Isbell Duality and a Dual Pair of Double Dualization Codensity Monads

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Introduction

There is a known Stone-type duality between the categories of poc-sets and median algebras [2] induced by the two-element Boolean algebra $\mathbb{2} \in$ **Bool**, which carries both structures in a compatible way. We show that this duality gives rise to a dual pair of codensity monads (see Theorem 1). Our example follows a pattern which is common in the literature, namely, that the endofunctor parts of these codensity monads are given by a form of double dualization. For instance, the linear double dual construction is the codensity monad induced by the inclusion of finite-dimensional k-vector spaces into $\mathbf{Vect}_k$ [4]. It happens that the ultrafilter monad, which is the codensity monad of the inclusion $i_{\mathbf{Set}} : \mathbf{FinSet} \longrightarrow \mathbf{Set}$ [3], can be viewed as a form of double dualization as well. This poster will emphasize the analogy between poc-sets/median algebras and Boolean algebras/sets by developing these examples in parallel. Along the way, other interesting relationships between these categories will be pointed out. In particular, we ultimately construct the double dualization codensity monads as the induced monads of a contravariant adjunction, which itself arises from a two-variable tensor-hom adjunction (see Lemma 2). This gives a formula for Booleanization, and strengthens the analogy between poc-sets/median algebras and vector spaces. A necessary condition implicit in the construction of the tensor product is the fact that Boolean algebras are exactly those sets with compatible structures of a poc-set and median algebra. That is to say, the structure maps of a poc-set are morphisms of median algebras exactly when the original poc-set is a Boolean algebra (Lemma 1). Following this line of reasoning, we prove that the category of finite poc-sets (resp. finite median algebras) is an essentially algebraic theory for the category of median algebras (resp. poc-sets) in the sense that the category of finitely continuous functors **FinPoc** $\longrightarrow$ **Set** can be identified with **Med** (and vice versa). This last result (Theorem 2) is related to Isbell duality. If you are curious to know more, feel free to inquire about the current draft of the manuscript.

Poc-sets and Median Algebras

Poc-sets and median algebras are simultaneous generalizations of both power sets and trees. For the purposes of this poster, the connections with Boolean algebras (a generalization of power sets) are more important. However, we note that median algebras derive their name from the example of $\mathbb{R}$ (viewed as a tree) wherein the (usual) median of three real numbers coincides with the ternary 'median operation'

Med := the category of median algebras.

Definition: A **median algebra** is a pair (M, m) where $M \in$ **Set** and $m : M^3 \longrightarrow M$ is a ternary operation on M satisfying: $\forall w, x, y, z \in M$

(M.i) $m(x, x, y) = x$

(M.ii) $m(x, y, z) = m(y, x, z) = m(x, z, y)$

(M.iii) $m(x, w, m(y, w, z)) = m(m(x, w, y), w, z)$.

Poc := the category of poc-sets.

Definition: A partially ordered set $(P, \to) \in$ **Poset** with an order-reversing involution $\neg : P \longrightarrow P$ is a **poc-set** provided:

(P.i) $\exists \perp \in P$ s.t. $\forall p \in P \quad \perp \to p$; and

(P.ii) $\forall p \in P \quad p \to \neg p$ implies $p = \perp$.

Morphisms of poc-sets are monotone functions that commute with the given involutions.

Definition: Given a Boolean algebra $B \in$ **Bool**, its **Boolean majority function**, denoted $m_{\mathbf{Bool}}$, has type $B^3 \longrightarrow B$, and is given by the following self-dual expression:

$$m_{\mathbf{Bool}}(x, y, z) := (x \wedge y) \vee (x \wedge z) \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \wedge (y \vee z).$$

There is a commutative diagram of forgetful functors:

$$\begin{tikzcd} & \mathbf{Med} \arrow[dl, "U^{\mathbf{Med}}"'] \arrow[dr, "U_{\mathbf{Med}}"] & \\ \mathbf{Bool} \arrow[dr, "U^{\mathbf{Poc}}"'] & & \mathbf{Set} \\ & \mathbf{Poc} \arrow[ur, "U_{\mathbf{Poc}}"'] & \end{tikzcd}$$

where $U^{\mathbf{Med}} : B \mapsto (B, m_{\mathbf{Bool}}) \in$ **Med**.

The presence of these forgetful functors gives $\mathbf{Hom}(-, \mathbb{2})$ four potential interpretations. We characterize each of these so-called **dualization** functors in the table below. Functors occupying the same row of the table are mutual right adjoints.

$\mathbf{Bool}^{op} \underset{\mathbf{Bool}(-,\mathbb{2})}{\overset{\mathcal{U}_{\mathbf{Bool}}}{\rightleftarrows}} \mathbf{Set}$ ($\cong$)	**Boolean Ultrafilters** $\mathcal{U}_{\mathbf{Bool}}(B) := \{S \subseteq B \mid$ a), b), & c)$\}$ a) $p \in S$ & $p \to q$ implies $q \in S$ b) $p \in S$ iff $\neg p \notin S$ c) $p, q \in S$ implies $p \wedge q \in S$	**Power Sets** $\mathcal{P}X := \{S \subseteq X\}$ The subsets of a set can be given the structure of a Boolean algebra. $(\mathcal{P}X, \varnothing, X, (-)^c, \cap, \cup, \subseteq) \in$ **Bool**	$\mathbf{Set}^{op} \underset{\mathbf{Set}(-,\mathbb{2})}{\overset{\mathcal{P}}{\rightleftarrows}} \mathbf{Bool}$ ($\cong$)
$\mathbf{Poc}^{op} \underset{\mathbf{Poc}(-,\mathbb{2})}{\overset{\mathcal{U}_{\mathbf{Poc}}}{\rightleftarrows}} \mathbf{Med}$ ($\cong$)	**Poc-set Ultrafilters** $\mathcal{U}_{\mathbf{Poc}}(P) := \{S \subseteq P \mid$ a) & b)$\}$ The median of three poc-ultrafilters is given by post-composition with the Boolean majority function on $\mathbb{2}$.	**Halfspaces** $\mathcal{H}M := \{$Halfspaces of $M\}$ **Definition:** A **halfspace** $S \subseteq M \in$ **Med** is a subalgebra s.t. $S^c := M \backslash S$ is a subalgebra.	$\mathbf{Med}^{op} \underset{\mathbf{Med}(-,\mathbb{2})}{\overset{\mathcal{H}}{\rightleftarrows}} \mathbf{Poc}$ ($\cong$)

Next, we discuss the functors whose right adjoints are the forgetful functors in the diamond above, and record useful descriptions of their actions on finite objects. We extend these formulas as a corollary of Lemma 2.

Free Objects

- The class of median algebras forms a variety in the sense of universal algebra. Therefore, the forgetful functor $U_{\mathbf{Med}}$ is monadic. In particular, it has a left adjoint $F_{\mathbf{Med}}$.
- The forgetful functor $U_{\mathbf{Poc}}$ is not monadic, however, the construction of $F_{\mathbf{Poc}}$ is straightforward from the universal property; $U_{\mathbf{Poc}} F_{\mathbf{Poc}} X$ has $2|X| + 2$ elements.

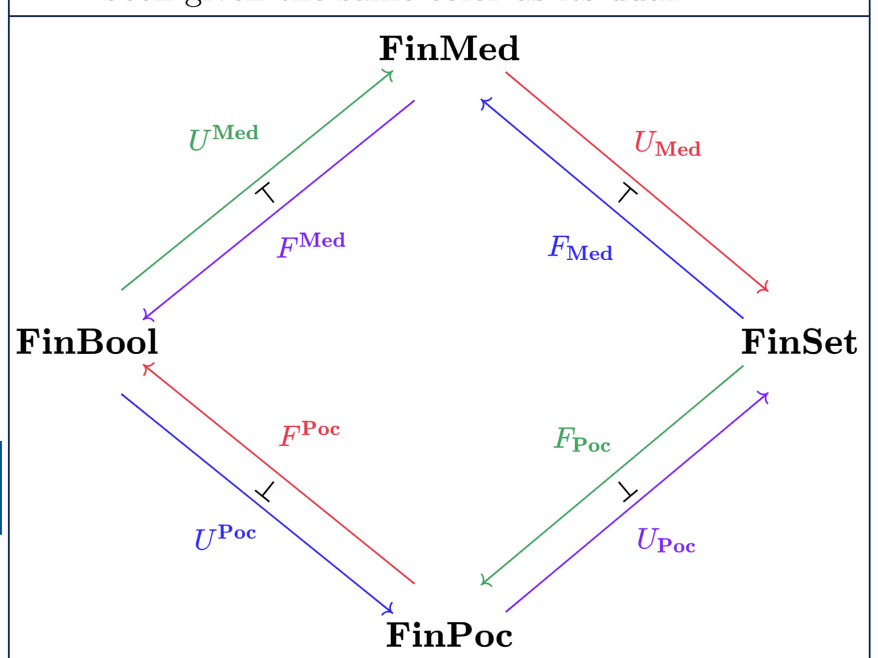
Booleanization

Left adjoints $F^{\mathbf{Med}} \dashv U^{\mathbf{Med}}$ and $F^{\mathbf{Poc}} \dashv U^{\mathbf{Poc}}$ exist, and are easy to compute for finite objects. If $\mathbb{C} \in \{\mathbf{Bool}, \mathbf{Med}, \mathbf{Poc}, \mathbf{Set}\}$, then $\mathbf{Hom}_{\mathbb{C}}(-, \mathbb{2})$ restricts to an anti-equivalence on the subcategory of finite objects. Thus $\mathbf{FinSet}^{op} \simeq \mathbf{FinBool}$ and $\mathbf{FinMed}^{op} \simeq \mathbf{FinPoc}$.

The restriction of the adjunction $F^{\mathbf{Med}} \dashv U^{\mathbf{Med}}$ is dual to the restriction of $F_{\mathbf{Med}} \dashv U_{\mathbf{Med}}$, and similarly for **Poc**.

$$\Longrightarrow \quad \begin{aligned} F^{\mathbf{Poc}}(P) &\cong \mathbf{Set}(U_{\mathbf{Med}}(\mathbf{Poc}(P, \mathbb{2})), \mathbb{2}) \\ F^{\mathbf{Med}}(M) &\cong \mathbf{Set}(U_{\mathbf{Poc}}(\mathbf{Med}(M, \mathbb{2})), \mathbb{2}) \\ F_{\mathbf{Poc}}(X) &\cong \mathbf{Med}(U^{\mathbf{Med}}(\mathbf{Set}(X, \mathbb{2})), \mathbb{2}) \\ F_{\mathbf{Med}}(X) &\cong \mathbf{Poc}(U^{\mathbf{Poc}}(\mathbf{Set}(X, \mathbb{2})), \mathbb{2}) \end{aligned}$$

The diagram below illustrates two separate dual pairs of adjunctions. Each functor has been given the same color as its dual.



The duality between bounded distributive lattices and posets behaves similarly [6].

Birkhoff and Kiss identify median algebras arising from bounded distributive lattices as those with a pair of elements such that the median operation becomes the identity when partially applied to those elements [1]. Our first lemma is a modest extension of this fact to allow for the identification of Boolean algebras (i.e., complemented distributive lattices) within **Med**. The lemma follows a motivating example, which requires we first introduce a certain functor.

A Representable Functor

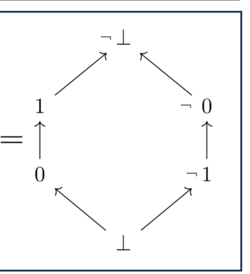
$$\mathbf{Poc} \xrightarrow{\mathbf{Arr}} \mathbf{Set}$$
$$(P, \perp, \neg, \to) \mapsto \{(p, q) \in P^2 \mid p \to q\}$$

The **arrows** functor sends a poc-set to its underlying poset $(P, \to)$ viewed as a subset of $(U_{\mathbf{Poc}} P)^2$.

$$\mathbf{Poc} \underset{\mathbf{Poc}(L_{\mathbf{Poc}}[2], -)}{\overset{\mathbf{Arr}}{\rightrightarrows}} \mathbf{Set} \quad (\cong)$$

The Representing Object

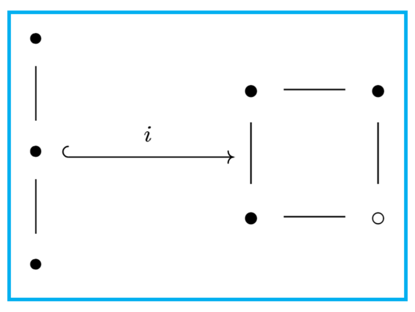
$L_{\mathbf{Poc}}[2] :=$



A Model Example: Set $P = \mathbb{2}$

In this case, the set of arrows inherits the structure of a median algebra either as a subalgebra of $(\mathbb{2}, m_{\mathbf{Bool}}) \times (\mathbb{2}, m_{\mathbf{Bool}})$, or equivalently as the dual of $L_{\mathbf{Poc}}[2]$.

$$\mathbf{Arr}(\mathbb{2}) \cong \mathbf{Poc}(L_{\mathbf{Poc}}[2], \mathbb{2}) \in \mathbf{Med}$$
$$\{(0,0), (0,1), (1,1)\} \subseteq \mathbb{2} \times \mathbb{2} \in \mathbf{Med}$$



the canonical inclusion $i : \mathbf{Arr}(\mathbb{2}) \longrightarrow \mathbb{2} \times \mathbb{2}$ is a morphism in **Med**.

and in general,

$$\mathbf{Arr}(P) \cong \mathbf{Poc}(\mathbf{Med}(\mathbf{Arr}(\mathbb{2}), \mathbb{2}), P) \quad \forall (P, \perp, \neg, \to) \in \mathbf{Poc}.$$

Lemma 1 (K. - Mine)

Our first result states that a set which is simultaneously a poc-set and median algebra is a Boolean algebra if and only if the two structures are compatible in the sense that 1) the arrows of the poc-set form a median algebra; and 2) $\neg$ is an endomorphism of B as a median algebra.

If $(B, m) \in$ **Med** & $(B, \perp, \neg, \to) \in$ **Poc**, then $(B, \perp, \neg\perp, \neg, \wedge_m, \vee_m, \to) \in$ **Bool** iff $\mathbf{Arr}(B) \in$ **Med** and $\neg : (B, m) \longrightarrow (B, m)$ is a morphism in **Med**, i.e., $\neg \circ m = m \circ \neg^3$.

where $x \wedge_m y := m(x, \perp, y)$ and $x \vee_m y := m(x, \neg\perp, y) \quad \forall x, y \in B$.

Definition: A triple of bifunctors $F : \mathbb{C} \times \mathbb{D} \longrightarrow \mathbb{B}$, $G : \mathbb{C}^{op} \times \mathbb{B} \longrightarrow \mathbb{D}$, $H : \mathbb{D}^{op} \times \mathbb{B} \longrightarrow \mathbb{C}$ equipped with natural isomorphisms $\mathbb{C}(C, H(D, B)) \cong \mathbb{B}(F(C, D), B) \cong \mathbb{D}(D, G(C, B))$ defines a **two-variable adjunction** [5].
We show that there is a two-variable adjunction with $\mathbb{B} :=$ **Bool**, $\mathbb{C} :=$ **Med**, $\mathbb{D} :=$ **Poc**, $H :=$ **Poc**$(-, -)$, and $G :=$ **Med**$(-, -)$.
The construction of the third bifunctor can be understood by analogy with the construction of tensor products in $\mathbf{Vect}_k$, which is the corresponding bifunctor of another famous two-variable adjunction.

Currying $\quad \mathbf{Set}(X, \mathbf{Set}(Y, Z)) \cong \mathbf{Set}(X \times Y, Z) \cong \mathbf{Set}(Y, \mathbf{Set}(X, Z))$

Tensor-Hom $\quad \mathbf{Vect}_k(U, \mathbf{Vect}_k(V, W)) \cong \mathbf{Vect}_k(U \otimes V, W) \cong \mathbf{Vect}_k(V, \mathbf{Vect}_k(U, W))$

Lemma 2 (K. - Mine)

A Tensor Product of Poc-sets and Median Algebras

$\exists$ a bifunctor $\otimes : \mathbf{Med} \times \mathbf{Poc} \longrightarrow \mathbf{Bool}$, and natural isomorphisms

$$\mathbf{Med}(M, \mathbf{Poc}(P, B)) \cong \mathbf{Bool}(M \otimes P, B) \cong \mathbf{Poc}(P, \mathbf{Med}(M, B)).$$

Remark: The general definition of two-variable adjunction involves three categories, yet in many of the examples from nature, two or more of the categories coincide. Lemma 2 above is the only example of a two-variable adjunction that the authors are aware of that takes advantage of the full generality afforded by the definition.
Question: What other examples of two-variable adjunctions involve three distinct categories?

Proof Sketch:

By the universal property of the tensor product, there is a bijection between Boolean homomorphisms and **bicompatible maps**:

$$\mathbf{Bool}(M \otimes P, B) \cong \mathbf{Bicomp}(M \times P, B) := \{f : M \times P \longrightarrow B \mid \text{(I) \& (II)}\}$$

(I) $\forall x \in M \quad p \mapsto f(x, p)$ is a morphism in **Poc**
(II) $\forall p \in P \quad x \mapsto f(x, p)$ is a morphism in **Med**

Currying

$$\mathbf{Med}(M, \mathbf{Poc}(P, B)) \cong \mathbf{Bicomp}(M \times P, B) \cong \mathbf{Poc}(P, \mathbf{Med}(M, B)).$$

It remains to give a construction of the tensor product. Let $F_{\mathbf{Bool}} : \mathbf{Set} \longrightarrow \mathbf{Bool}$ be the free Boolean algebra functor (i.e., $\mathcal{PP}$), then the tensor product of $(M, m) \in$ **Med** and $(P, \perp, \neg, \to) \in$ **Poc** is

$$M \otimes P \cong F_{\mathbf{Bool}}(M \times P)/\sim$$

where $\sim$ is the congruence generated by

$$(m(x, y, z), p) \sim m_{\mathbf{Bool}}((x, p), (y, p), (z, p))$$
$$(x, p) \wedge (x, q) \sim (x, p) \quad \forall p \to q$$
$$(x, \neg p) \sim (x, p)^c$$
$$(x, \perp) \sim \varnothing$$

Corollary 1: The Dualization Adjunction

In particular, taking $B = \mathbb{2}$ we find that $\mathbf{Med}(-, \mathbb{2})$ and $\mathbf{Poc}(-, \mathbb{2})$ are mutual right adjoints. The induced monads are a dual pair of double dualization monads.

$$\mathbf{Med}(M, \mathbf{Poc}(P, \mathbb{2})) \cong \mathbf{Poc}(P, \mathbf{Med}(M, \mathbb{2}))$$

Specializing further to the case where both M and P are free, we see that $\mathcal{P}$ is its own right adjoint.

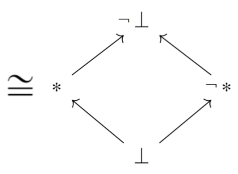
Corollary 2: Booleanization

Let $\mathbb{1} := \{*\} \in$ **Set** be a singleton.
By the universal property of $\otimes$, $- \otimes F_{\mathbf{Med}}\mathbb{1} \dashv U^{\mathbf{Med}} : \mathbf{Bool} \longrightarrow \mathbf{Med}$, and similarly $- \otimes F_{\mathbf{Poc}}\mathbb{1} \dashv U^{\mathbf{Poc}}$.

$$\Longrightarrow \quad F^{\mathbf{Med}} \cong F_{\mathbf{Med}}\mathbb{1} \otimes - \qquad F^{\mathbf{Poc}} \cong - \otimes F_{\mathbf{Poc}}\mathbb{1}$$

Note:

$F_{\mathbf{Med}}\mathbb{1} \cong \mathbb{1}$ $\qquad$ $F_{\mathbf{Poc}}\mathbb{1} \cong$



A similar construction can be used to build a tensor product of sets and Boolean algebras for which $X \otimes B$ is isomorphc to the X-fold copower of B, so $\mathbf{Set}(X, \mathbf{Bool}(B, B')) \cong \mathbf{Bool}(X \otimes B, B') \cong \mathbf{Bool}(B, \mathbf{Set}(X, B))$ is a two-variable adjunction.

Theorem 1 - A Dual Pair of Codensity Monads

The codensity monad of a functor $R : \mathcal{C} \longrightarrow \mathcal{D}$ is, by definition, the right Kan extension of R along itself. In the presence of a left adjoint L, the codensity monad $\mathbf{Ran}_R R$ agrees with the adjunction-induced monad. However, if $\mathcal{C}$ is the subcategory of finitely presented objects in $\mathcal{D}$, and R is the associated inclusion, then some basic size considerations suggest that an adjunction $L \dashv R$ is unlikely to exist. Nevertheless, R may still induce a codensity monad, and in fact, this much is guaranteed whenever $\mathcal{C}$ is essentially small and $\mathcal{D}$ has limits of a particular form. In the table below, we review relevant examples of codensity monads (and their algebras) induced by inclusions, then state our contributions to this list. We also say a few words about the proof and note that, up to a point, all examples listed below can be treated similarly.

Inclusion	Codensity Monad (endofunctor part)	Algebras
$\mathbf{FinSet} \overset{i_{\mathbf{Set}}}{\hookrightarrow} \mathbf{Set}$	$\mathbf{Bool}(\mathbf{Set}(-, \mathbb{2}), \mathbb{2})$	Compact Hausdorff spaces
$\mathbf{FinVect}_k \overset{i_{\mathbf{Vect}_k}}{\hookrightarrow} \mathbf{Vect}_k$	$\mathbf{Vect}_k(\mathbf{Vect}_k(-, k), k)$	Linearly compact vector spaces
⋮	⋮	⋮

Theorem 1 (K. - Mine)

The dual pair of inclusions $\mathbf{Fin}\mathbb{C} \overset{i_{\mathbb{C}}}{\hookrightarrow} \mathbb{C}$ where, $\mathbb{C} \in \{\mathbf{Med}, \mathbf{Poc}\}$, induces a dual pair of codensity monads: $\mathbf{Hom}_{\mathbb{C}^\circ}(\mathbf{Hom}_{\mathbb{C}}(-, \mathbb{2}), \mathbb{2})$ where, $\mathbf{Poc}^\circ := \mathbf{Med}$ & $\mathbf{Med}^\circ := \mathbf{Poc}$.

Conjecture

The objects in the image of the double dualization functors on **Poc** and **Med** are compact, Hausdorff, and totally disconnected topological spaces (i.e., **Stone spaces**). We conjecture that the categories of algebras for the codensity monads of the inclusions $i_{\mathbb{C}}$ are equivalent to **StonePoc** and **StoneMed**.

Proof Strategy

Let $\mathbb{D} \in \{\mathbf{Med}, \mathbf{Poc}, \mathbf{Set}, \mathbf{Vect}_k\}$, $X \in \mathbb{D}$, and $F \in \mathbb{C} := \mathbf{Fin}\mathbb{D}$. Then the **evaluation map** $\mathrm{ev}_F : F \longrightarrow \mathbf{Hom}_{\mathbb{C}^{op}}(\mathbf{Hom}_{\mathbb{C}}(F, \mathbb{2}), \mathbb{2})$ is an isomorphism. We follow the proof given for vector spaces in [4], but define our integration operations using the pushforward: given $\mu \in X^{\circ\circ} := \mathbf{Hom}(\mathbf{Hom}(X, \mathbb{2}), \mathbb{2})$ and $g \in \mathbf{Hom}(X, F)$

$$\int_X g\, d\mu := \mathrm{ev}_F^{-1}(g_*(\mu))$$

$$\begin{tikzcd} X^{\circ\circ} \arrow[r, "g^{\circ\circ}"] & F^{\circ\circ} \arrow[r, "\mathrm{ev}_F^{-1}"] & F \\ \mu \arrow[r, mapsto] & g_*(\mu) \arrow[r, mapsto] & \int_X g\, d\mu \end{tikzcd}$$

The properties required of these integration operators then follow formally. This gives a map from the double dual of X to the codensity monad of the appropriate inclusion. The map in the other direction $\mathbf{Nat}(\mathbf{Hom}(X, i_{\mathbb{D}}(-)), U) \longrightarrow X^{\circ\circ}$, where $U : \mathbb{C} \longrightarrow \mathbf{Set}$ is forgetful, sends a natural transformation to its component at the dualizing object ($\mathbb{2}$ or k). This requires special considerations for each separate category; to show that the component is an element of the double dual, one invokes naturality on special maps.

Theorem 2 - Isbell Duality

The connection to Isbell duality begins with the observation that the structure maps for $\mathbb{2}$ (either as a poc-set or median algebra) are morphisms in the opposite category. For instance, we saw in Lemma 1 that $\mathbf{Arr}(\mathbb{2}) \in$ **Med**. On the other hand, its Boolean majority function is the unique non-principal poc-ultrafilter in $\mathbb{2}^3$. Let Δ_X be the diagonal map on X and π_j, $j \in \{1, 2\}$, denote the j^{th}-projection map, then

$$\begin{tikzcd} \mathbb{2}^2 \arrow[rr, "\pi_1"] \arrow[dr, "\Delta_{\mathbb{2}} \times \mathrm{id}_{\mathbb{2}}"'] & & \mathbb{2} \\ & \mathbb{2}^3 \arrow[ur, "m_{\mathbf{Bool}}"'] & \end{tikzcd}$$

The triangle above is a commutative diagram in **Poc**. It expresses the fact that $m_{\mathbf{Bool}}$ satisfies axiom (M.i).

$$\begin{tikzcd} \mathbb{1} \arrow[r, "0"] \arrow[d, "{(0,1)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{2} \arrow[d, "{\langle \mathrm{id}_{\mathbb{2}}, \neg \rangle}"] \\ \mathbf{Arr}(\mathbb{2}) \arrow[r, "i"] & \mathbb{2} \times \mathbb{2} \end{tikzcd}$$

The diagram to the left is a pullback square in **Med**. Commutativity says $0 \leqslant \neg 0$ in the poc-set $(\mathbb{2}, 0, \neg, \to)$, while universality says that 0 is the unique element less than its own negation. In other words, the universal property verifies (P.ii) for the initial poc-set.

Isbell duality refers to a duality between presheaves and copresheaves on a category. In this case, we restrict to the subcategories of finitely continuous functors. We write $[\mathbb{C}, \mathbb{D}]_{\text{f.c.}}$ for the category of finitely continuous functors $\mathbb{C} \longrightarrow \mathbb{D}$.

Theorem 2 (K. - Mine)

$$[\mathbf{FinPoc}, \mathbf{Set}]_{\text{f.c.}} \simeq \mathbf{Med}, \text{ and } [\mathbf{FinMed}, \mathbf{Set}]_{\text{f.c.}} \simeq \mathbf{Poc}.$$

Remark: The Isbell perspective leads to an alternate proof of Theorem 1.

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