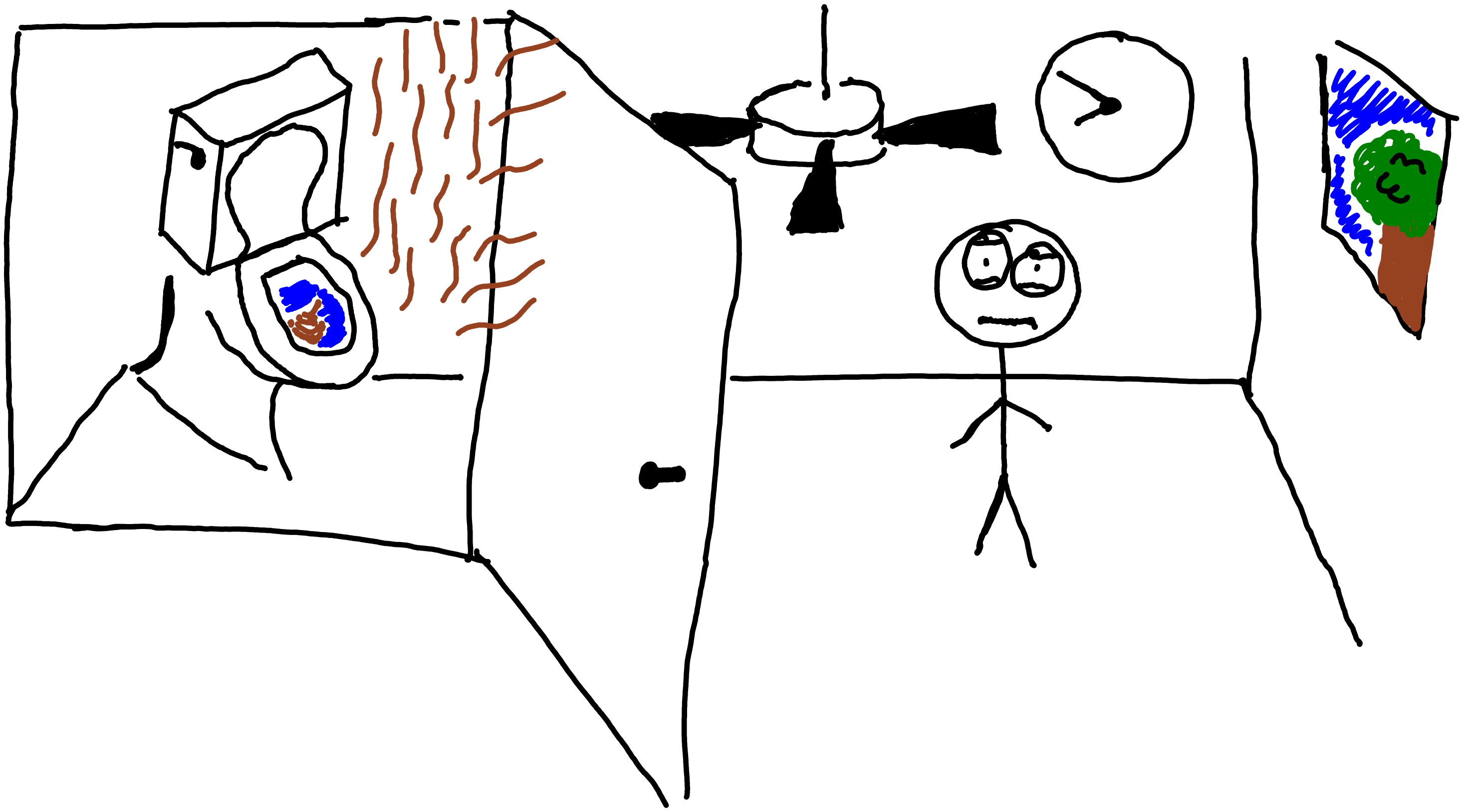


Generic Taco FastFood Chain
+
Airport Sushi = offensive smells filling the apartment

offensive smells filling the apartment
+
that special someone arriving soon = Disaster ⚠

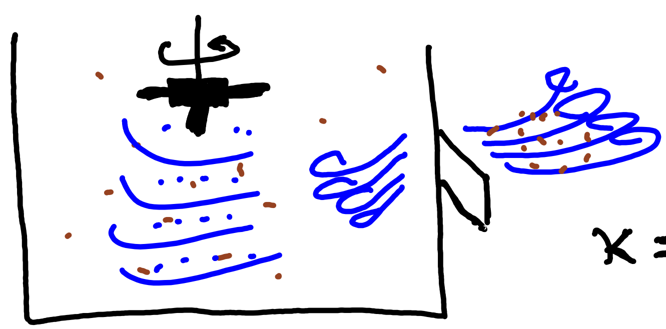
after noseblindness sets in all we will have to rely on is our calculations - we can't afford to make any mistakes!

Q: How smelly will it be in the apartment in t minutes from this moment?

Assumptions:

(i) The ceiling fan is turned on. The room is small enough that the ceiling fan thoroughly mixes the air in the room - evenly dispersing all smells.

(ii) The window is open allowing fresh air in at exactly the same rate the well-mixed room air escapes.



t = time measured in min

P = initial # of smelly particles

N = total # of particles in room

K = # particles entering/exiting every min

$p(t)$ = # smelly particles @ time t

$c(t)$ = concentration of the smell $= \dfrac{p(t) \text{ smelly part}}{N \text{ particles}}$

measures changes in # smelly particles as time progresses

units of $\dfrac{\text{smelly particles}}{\text{min}}$

$$\frac{dp}{dt} = K\,\frac{\text{particles}}{\text{min}} \cdot \frac{0 \text{ smelly particles}}{K \text{ fresh air particles}} - K\,\frac{\text{particles}}{\text{min}}\, c(t)\, \frac{\text{smelly particles}}{\text{particles}}$$

$$= -Kc(t)$$

Smell gets better because # smelly particles decreases

smelly particles that leaves each min is the exact proportion you would expect (Average concentration)

$$C(t) = \frac{p(t)}{N}$$

therefore $\frac{dp}{dt} = -\frac{K}{N} p(t)$ since K & $N > 0$

this equation results in an exponential decay

$$p(t) = e^{-\frac{Kt}{N}}$$

<u>proof:</u> $$\frac{dp}{dt} = -\frac{K}{N} p(t)$$

$$\frac{dp}{p(t)} = -\frac{K}{N} dt$$

$$\int \frac{dp}{p} = \int -\frac{K}{N} dt$$

$$\ln|p| = -\frac{K}{N} t + c$$

$$p = |p| = e^{\ln|p|} = e^{-\frac{Kt}{N} + c} = e^{-\frac{Kt}{N}} e^{c} = \tilde{C} e^{-\frac{Kt}{N}}$$

$$\boxed{p(t) = C e^{-\frac{Kt}{N}}}$$

$$P = p(0) = Ce^{0} = C \cdot 1 = C \Rightarrow \boxed{p(t) = P e^{-\frac{Kt}{N}}}$$

OK, looks like our assumptions mean that the smell will exponentially decay at a rate depending on the proportion of the room's air that can be exchanged with fresh air in 1 unit time.

Q: What if, after plugging in particular values of ℓ, K, & N we realize that the smell will still be too pervasive to avoid Disaster ⚠

Can we speed up the process by drawing in more fresh air $\Rightarrow$ lower overall concentration

OR perhaps

by pushing out the toxic air at a faster rate?

(I)

point a fan inside the room & set up against the window

Fresh Air Particles Coming In > mixed room air going out = K

(II)

point a fan outside the window

Fresh Air Particles Coming in < mixed room air going out = K

In both cases (I) & (II) above N is no longer constant. If we call the # fresh air particles coming in F then N as a function of t is

$$\underbrace{N(t)}_{\substack{\text{\# particles in} \\ \text{room @ time } t \\ \text{measured in min}}} = \underbrace{N(0)}_{\substack{\text{\# particles} \\ \text{there were} \\ \text{to begin with}}} + (F - K)t$$

time in min

$\dfrac{\text{\# fresh air particles}}{\text{min}}$

$\dfrac{\text{\# room particles}}{\text{min}}$

<u>Note:</u> the units check out ✓ LHS units = RHS units

Now our differential equation becomes

$$\frac{dp}{dt} = -K\,c(t) = -K \frac{p(t)}{N(0) + (F-K)t}$$

$$\boxed{\frac{dp}{dt} = -K \frac{p(t)}{N(0) + (F-K)t}}$$

the rate of change of poop in the air as time progresses

poop particles

particles in room @ start

air exchange rate

problem 1

$$\frac{dp}{dt} = \frac{-K}{N} p(t) \qquad p(t) = p_0 e^{\frac{-kt}{N}}$$

Method: Separate Variables

$\boxed{\text{In General: Separable equations} \quad \frac{dy}{dx} = f(x)g(y)}$

Separate variables $\quad \frac{dy}{g(y)} = f(x)dx$

integrate both sides

$$\boxed{\int \frac{dy}{g(y)} = \int f(x)dx}$$

← this is the best we can say in general.

our example from before

$$\frac{dp}{dt} = -\frac{K}{N} p(t) = f(t)g(p) \qquad f(t) = -\frac{K}{N}$$

$$\frac{dp}{p} = f(t)dt \qquad g(p) = p$$

$$\int \frac{dp}{p} = \int f(t)dt = \int -\frac{K}{N}dt = -\frac{K}{N}t$$

$$\parallel$$

$\ln|p| = \ln|p(t)|$ ← can then solve for $p(t)$

our new example

$$\frac{dp}{dt} = \frac{-kp(t)}{N(0)+(F-k)t}$$

Q: is this a separable differential equation? i.e. can the RHS be written as $f(t)g(p)$?

Yes!

$$f(t) = -\frac{k}{N(0)+(F-k)t}$$

$$g(p) = p$$

Separate variables

$$\frac{dp}{dt} = -\frac{k}{N(0)+(F-k)t}\cdot p$$

$$\frac{dp}{p} = \frac{-k\,dt}{N(0)+(F-k)t}$$

integrate

$$\int \frac{dp}{p} = \int \frac{-k\,dt}{N(0)+(F-k)t}$$

$$u = (F-k)t + N(0)$$

$$du = (F-k)dt$$

$$\frac{du}{F-k} = dt$$

$$\ln|p| = \frac{k}{k-F}\int\frac{du}{u} = \frac{k}{k-F}\ln|u| + c = \frac{k}{k-F}\ln|(F-k)t+N(0)| + c$$

$$e^{\ln|p|} = e^{\frac{k}{k-F}\ln|(F-k)t+N(0)|+c}$$

$$= \left(e^{\ln|(F-k)t+N(0)|}\right)^{k/k-F} e^{c}$$

$$= e^{c}\,|(F-k)t+N(0)|^{\frac{k}{k-F}}$$

$$e^{\ln|p|} = |p| = p$$

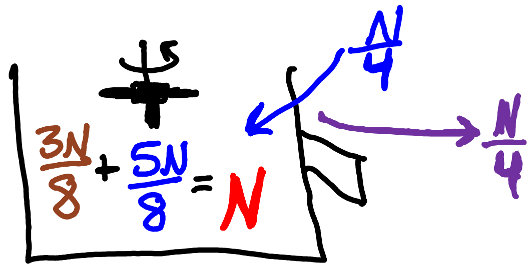
$$p(t) = e^{c}\,|(F-k)t+N(0)|^{\frac{k}{k-F}}$$

constant depending on air exchange

constant value of particles in the room

Testing these results

1/4 particles in room get exchanged for fresh air particles every minute. The initial concentration of bad smell is 3/8 i.e. after turning on the ceiling fan every region of the room is exactly 3/8 smelly particles & 5/8 fresh air. How many minutes does it take for the smell to fall below a concentration of 1/10 ?



$$\frac{dp}{dt} = -\frac{N}{4} \cdot \frac{p(t)}{N}$$

mixed particles out

Concentration of smell in the mixture

$$\Rightarrow p(t) = \frac{3N}{8} e^{-t/4}$$

<u>Goal:</u> $\frac{1}{10} \geq c(t) = \frac{p(t)}{N} = \frac{3}{8} e^{-t/4}$

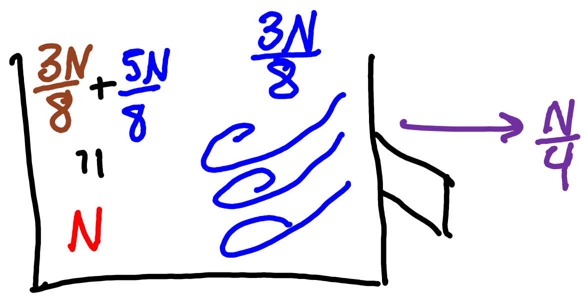
$$\Leftrightarrow \frac{8}{30} \geq e^{-t/4} \quad \Leftrightarrow \ln(8/30) \geq -t/4$$

$$\Leftrightarrow 4 \ln(30/8) \leq t$$

$$\Leftrightarrow t \geq 5.287023359\ldots$$

If you've got 6 min you're good :)

(I) What if you put a fan facing inside your room in order to blow more fresh air in. How would this change the concentration?



$$\frac{dp}{dt} = \frac{-N}{4}\frac{p(t)}{N+\left(\frac{3N}{8}-\frac{N}{4}\right)t} = \frac{-N}{4}\frac{p(t)}{N+Nt/8}$$

$$= \frac{-p(t)}{4(1+t/8)}$$

$$-\int \frac{dp}{p} = \int \frac{dt}{4(1+t/8)}$$

$$-\ln|p| = 2\ln|1+t/8| + c$$

$$1/p = e^{2\ln|1+t/8|+c}$$

$$= e^{2\ln|1+t/8|}e^{c}$$

$$= \left(e^{\ln|1+t/8|}\right)^2 e^{c}$$

$$= |1+t/8|^2 e^{c} = (1+t/8)^2 e^{c}$$

$$\Longrightarrow p(t) = \frac{3}{8}\frac{N}{(1+t/8)^2} \quad \Rightarrow c(t) = \frac{p(t)}{N(t)} = \frac{3}{8}\frac{1}{(1+t/8)^3}$$

IN

1/10

$$\Longleftrightarrow (1+t/8)^3 \geq 30/8 \Longleftrightarrow 1+t/8 \geq \sqrt[3]{\frac{30}{8}}$$

$$\Longleftrightarrow t \geq 8\sqrt[3]{30/8} - 8$$

$\parallel$

$4\sqrt[3]{30} - 8$

$\approx$

$4.42893\ldots$

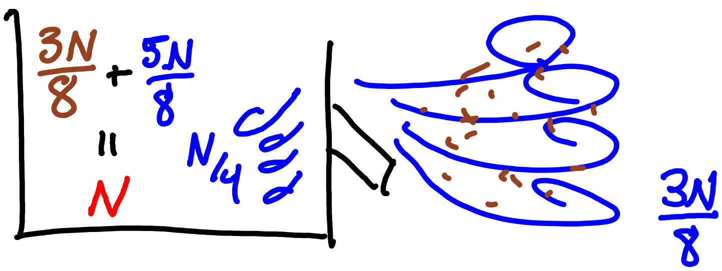
~ 6 min w/ no fan pointed in

~ 5 min w/ fan pointed in

About 1 min improvement

(II) what about in the other situation

where we use the second fan to force an extra $N/8$ well-mixed particles out?



$$\frac{dp}{dt} = -\frac{3N}{8} \cdot \frac{p(t)}{N - Nt/8} = -\frac{3}{8}\frac{p(t)}{1 - t/8}$$

$$\Longrightarrow \quad \frac{dp}{p} = \frac{3}{8}\int \frac{dt}{t/8 - 1} \qquad u = t/8 - 1, \quad du = \frac{dt}{8}$$

$$\ln|p| = 3\int \frac{du}{u} = 3\ln|u| + c = 3\ln|1 - t/8| + c$$

$$p(t) = (1 - t/8)^3 e^c$$

$$\frac{3N}{8} = p(0) = e^c$$

$$\Rightarrow c(t) = \frac{p(t)}{N(t)} = \frac{3}{8}\frac{N(1-t/8)^3}{N - Nt/8}$$

$$= \underbrace{\frac{3}{8}(1 - t/8)^2}$$

Concentration increases!

Conclusion: In this case the options rank as follows

Worst

use second fan to blow more air out

more air leaves than enters but the concentration of the bad smell in the well-mixed air is low enough that the new air increases the overall concentration of the smell!

Pretty good

don't use second fan

Smell exponentially decays

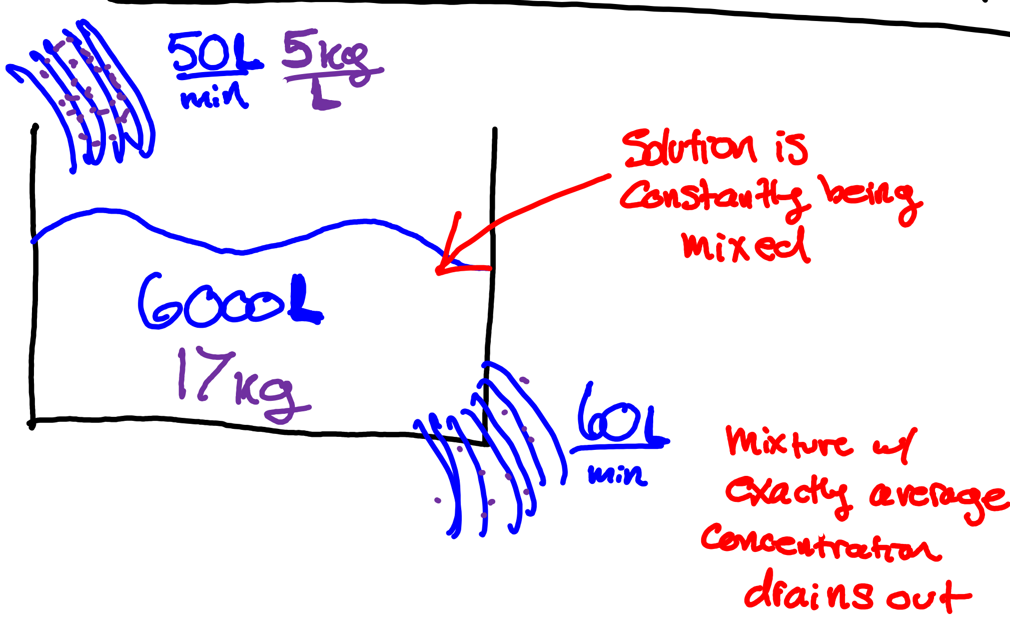
best

use second fan to pull in more fresh air

Smell decays like $\frac{1}{t^n}$ but * fresh particles increases & dilutes the smell even more

You now have the know-how to do your own estimations. Thank me later ;)

The Most General Tank Problem



Q: What is the concentration of Solute in Solvent @ time t?

Hint: Keeping track of units helps you avoid mistakes

$$\text{Concentration } \frac{kg}{L} = \frac{\text{raw amount of Solute}}{\text{total volume Solvent}} = \frac{S(t)\ kg}{V(t)\ L}$$

$$V(t)_L = (\text{Starting Volume})_L + \left(\frac{\text{volume in } L}{\text{min}} - \frac{\text{Volume out } L}{\text{min}}\right)(\#\text{min})$$

$$= 6000L + \left(\frac{50L}{\text{min}} - \frac{60L}{\text{min}}\right) t \text{ min}$$

$$= (6000 - 10t)_L$$

Don't know much about $S(t)$ other than $S(0) = 17_{kg}$
but we are given information about how $S(t)$ changes
as the minutes pass

$$\frac{ds}{dt}\frac{kg}{min} = \frac{\text{Solute added } kg}{\text{min}} - \frac{\text{Solute draining out in solution } kg}{\text{min}}$$

$$= \underbrace{\begin{pmatrix}\text{volume}\\\text{Solvent}\\\text{added}\end{pmatrix}}_{L/min}\underbrace{\begin{pmatrix}\text{concentration}\\\text{of solute}\\\text{in solution}\end{pmatrix}}_{kg/L} - \underbrace{\begin{pmatrix}\text{volume}\\\text{Solvent}\\\text{drained}\\\text{out}\end{pmatrix}}_{L/min}\underbrace{\begin{pmatrix}\text{concentration}\\\text{of solute}\\\text{in solution}\end{pmatrix}}_{kg/L}$$

$$= \frac{50L}{min} \cdot \frac{5kg}{L} - \frac{60L}{min} \cdot \frac{S(t)\ kg}{V(t)\ L}$$

$$\boxed{\frac{ds}{dt} = \frac{250\ kg}{min} - 60\frac{S(t)}{6000-10t}\ \frac{kg}{min}}$$

Notice: This differential equation is not separable

$$\underbrace{\frac{ds}{dt} + f(t)s}_{\text{the key idea is to treat this part as if it came from the product rule}} = 250$$

this is a new type of equation
$\Rightarrow$ will need a new idea to solve it

$$\frac{d}{dt}\left[e^{\int f(t)dt} S(t)\right] = S(t)\frac{d}{dt}e^{\int f(t)dt} + e^{\int f(t)dt}\frac{ds}{dt}$$

$$= S(t)\, f(t) e^{\int f(t)dt} + \frac{ds}{dt}e^{\int f(t)dt} = 250\, e^{\int f(t)dt}$$

Integrating Both Sides

$$\underbrace{\int \frac{d}{dt}}_{\text{cancel out by FToC}}\left[e^{\int f(t)dt} S(t)\right] = \int 250\, e^{\int f(t)dt}\, dt$$

$$\boxed{S(t) = \frac{\int 250\, e^{\int f(t)dt}\, dt}{e^{\int f(t)dt}}}$$

$$\frac{ds}{dt} + \frac{6}{600-t}s = 250$$

multiply by the "integrating factor

$$\frac{d}{dt}\left[e^{\int \frac{6}{600-t}dt} S(t)\right] = e^{\int \frac{6}{600-t}dt}\left(\frac{ds}{dt} + \frac{6}{600-t}s\right) = 250 e^{\int \frac{6dt}{600-t}}$$

product rule

integrate both sides

$$e^{\int \frac{6dt}{600-t}} S(t) = \int \underbrace{\frac{d}{dt}}_{\text{FToC}}\left[e^{\int \frac{6}{600-t}dt} S(t)\right] = \int 250\, e^{\int \frac{6dt}{600-t}} dt$$

$$= 250 \int e^{-6\ln|600-t|} dt$$

$$= 250 \int |600-t|^{-6} dt$$

$$= 250 \int \frac{dt}{(600-t)^6}$$

$$\frac{50}{(600-t)^5} + c = \frac{250}{5}\frac{1}{u^5} + c = 250 \int \frac{-du}{u^6}$$

$u = 600 - t$

$du = -dt$

$-du = dt$

Conclusion: $\frac{1}{(600-t)^6} S(t) = \frac{50}{(600-t)^5} + C$

$$\boxed{S(t) = 50(600-t) + C(600-t)^6}$$

Review

Pick a function @ random

$\Downarrow$

You probably won't know how to integrate it

Integration is Hard

Integration is the simplest example of a differential equation (by FToC)

therefore

pick a random differential equation

$\Downarrow$

there's no hope of solving it

Differential Equations Are Super Hard

What Differential Equations Do you Need To be able to solve?

First Order – one derivative

Separable

$$\boxed{\frac{dy}{dx} = f(x)g(y)}$$

$$\frac{dy}{g(y)} = f(x)dx$$ Separate variables

$$\boxed{\int \frac{dy}{g(y)} = \int f(x)dx}$$ integrate

First order Linear

$$\frac{dy}{dt} + f(t)y = g(t)$$

multiply by the integrating factor $e^{\int f(t)dt}$

$$\underbrace{\frac{dy}{dt} e^{\int f(t)dt} + f(t)y e^{\int f(t)dt}}_{\text{this part came from product rule}} = g(t)e^{\int f(t)dt}$$

$$\frac{d}{dt}\left[ye^{\int f(t)dt}\right] = g(t)e^{\int f(t)dt}$$

integrate

$$ye^{\int f(t)dt} = \int g(t)e^{\int f(t)dt} dt$$

$$\boxed{y(t) = \frac{\int g(t) e^{\int f(t)\,dt}\, dt}{e^{\int f(t)\,dt}}}$$

<u>Note:</u> This process only works if you isolate $\frac{dy}{dt}$ by itself.

given $$a(t)\frac{dy}{dt} + b(t)y = c(t)$$

we first divide by $a(t)$ to bring the equation into the standard form above (with $\frac{dy}{dt}$ isolated)

$$\frac{dy}{dt} + \frac{b(t)}{a(t)}y = \frac{c(t)}{a(t)}$$

then the integrating factor is $e^{\int \frac{b(t)}{a(t)}dt}$