{6x^3 + 6x}{-4x^2 - 5x + 5}$

$\frac{-4x^2 - 4}{-5x + 9}$

$\text{Remainder Has Degree 1}$

$(x^2 + 1)(6x - 4) - 5x + 9 = 6x^3 - 4x^2 + x + 5$

\[ \sum \text{multiplicities of zeros} = \text{degree of polynomial} \]
<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Degree</th>
<th>Roots/Zeros</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x - 8$</td>
<td>1</td>
<td>$x = 8$</td>
</tr>
<tr>
<td>$3x$</td>
<td>1</td>
<td>$x = 0$</td>
</tr>
<tr>
<td>$7x - 2$</td>
<td>1</td>
<td>$x = \frac{2}{7}$</td>
</tr>
<tr>
<td>$ax + b$</td>
<td>1</td>
<td>$x = -\frac{b}{a}$</td>
</tr>
<tr>
<td>$(x+1)(x-2)$</td>
<td>2</td>
<td>$x = -1, 2$</td>
</tr>
<tr>
<td>$(x-3)^2$</td>
<td>2</td>
<td>$x = 3$ Multiplicity 2</td>
</tr>
<tr>
<td>$(x-3)(x+4)$</td>
<td>4</td>
<td>$x = 3, -4$ Multiplicity 3</td>
</tr>
<tr>
<td>$x^2 + 1$</td>
<td>2</td>
<td>$x = \pm i$ Complex *</td>
</tr>
<tr>
<td>Cannot be factored further over the real numbers.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(x^2 + x + 1)^2(x+2)^2$</td>
<td>6</td>
<td>$x = -2, \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$ all have multiplicity 2</td>
</tr>
<tr>
<td>Irreducible quadratic (complex zeros)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$ax^2 + bx + c$</td>
<td>2</td>
<td>??</td>
</tr>
</tbody>
</table>
\( Q(x) = ax^2 + bx + c = \) the most general polynomial of degree 2.

\( = \) the most general quadratic equation

\[ \frac{Q(x)}{a} = x^2 + \frac{b}{a}x + \frac{c}{a} \]

\( = (x + z_1)(x + z_2) \)

does this polynomial factor as a product of two degree 1 polynomials as so?

\[ \begin{align*}
Q(x) &= x^2 + (z_1 + z_2)x + z_1z_2 \\
\text{If so,} \quad \frac{b}{a} &= z_1 + z_2 \\
&\quad \frac{c}{a} = z_1z_2
\end{align*} \]

This is how we usually think of factoring quadratics

Another way is to realize that

\[ \frac{Q(x)}{a} = (x + z_1)(x + z_2) \]

can be thought of as an equality of functions of \( x \)

& therefore must agree for all inputs
\[
\frac{Q(-z_1)}{a} = (-z_1 + z_1)(x + z_2) = 0
\]

therefore \[Q(-z_1) = 0 \quad \text{& Similarly} \quad Q(-z_2) = 0\]

So by $F \setminus \text{To}\, A$, $z_1$ & $z_2$ are the only 2 zeros of this degree 2 polynomial.

Rewriting \[\frac{Q(x)}{a} = (x - z_1)(x - z_2)\] is equivalent to solving \[\frac{Q(x)}{a} = 0\] which we know can be done by completing the square.

\[
0 = \frac{Q(x)}{a} = x^2 + \frac{b}{a}x + \frac{c}{a}
\]

Represent as an area

\[
\begin{align*}
x^2 & \quad \frac{b}{a}x \\
\hline
\frac{b}{a}x & \quad \frac{c}{a}
\end{align*}
\]

\[
x^2 + \frac{b}{a}x = x \cdot x + x \cdot \frac{b}{a}
\]
\[
\frac{Q(x)}{a} = x^2 + \frac{b}{a}x + \frac{c}{a} = (x + \frac{b}{2a})^2 - (\frac{b}{2a})^2 + \frac{c}{a}
\]

but we assumed \( \frac{Q(x)}{a} = 0 \) so we could
Solve for \( x \). The advantage of completing the square was that our new representation of \( \frac{Q(x)}{a} \) has only a single occurrence of \( x \) thus the “solving” step is easy

\[
(x + \frac{b}{2a})^2 - (\frac{b}{2a})^2 + \frac{c}{a} = 0
\]

\[\iff (x + \frac{b}{2a})^2 = (\frac{b}{2a})^2 - \frac{c}{a}\]

\[\iff x + \frac{b}{2a} = \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}\]
\[ x = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a} \]

\[ x = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a} \]

If \( ac < 0 \)

\[ x = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a} \]

If \( a > 0 \)

\[ x = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a} \]

**Definition**

\[ Q(x) = ax^2 + bx + c = a(x + \frac{b}{2a})^2 - \frac{b^2}{4a} + c \]

\[ = a(x + \frac{1}{2}(b + \sqrt{b^2-4ac})))(x + \frac{1}{2}(b - \sqrt{b^2-4ac})) \]

**FTBA**

\( Q(z_1) = 0 \)

\( Q(z_2) = 0 \)

\[ z_1, z_2 = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \]

**The quadratic formula**
Looking at this formula, we see that 

The discriminant: \( b^2 - 4ac \) is an important value.

If \( b^2 - 4ac < 0 \) (is negative), then \( \sqrt{b^2 - 4ac} \) (its square root) is an imaginary number.

\[
\sqrt{-1 \cdot (4ac - b^2)} = \sqrt{-1} \cdot \sqrt{4ac - b^2} = i \sqrt{4ac - b^2}
\]

This is now positive, so some real \( \alpha > 0 \).

Let \( z_1 = \frac{b}{2a} + \sqrt{4ac - b^2}i \) \( z_2 = \frac{b}{2a} - \sqrt{4ac - b^2}i \) \( \) A pair of complex numbers.

(Definition: if \( z \) is the complex number \( x + iy \), then its conjugate \( \overline{z} \) is \( x - iy \)).

If \( b^2 - 4ac = 0 \) then \( (by \ FTC or A) \) \( z_1 = \frac{b}{2a} + 0 = \frac{b}{2a} = \frac{b}{2a} - 0 = z_2 \).
& we get a repeated real root \[ \frac{b}{2a} \]
(i.e. a real root of multiplicity 2)

If \[ b^2 - 4ac > 0 \]
then
\[ \sqrt{b^2 - 4ac} > 0 \]
(cits square root)
is a positive real number

\[ z_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \quad z_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \]

two different real roots

These different possibilities correspond
to the following pictures

Conjugate pair of complex roots

\[ b^2 - 4ac < 0 \]

A single (repeated) real root

\[ b^2 - 4ac = 0 \]
two distinct real roots
\[ b^2 - 4ac > 0 \]

Summary

\[ p(z) = 0 \iff p(x) = (x-z)q(x) \]

Polynomial Root A.K.A. Zero Degree N Can factor this out Degree N-1

Factoring Quadratic Polynomials
(complete the square \( \Rightarrow \) quadratic formula)

Discriminant

\[ a(x-(x+8i))(x-(x-8i)) \]
\[ a(x-(r))^2 \]
\[ (x-r_1)(x-r_2) \]
For today's Lesson

We will only be using real numbers in our factorization meaning discriminant \((b^2-4ac)\)

\[ x^2 + 1 \quad -4 < 0 \quad (x+i)(x-i) \quad \text{complex} \times s \]

\[ x^2 + x + 1 \quad -3 < 0 \quad (x+(\frac{1}{2}+\frac{\sqrt{3} i}{2}))(x+(\frac{1}{2}-\frac{\sqrt{3} i}{2})) \]

\[ x^2 + 2x + 1 \quad = 0 \quad (x+1)^2 \quad \text{repeated real root} \]

\[ x^2 - 1 \quad 4 > 0 \quad (x+1)(x-1) \quad \text{two distinct real roots} \]

\[ x - 7x + 12 \quad 1 > 0 \quad (x-4)(x-3) \quad \text{two distinct real roots} \]
Now that we have reviewed the algebra topics relevant to today's topic. Let's invent a new technique of integration.

Integral we know + Derivative we know + Chain Rule

⇒ Learn New U-sub Integrals

\[ \int \frac{du}{u} = \ln|u| + C \quad \text{and also that} \quad \frac{d}{dx} x^n = n \cdot x^{n-1} \]

The power rule allows us to find the derivative of any polynomial.

Therefore, the result of differentiating

\[ \ln|\text{polynomial}| \quad \text{should be integrable by u-substitution} \]

\[ \ln|u \cdot v| = \ln|u| + \ln|v| \]

Set \( u = 3x+2 \) and \( v = x-1 \)

\[ 3 \int \frac{dx}{3x+2} + \int \frac{dx}{x-1} \]

Express the rational function as a sum of partial fractions:

\[ \int \frac{6x-1}{3x^2-x-2} \, dx \]

\[ \int \frac{6x-1}{3x^2-x-2} \, dx = \int \frac{6x}{3x^2-x-2} \, dx + \int \frac{-1}{3x^2-x-2} \, dx \]

Find common denominators:

\[ \int \frac{6x-1}{3x^2-x-2} \, dx \]

\[ \frac{6x-1}{3x^2-x-2} \]
Partial fractions

Rational Function \[ \frac{a(x)}{b(x)} = \frac{\text{Polynomial}}{\text{Polynomial}} \]

\[ a(x) = q(x) \cdot b(x) + r(x) \]

\[ \Rightarrow \quad \frac{a(x)}{b(x)} = \frac{q(x) \cdot b(x) + r(x)}{b(x)} = q(x) + \frac{r(x)}{b(x)} \]

 Completely factor \( b(x) \)

\[ b(x) = (x-z_1)(x-z_2)^3(x-z_3)(x^2+1)^2 \]

\[ b(z_1) = 0 \quad b(z_2) = 0 \quad b(z_3) = 0 \]

cannot be factored any further over the reals \( \mathbb{R} \)

multiplicity of this zero is 2

Then,

\[ \frac{f(x)}{b(x)} = \frac{A}{x-z_1} + \frac{B}{x-z_2} + \frac{C}{(x-z_2)^2} + \frac{D}{(x-z_2)^3} + \frac{E}{x-z_3} + \frac{F(x+G)}{x^2+1} + \frac{H(x+I)}{(x^2+1)^2} \]

we can always find constants \( A, B, C, D, E, F, G, H, I \) to make this equality true. Each of the 7 fractions on the right hand side are called “partial fractions” of \( \frac{f(x)}{b(x)} \).
Reminder: This "partial fraction decomposition" comes from reversing the process of finding common denominators.

Finding The Partial Fraction Decomposition

1. polynomial long division (just focus on the remainder)
2. Completely factor the denominator (by finding all zeros & multiplicities & leaving irreducible quadratics unfactored)
3. Set up 1 partial fraction for every zero in the denominator. The numerators of these partial fractions need to be solved for while the denominators just depend on the factorization of the original denominator \( b(x) \).

Exactly how to carry out step 3 is what we hope to explain next.

**two distinct real roots case:**

division by \( x^2 - 7x + 12 \) will result in some remainder of degree less than or equal to 1

\[ r(x) = \alpha x + \beta \]
division by \(x-3\) will always result in a remainder polynomial of degree 0 (i.e. a number) & the same is true of \(x-4\)

We therefore expect

\[
\frac{\alpha x + \beta}{x^2 - 2x + 12} = \frac{A}{x-3} + \frac{B}{x-4}
\]

\[
\alpha x + \beta = \frac{\alpha x + \beta}{(x-3)(x+4)} = \frac{A}{x-3} + \frac{B}{x-4}
\]

\[
\text{two polynomials are equal if \& only if their coefficients agree}
\]

\[
\alpha x + \beta = A(x-4) + B(x-3) = (A+B)x - (4A+3B)
\]

So given a value for \(\alpha\) & \(\beta\) (known) we could solve the system of equations below for \(A\) & \(B\) (unknown)

\[
\begin{cases}
\alpha = A + B \\
\beta = -4A - 3B
\end{cases}
\]
Repeated Root Case: \[ \frac{\alpha x + \beta}{(x-r)^2} = \sum \text{Partial Fractions} \]

**Most general possible remainder**

\[ \frac{x + \beta}{\alpha x^2 + bx + c} = \frac{\alpha x + \beta}{(x-r)^2} \]

**Most general possible quadratic polynomial**

Factoring assumption:

\[ b^2 - 4ac = 0 \]

Multiply by \((x-r)^2\)

Solve for numerators

\[ \frac{\alpha x + \beta}{(x-r)^2} = \frac{A}{x-r} + \frac{B}{x-r} \]

Multiply by \((x-r)^2\)

\[ \alpha x + \beta = (x-r)A + (x-r)B \]

\[ = (A+B)(x-r) \]

\[ = (A+B)x - (A+B)r \]

We need to solve the system of equations

\[ \begin{align*}
\sum \alpha &= A + B \\
\sum \beta &= -(A+B)r
\end{align*} \]

But this is only possible when

\[ \alpha = -\frac{1}{r} \beta \]
For example,

Set $\alpha = 1, \beta = 2, r = 3$ & let's try to
Solve for $A$ & $B$ as in the above situation

$$\frac{x + 2}{(x - 3)^2} = \frac{A}{x - 3} + \frac{B}{(x - 3)} = \frac{A + B}{x - 3}$$

$\iff$ multiply by $(x - 3)^2$

$$x + 2 = (A + B)(x - 3) (A + B)$$

$\iff$ equate the constant & coefficient of $x$

$$\begin{cases} 1 = A + B \\ 2 = -3A - 3B \end{cases}$$

but these equations contradict each other

However, nothing but $x - r$ can be introduced in the denominators because we could not end up with a common denominator of $(x - r)^2$

Other options? $\frac{Ax + B}{(x - r)^2}$

Only one where we NEED to find a common denominator
\[
\frac{\alpha x + \beta}{(x-r)^2} = \frac{A}{(x-r)} + \frac{B}{(x-r)^2}
\]

\[\leftrightarrow \text{ multiply by } (x-r)^2\]

\[
\alpha x + \beta = A(x-r) + B
\]

\[= A x - A r + B\]

\[\leftrightarrow \text{ equate coefficient of } x \text{ & constant}\]

\[\alpha = A \quad \text{?? knowing } \alpha \text{ immediately tells you } A\]

\[\beta = B - A r \quad \text{?? knowing } A \text{ & } \beta \text{ lets you solve for } B\]


\[
\frac{p(x)}{(x-r)^n} = \frac{A_1}{(x-r)^1} + \frac{A_2}{(x-r)^2} + \ldots + \frac{A_n}{(x-r)^n}
\]

\[p(x) \text{ polynomial of degree } n-1\]

\underline{Irreducible Quadratic Case}

\[
\frac{\alpha x + \beta}{x^2+2x+1} \quad \text{?? this is a partial fraction by definition}\]
Summary: Partial Fractions

Reverse Common Denominators

\[ R(x) = \frac{P(x)}{Q(x)} = \frac{\text{numerator}}{\text{denominator}} \]

\[ P(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_M x^M \]

\[ Q(x) = b_0 + b_1 x + b_2 x^2 + \ldots + b_N x^N \]

\[ \text{degree of } P(x) = M \]

\[ \text{degree of } Q(x) = N \]

\[ R(x) = \frac{P(x)}{Q(x)} = \frac{Q(x) + \frac{r(x)}{Q(x)}}{Q(x)} = \frac{Q(x)}{Q(x)} + \frac{r(x)}{Q(x)} = Q(x) + \frac{r(x)}{Q(x)} = Q(x) + \frac{r(x)}{Q(x)} \]

\[ \text{degree of } r(x) < \text{degree of } Q(x) \]

\[ r(x) = \sum_{i=1}^{\text{multiplicities}} m_i (x - z_i)^{m_i} \]

\[ \sum_{i=1}^{\text{multiplicities}} m_i = \text{degree of } r(x) < \text{degree of } Q(x) = M = \sum_{i=1}^{\text{multiplicities}} n_i \]

\[ \text{Here } P(z_i) = 0 \quad \forall \quad i = 1, 2, \ldots, u \]

\[ \& \quad Q(z_i) = 0 \quad \forall \quad i = 1, 2, \ldots, v \]

\[ \text{FToC} \]

\[ \text{Sum of multiplicities} \]

\[ \text{FToC} \& \text{Notation} \]
$$R(x) = \frac{P(x)}{Q(x)} = Q(x) + \frac{n(x)}{Q(x)}, \quad \text{deg}(P(x))=M, \quad \text{deg}(Q(x))=N$$

\[\begin{align*}
\frac{b_0 + p_1 x + p_2 x^2 + \ldots + p_m x^m}{b_0 + q_1 x + q_2 x^2 + \ldots + q_n x^n} &= \frac{(x-z_1)^{m_1}(x-z_2)^{m_2}\ldots(x-z_u)^{m_u}}{(x-\tilde{z}_1)^{n_1}(x-\tilde{z}_2)^{n_2}\ldots(x-\tilde{z}_v)^{n_v}} \\
&= Q(x) + \frac{(x-z_1)^{m_1}(x-z_2)^{m_2}\ldots(x-z_u)^{m_u}}{(x-\tilde{z}_1)^{n_1}(x-\tilde{z}_2)^{n_2}\ldots(x-\tilde{z}_v)^{n_v}} \\
&= Q(x) + \frac{(x-z_1)^{m_1}(x-z_2)^{m_2}\ldots(x-z_u)^{m_u}}{(x-\tilde{z}_1)^{n_1}(x-\tilde{z}_2)^{n_2}\ldots(x-\tilde{z}_v)^{n_v}}
\end{align*}\]

- Divide & factor
- Same numerator
- Real roots to their multiplicities
- Irreducible quadratic factors to powers
- Partial fraction decomposition

\[\begin{align*}
= Q(x) + \left(\frac{C_1}{x-z_1} + \ldots + \frac{C_{n_1}}{(x-z_1)^{n_1}}\right) + \ldots + \left(\frac{C_{k+1}}{x-z_{k+1}} + \ldots + \frac{C_{n_{k+1}}}{(x-z_{k+1})^{n_{k+1}}}\right) + \ldots + \left(\frac{A_{n_k+\ldots+n_{k+1}}}{x^{n_k+\ldots+n_{k+1}}} + \ldots + \frac{A_{n_k+\ldots+n_{k+1}}}{x^{n_k+\ldots+n_{k+1}}}\right)
&+ \ldots + \left(\frac{A_{n_k+\ldots+n_{k+1}}}{x^{n_k+\ldots+n_{k+1}}} + \ldots + \frac{A_{n_k+\ldots+n_{k+1}}}{x^{n_k+\ldots+n_{k+1}}}\right)
&+ \ldots + \left(\frac{A_{n_k+\ldots+n_{k+1}}}{x^{n_k+\ldots+n_{k+1}}} + \ldots + \frac{A_{n_k+\ldots+n_{k+1}}}{x^{n_k+\ldots+n_{k+1}}}\right)
= Q(x) + \sum \text{Partial Fractions}
\end{align*}\]
Integrating Rational Functions

"The Method of Partial Fractions"

\[
\int R(x) \, dx = \int \frac{P(x)}{Q(x)} \, dx = \int \left[ \frac{P(x)}{Q(x)} + \frac{r(x)}{Q(x)} \right] \, dx
\]

\[
= \int \frac{Q(x)}{Q(x)} \, dx + \int \frac{r(x)}{Q(x)} \, dx
\]

\[
= \int \frac{Q(x)}{Q(x)} \, dx + \int \left[ \frac{A_1x + B_1}{(a_1x^2 + b_1x + c_1)^{m_1}} + \frac{A_2x + B_2}{(a_2x^2 + b_2x + c_2)^{m_2}} + \frac{C_1}{(x - r_1)^{n_1}} + \frac{C_2}{(x - r_2)^{n_2}} \right] \, dx
\]

Linearity reduces this integral into

Problems of the type

\[
\int \frac{Ax + B}{ax^2 + bx + c} \, dx \quad \& \quad \int \frac{C}{x - r} \, dx
\]

\[
= \frac{A}{a} \int \frac{x + \frac{B}{a}}{x^2 + \frac{b}{a}x + c/a} \, dx
\]

\[
= \frac{A}{a} \int \frac{x + \frac{B}{a}}{(x + \frac{b}{2a})^2 + \frac{b^2}{4a^2} + \frac{c}{a}} \, dx
\]

Call this constant \( t \)

\[
= \frac{A}{a} \int \frac{(x + \frac{b}{2a}) + \left( \frac{A}{a} - \frac{b}{2a} \right)}{(x + \frac{b}{2a})^2 + t} \, dx
\]

\[
= \frac{A}{a} \int \left( x + \frac{b}{2a} \right) + (\frac{A}{a} - \frac{b}{2a}) \, dx
\]

\[
\text{Call this constant} \quad s
\]

\[
U = x + \frac{b}{2a} \quad du = dx
\]

\[
\int \frac{C}{u} \, du = C \ln |u|
\]

\[
= C \ln |x - r|
\]

\[
U = x - r \quad du = dx
\]

\[
du = dx
\]

Done with this one
\[ = \frac{A}{a} \int \frac{u+s}{u^2+t} \, du = \frac{A}{a} \int \frac{u}{u^2+t} \, du + \frac{As}{a} \int \frac{du}{u^2+t} \]

\[ u^2+t = x \]
\[ 2udu = dx \]
\[ u \, du = \frac{dx}{2} \]
\[ = \frac{A}{2a} \int \frac{dx}{x} \]
\[ = \frac{A}{2a} \ln |x| \]
\[ = \frac{A}{2a} \ln |u^2+t| \]
\[ = \frac{A}{2a} \ln \left( \left( x + \frac{b}{2a} \right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} \right) \]

\[ \frac{As\sqrt{t}}{a\sqrt{x}} \int \frac{du}{u^2+(\sqrt{x})^2} \]
\[ = \frac{As}{a\sqrt{x}} \int \frac{\sqrt{t} \, du}{u^2+(\sqrt{x})^2} \]
\[ = \frac{As}{a\sqrt{x}} \arctan \left( \frac{u}{\sqrt{x}} \right) \]
\[ = \frac{A \left( \frac{b}{a} - \frac{b}{2a} \right)}{a\sqrt{\frac{c}{a} - \frac{b^2}{4a^2}}} \arctan \left( \frac{\sqrt{\frac{c}{a} - \frac{b^2}{4a^2}}}{\sqrt{x}} \right) \]

Done

So, just memorize these formulas & you're good!

OR, just remember the big idea

1. Polynomial Long Division
2. Factor the denominator
3. Find the partial fractions decomposition of the remainder term
4. \[ \int \frac{du}{u} = \ln |u| + C \]
   \[ \int \frac{adu}{u^2+a^2} = \arctan \left( \frac{u}{a} \right) + C \]