Generic Taco FastFood Chain  
\[ + \]  
Airport Sushi  
\[ = \]  
offensive smells filling the apartment

offensive smells filling the apartment  
\[ + \]  
That Special Someone arriving soon  
\[ = \]  
Disaster !

After nosoblindness sets in, all we will have to rely on is our calculations - we can't afford to make any mistakes!
Q: How smelly will it be in the apartment in $t$ minutes from this moment?

Assumptions:

(i) The ceiling fan is turned on. The room is small enough that the ceiling fan thoroughly mixes the air in the room - evenly dispersing all smells.

(ii) The window is open allowing fresh air in at exactly the same rate the well-mixed room air escapes.

\[
\frac{dp}{dt} = K \frac{\text{Particles per min}}{\text{K fresh air particles}} \cdot 0 \frac{\text{Smelly particles}}{\text{K fresh air particles}} - K \frac{\text{particles}}{\text{min}} \frac{C(t)}{N} \frac{\text{Smelly particles}}{\text{particles}}
\]

= $-Kc(t)$
\[ C(t) = \frac{p(t)}{N} \]

Therefore \[ \frac{dp}{dt} = -\frac{K}{N} p(t) \] since \[ K \& N > 0 \]

This equation results in an exponential decay

\[ p(t) = e^{-\frac{Kt}{N}} \]

Proof:

\[ \frac{dp}{dt} = -\frac{K}{N} p(t) \]

\[ \int \frac{dp}{p} = -\frac{K}{N} \int dt \]

\[ \ln|p| = -\frac{K}{N} t + C \]

\[ p = |p| = e^{\ln|p|} = e^{-\frac{Kt}{N} + C} = e^{-\frac{Kt}{N}} e^C = Ce^{-\frac{Kt}{N}} \]

\[ p(t) = Ce^{-\frac{Kt}{N}} \]

\[ P = p(0) = Ce^0 = C \cdot 1 = C \Rightarrow p(t) = Pe^{-\frac{Kt}{N}} \]
OK, looks like our assumptions mean that the smell will exponentially decay at a rate depending on the proportion of the room's air that can be exchanged with fresh air in 1 unit time.

Q: What if, after plugging in particular values of $P$, $K$, & $N$ we realize that the smell will still be too pervasive to avoid Disaster ⚠️

Can we speed up the process by drawing in more fresh air $\Rightarrow$ lower overall concentration?

OR perhaps by pushing out the toxic air at a faster rate?

(I) Point a fan inside the room & set up against the window

(II) Point a fan outside the window

\[
\text{Fresh Air Particles Coming In} \quad > \quad \text{Mixed room air going out} = K
\]

\[
\text{Fresh Air Particles Coming in} < \quad \text{Mixed room air going out} = K
\]
In both cases (I) & (II) above \( N \) is no longer constant. If we call the "fresh air particles coming in \( F \) then \( N \) as a function of \( t \) is

\[
N(t) = N(0) + (F - K)t
\]

\( \text{Note: the units check out } \checkmark \text{ LHS units = RHS units} \)

Now our differential equation becomes

\[
\frac{dp}{dt} = -K \cdot c(t) = -K \cdot \frac{p(t)}{N(0) + (F - K)t}
\]

\( \text{the rate of change of poop in the air as time progresses} \)

\( \text{poop particles} \)

\( \text{particles in room @ start} \)

\( \text{air exchange rate} \)
Problem 1

\[ \frac{dp}{dt} = -\frac{K}{N} p(t) \]

\[ p(t) = P e^{-\frac{Kt}{N}} \]

Method: Separate Variables

In General: Separable equations

\[ \frac{dy}{dx} = f(x)g(y) \]

Separate variables

\[ \frac{dy}{g(y)} = f(x)dx \]

Integrate both sides

\[ \int \frac{dy}{g(y)} = \int f(x)dx \]

This is the best we can say in general.

Our example from before

\[ \frac{dp}{dt} = -\frac{K}{N} p(t) = f(t)g(p) \]

\[ \frac{dp}{p} = f(t)dt \]

\[ \int \frac{dp}{p} = \int f(t)dt = \int -\frac{K}{N} dt = -\frac{K}{N} t \]

\[ \ln|p| = \ln|p(t)| = \ln|p(t)| \quad \text{can then solve for } p(t) \]
\[ \frac{dp}{dt} = \frac{-Kp(t)}{N_0 + (F-K)t} \]

Q: Is this a separable differential equation? i.e., can the RHS be written as \( \frac{f(t)}{g(p)} \)?

Yes!

\[ f(t) = \frac{-K}{N_0 + (F-K)t} \]
\[ g(p) = p \]

Separate variables

\[ \frac{dp}{p} = \frac{-Kdt}{N_0 + (F-K)t} \]

\[ \int \frac{dp}{p} = \int \frac{-Kdt}{N_0 + (F-K)t} \]
\[ \ln(p) = \frac{K}{K-F} \int \frac{du}{u} = \frac{K}{K-F} \ln|u| + C = \frac{K}{K-F} \ln|(F-K)t + N_0| + C \]

\[ e^{\ln|p|} = e^{\frac{K}{K-F} \ln|(F-K)t + N_0| + C} \]
\[ |p| = e^{\frac{K}{K-F} \ln|(F-K)t + N_0| + C} \]
\[ p(t) = e^{e^{\frac{K}{K-F} \ln|(F-K)t + N_0| + C}} \]

\[ \frac{K}{K-F} \] constant depending on air exchange

\[ \text{constant value of particles in the room} \]
1/4 particles in room get exchanged for fresh air particles every minute. The initial concentration of bad smell is 3/8, i.e., after turning on the ceiling fan every region of the room is exactly 3/8 smelly particles & 5/8 fresh air. How many minutes does it take for the smell to fall below a concentration of 1/10?

\[
\frac{dN}{dt} = -\frac{N}{4} \cdot \frac{P(t)}{N}
\]

\[\Rightarrow P(t) = \frac{3N}{8} e^{-\frac{t}{4}}\]

Goal: \(\frac{1}{10} \geq C(t) = \frac{P(t)}{N} = \frac{3}{8} e^{-\frac{t}{4}}\)

\[\iff \frac{8}{30} \geq e^{-\frac{t}{4}} \iff \ln(\frac{8}{30}) \geq -\frac{t}{4}\]

\[\iff 4 \ln(\frac{8}{30}) \leq t\]

\[\iff t \geq 5.287023359\ldots\]

If you've got 6 min, you're good 😊
What if you put a fan facing inside your room in order to blow more fresh air in. How would this change the concentration?

\[
\frac{dp}{dt} = -\frac{N \cdot P(t)}{4} \left( N + \frac{3N}{8} - \frac{N}{4} \right) = -\frac{N \cdot P(t)}{4} \left( N + \frac{3N}{8} \right)
\]

\[
= -\frac{P(t)}{4(1+t/8)}
\]

\[
-\int \frac{dp}{P} = \int \frac{dt}{4(1+t/8)} \quad \Rightarrow \quad -\ln|P| = 2\ln|1+t/8| + C
\]

\[
\frac{1}{P} = e^{2\ln|1+t/8|} + C
\]

\[
= e^{2\ln|1+t/8|} e^C
\]

\[
= (e^{\ln|1+t/8|})^2 e^C = (1+t/8)^2 e^C
\]

\[
\Rightarrow P(t) = \frac{3N}{8 (1+t/8)^2}
\]

\[
\Rightarrow C(t) = \frac{P(t)}{N(t)} = -\frac{1}{8} (1+t/8)^3
\]

\[
\Leftrightarrow (1+t/8)^3 > \frac{3}{8} \quad \Leftrightarrow \quad 1+t/8 > \sqrt[3]{\frac{3}{8}}
\]

\[
\Leftrightarrow \quad t > 8\sqrt[3]{\frac{3}{8}} - 8
\]
~ 6 min w/ no fan pointed in
~ 5 min w/ fan pointed in

About 1 min improvement

(II) What about in the other situation

where we use the second fan to force an extra \( N/8 \) well-mixed particles out?

\[
\frac{dp}{dt} = -\frac{3N}{8} \cdot \frac{p(t)}{N-N_{d}/8} = -\frac{3}{8} \cdot \frac{p(t)}{1-t/8}
\]

\[
\Rightarrow \quad \frac{dp}{p} = \frac{3}{8} \int \frac{dt}{t/8-1} \quad u = \frac{t}{8} - 1
\]

\[
\ln(p) = 3 \int \frac{du}{u} = 3 \ln|u| + C = 3 \ln|1-t/8| + C
\]

\[
p(t) = (1-t/8)^3 e^C
\]

\[
\frac{3N}{8} = p(0) = C e^C \quad \Rightarrow C(t) = \frac{p(t)}{N(t)} = \frac{3}{8} \frac{N(1-t/8)^3}{N-N_{d}/8}
\]

\[
= \frac{3}{8} \frac{(1-t/8)^2}{N-N_{d}/8}
\]

Concentration increases!
Conclusion: In this case the options rank as follows

**Worst**
- use second fan to
- blow more air out
- More air leaves than enters but the concentration of the bad smell in the well-mixed air is low enough that the new air increases the overall concentration of the smell!

**Pretty good**
- don't use second fan
- Smell exponentially decays

**Best**
- Use second fan to pull in more fresh air
- Smell decays like $\frac{1}{2^n}$ but fresh particles increases & dilutes the smell even more

You now have the know-how to do your own estimations. Thank me later ;)

The Most General Tank Problem

![Diagram of a tank system with inflow and outflow rates and concentration levels]
Q: What is the concentration of Solute in Solvent at time t?

**Hint:** Keeping track of units helps you avoid mistakes.

Concentration \( \frac{\text{kg}}{\text{L}} = \frac{\text{raw amount of Solute}}{\text{total volume Solvent}} = \frac{S(t) \text{ kg}}{V(t) \text{ L}} \)

\[
V(t) \text{ L} = \text{Starting Volume} \text{ L} + \left( \frac{\text{volume in L}}{\text{min}} \cdot \frac{\text{volume out L}}{\text{min}} \right) \cdot \text{t \ min}
\]

\[
= 6000 \text{ L} + \left( \frac{50 \text{ L}}{\text{min}} - \frac{60 \text{ L}}{\text{min}} \right) \cdot t \text{ min}
\]

\[
= (6000 - 10t) \text{ L}
\]

Don't know much about \( S(t) \) other than \( S(0) = 17 \text{ kg} \) but we are given information about how \( S(t) \) changes as the minutes pass.

\[
\frac{ds}{dt} \text{ kg} = \frac{\text{solute added kg}}{\text{min}} - \frac{\text{solute draining out in solution kg}}{\text{min}}
\]

\[
= \left( \frac{\text{volume Solvent added L/min}}{\text{min}} \right) \cdot \left( \frac{\text{concentration of Solute in Solution kg/L}}{\text{L/min}} \right) - \left( \frac{\text{volume Solvent drained out L/min}}{\text{min}} \right) \cdot \left( \frac{\text{concentration of Solute in Solution kg/L}}{\text{L/min}} \right)
\]

\[
= 50 \text{ L/min} \cdot \frac{5 \text{ kg}}{\text{L}} - 60 \text{ L/min} \cdot \frac{S(t) \text{ kg}}{V(t) \text{ L}}
\]

\[
\frac{ds}{dt} = \frac{250 \text{ kg}}{\text{min}} - 60 \frac{S(t)}{6000 - 10t} \frac{\text{kg}}{\text{min}}
\]
Notice: This differential equation is not separable

\[ \frac{ds}{dt} + f(t)s = 250 \]

the key idea is to treat
this part as if it came
from the product rule

\[ \frac{d}{dt} \left[ e^{\int f(t) dt} S(t) \right] = S(t) \frac{d}{dt} e^{\int f(t) dt} + e^{\int f(t) dt} \frac{ds}{dt} \]

\[ = S(t) f(t) e^{\int f(t) dt} + \frac{ds}{dt} e^{\int f(t) dt} = 250 e^{\int f(t) dt} \]

Integrating Both Sides

\[ \int \frac{d}{dt} \left[ e^{\int f(t) dt} S(t) \right] dt = \int 250 e^{\int f(t) dt} dt \]

\[ \text{Cancel out by FTC} \]

\[ S(t) = \frac{\int 250 e^{\int f(t) dt} dt}{e^{\int f(t) dt}} \]
\[
\frac{ds}{dt} + \frac{6}{600-t} s = 250
\]

Multiply by the integrating factor

\[
\frac{d}{dt} \left[ e^{\int \frac{6}{600-t} dt} s(t) \right] = e^{\int \frac{6}{600-t} dt} \left( \frac{ds}{dt} + \frac{6}{600-t} s \right) = 250 e^{\int \frac{6dt}{600-t}}
\]

Product rule

Integrate both sides

\[
e^{\int \frac{6dt}{600-t}} s(t) = \int \frac{d}{dt} \left[ e^{\int \frac{6}{600-t} dt} s(t) \right] dt = \int 250 e^{\int \frac{6dt}{600-t}} dt
\]

\[
= 250 \int e^{-6 \ln(600-t)} dt
\]

\[
= 250 \int 1600-t^{-6} dt
\]

\[
= 250 \int \frac{dt}{(600-t)^6}
\]

\[
\frac{50}{(600-t)^5} + c = \frac{250}{5} \frac{1}{u^5} + c = 250 \int \frac{-du}{u^6}
\]

\[
u = 600 - t
\]

\[
du = -dt
\]

\[-du = dt\]
Conclusion: \[ \frac{1}{(600-t)^6} S(t) = \frac{50}{(600-t)^5} + C \]

\[ S(t) = 50(600-t) + C(600-t)^6 \]

Review

Pick a function @ random

\[ \downarrow \]

You probably won’t know how to integrate it

Integration is the simplest example of a differential equation (by FTC)

Therefore

pick a random differential equation

\[ \downarrow \]

there’s no hope of solving it

Integration is Hard

Differential Equations Are Super Hard
What Differential Equations Do you Need
To be able to solve?

**First Order** - one derivative

**Separable**

\[
\frac{dy}{dx} = f(x)g(y)
\]

\[
\frac{dy}{g(y)} = f(x)dx
\]

Separate variables

\[
\int \frac{dy}{g(y)} = \int f(x)dx
\]

Integrate

**First order Linear**

\[
\frac{dy}{dt} + f(t)y = g(t)
\]

Multiply by the integrating factor \( e^{\int f(t)dt} \)

\[
\frac{dy}{dt} e^{\int f(t)dt} + f(t)y e^{\int f(t)dt} = g(t)e^{\int f(t)dt}
\]

\[
\text{this part came from product rule}
\]

Integrate

\[
\frac{d}{dt} \left[ y e^{\int f(t)dt} \right] = g(t)e^{\int f(t)dt}
\]

\[
y e^{\int f(t)dt} = \int g(t)e^{\int f(t)dt} dt
\]
Note: This process only works if you isolate \( \frac{dy}{dt} \) by itself.

Given \( a(t) \frac{dy}{dt} + b(t)y = c(t) \)

we first divide by \( a(t) \) to bring the equation into the standard form above (with \( \frac{dy}{dt} \) isolated)

\[
\frac{dy}{dt} + \frac{b(t)}{a(t)} y = \frac{c(t)}{a(t)}
\]

Then the integrating factor is \( e^{\int \frac{b(t)}{a(t)} dt} \)