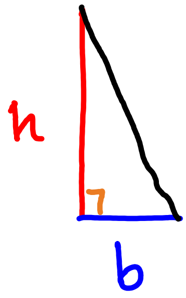
Q: What is the area of a right triangle whose height is equal to twice the length of the base?

A:



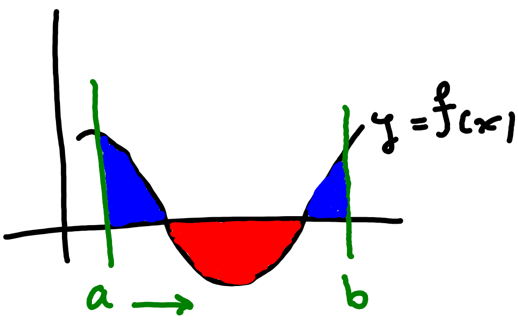
$$h = 2b$$

$$\boxed{\text{Area} = \tfrac{1}{2}(\text{base})\cdot(\text{height})}$$

$$= \tfrac{1}{2}bh = \tfrac{1}{2}b(2b) = \tfrac{2}{2}b^2 = b^2$$

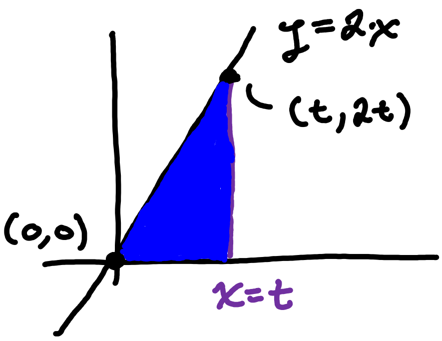
Q: Compute $\int_0^t 2x\,dx$.

A: Recall that



$$\underbrace{\int_a^b f(x)\,dx}_{\text{"the definite integral of the function } f \text{ on the interval from } a \text{ to } b\text{"}} = \overbrace{\text{Blue Area} - \text{Red Area}}^{\text{"Signed area between the graph of } f \text{ \& the } x\text{-axis"}}$$

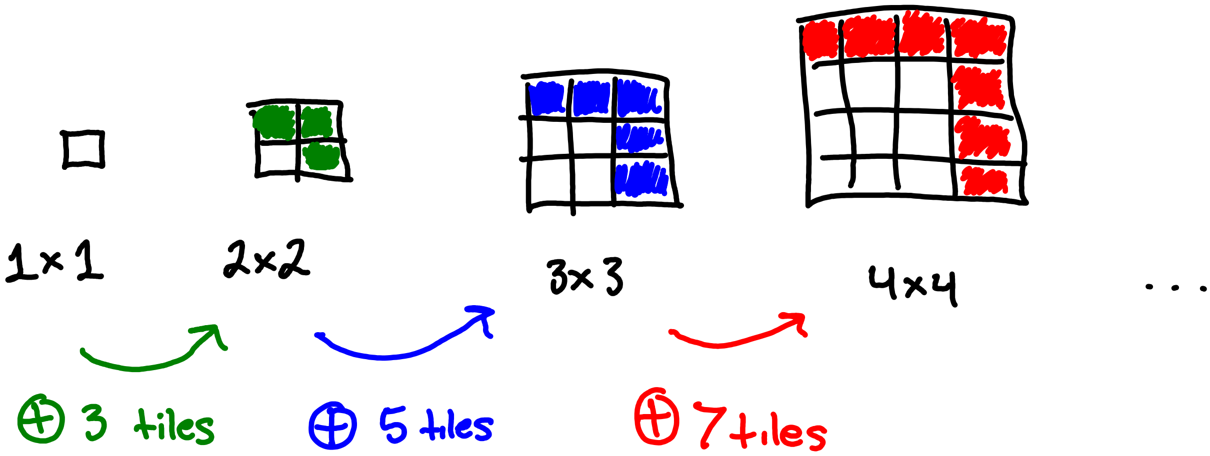
So we are being asked to find the area drawn below



$$\int_0^t 2x\,dx = \boxed{\text{Area of Right } \Delta \text{ with base length } t \text{ \& height } 2t}$$

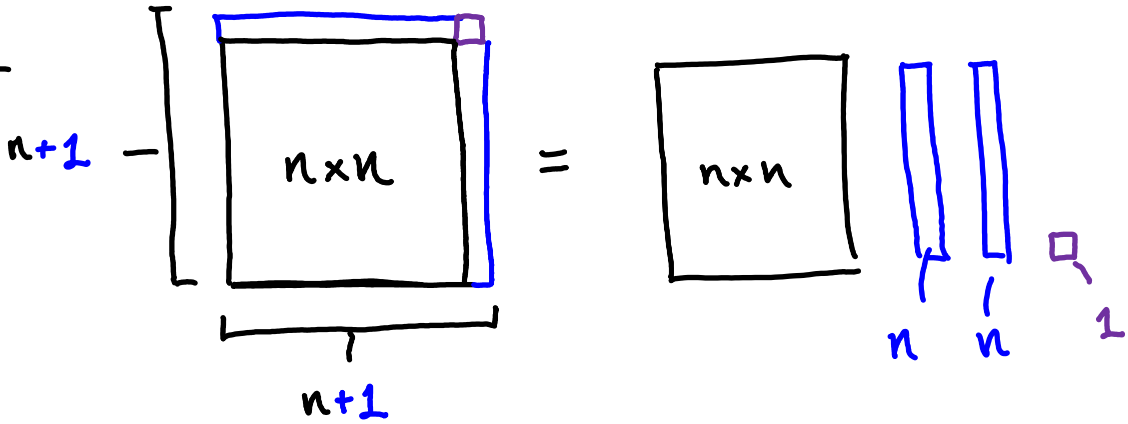
$$= t^2$$

Q:



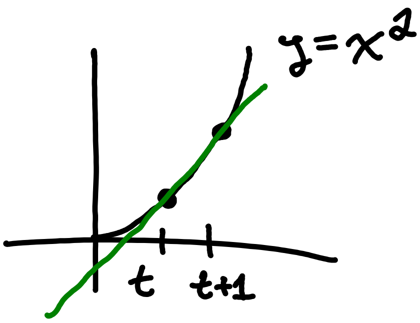
How many tiles do we need to add to make an $n+1$ by $n+1$ square if we already have an n by n square of tiles?

A:



$$= 2n+1$$

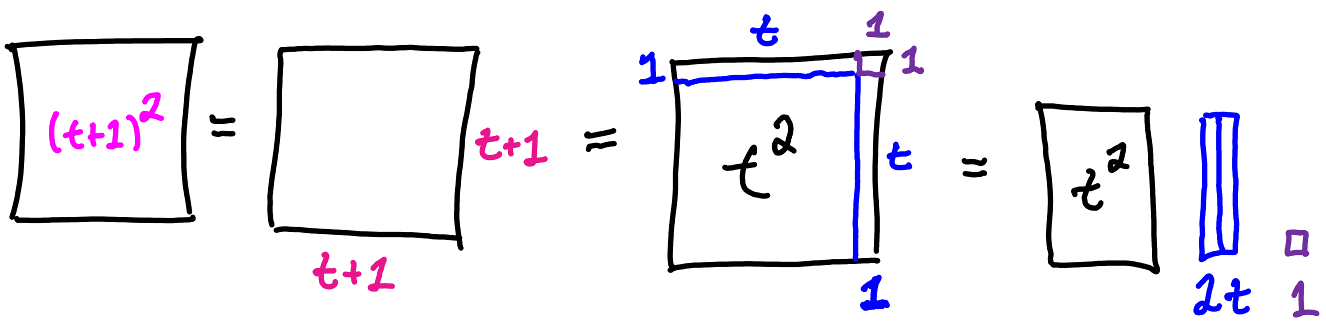
Q: Approximate the derivative of the function $g(x) = x^2$ at $x = t$ by finding the slope of the secant line drawn below.



$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{g(x_2) - g(x_1)}{x_2 - x_1}$$

$$= \frac{(t+1)^2 - t^2}{(t+1) - t}$$

$$(t+1)^2 = (t+1)(t+1) = t^2 + t + t + 1 = t^2 + 2t + 1$$



Subtracting off t^2 & dividing by the "run" to calculate the slope gives

in this case the number of tiles to be added to the side length of the square

$$\frac{t^2+2t+1-t^2}{(t+1)-t} = \frac{2t+1}{1} = 2t+1$$

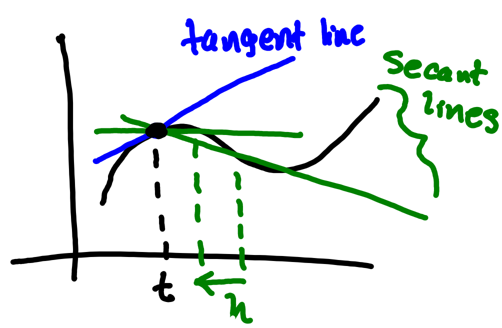
Replacing the 1 in the above computation by the variable h & taking the limit as $h \to 0$, we get the derivative of $g(x)$ at the point $x=t$.

$$\boxed{g'(t) = \lim_{h\to 0} \frac{g(t+h)-g(t)}{h}}$$

Slope of the tangent line to $g(x)$ @ $x=t$

take the limit as your approximation gets better & better.

Slope of secant line passing through $(t, g(t)), (t+h, g(t+h))$



$g(x) = x^2$ distribute

$$\lim_{h\to 0} \frac{g(t+h)-g(t)}{(t+h)-t} = \lim_{h\to 0} \frac{(t+h)^2 - t^2}{h} = \lim_{h\to 0} \frac{t^2+2th+h^2-t^2}{h}$$

$$= \lim_{h\to 0} \frac{2th+h^2}{h} = \lim_{h\to 0} 2t+h = 2t$$

now that each term in the numerator has a factor of h, the h in the denominator may be canceled out

"these now cancel out"

$$\boxed{\text{So} \quad g'(t) = 2t}$$

Moral: Calculus $\sim$ Approximation

IF you can approximate a quantity

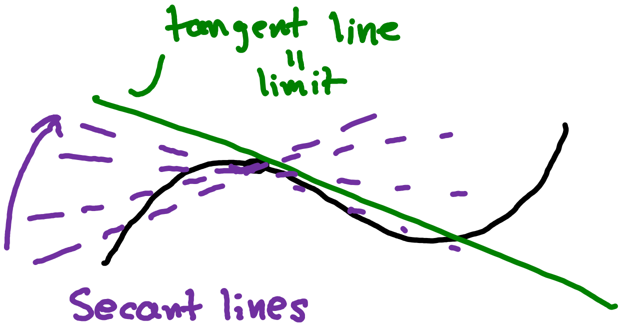
& you have a method of improving your approximations

THEN you can arrive at exact answers by taking the limit

eg.

The Derivative

$$\underbrace{\lim}_{\text{Successively improve the approximations}} \underbrace{\left(\begin{array}{c}\text{Slope of} \\ \text{Secant lines}\end{array}\right)}_{\text{approximate the slope of the tangent line}} = \underbrace{\begin{array}{c}\text{Slope of} \\ \text{tangent line}\end{array}}_{\text{the derivative}}$$



lim (Average Rate of Change) = "Instantaneous Rate of Change"

Similarly

The Definite Integral

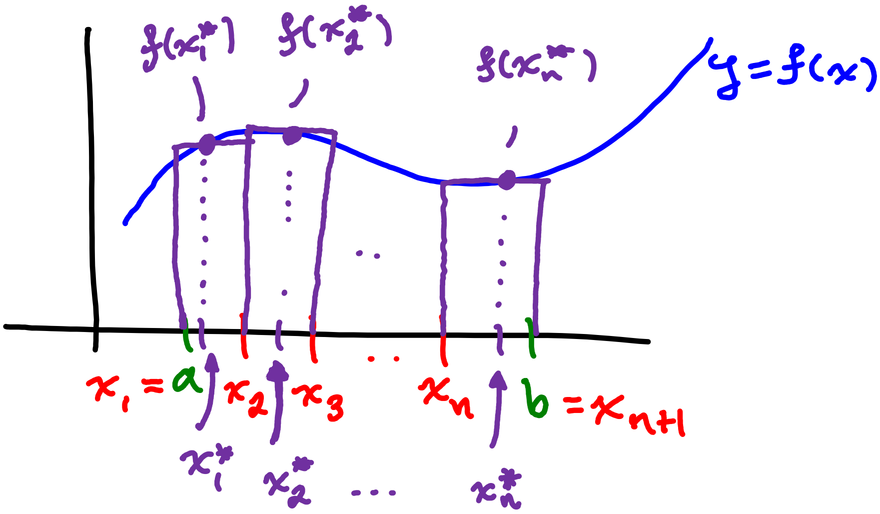
$$\lim \sum \left(\begin{matrix}\text{Areas of}\\ \text{Rectangles}\end{matrix}\right) = \begin{matrix}\text{Area under}\\ \text{the curve}\end{matrix}$$

Successively improve approximations

approximates the area under the graph of a function (or curve)

the definite integral

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$



a = Starting x-value

b = ending x-value

n = number of rectangles to use in the approximation

divide up the interval $[a,b]$ into n parts

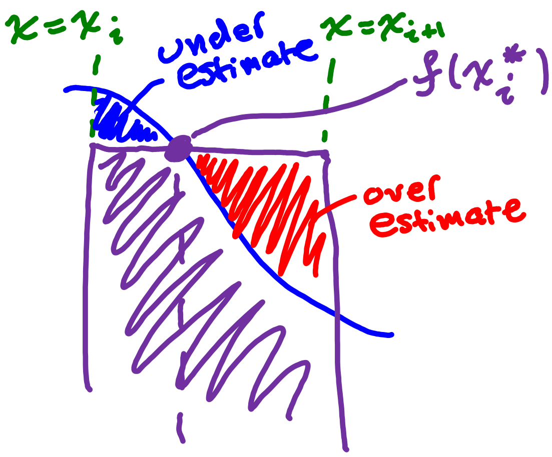
$a=x_1, x_2, x_3, \ldots, x_n, x_{n+1}=b$

(using the $n+1$ dividing lines above)

x_i^* = pick any point in the interval between dividing lines x_i to x_{i+1} & use the value $f(x_i^*)$ as the height of the i^{th} rectangle

Δx_i = width of the i^{th} approximating rectangle

Zooming in to the top of the i^{th} rectangle we see



$$\underbrace{\lim_{\substack{n \to \infty \\ \Delta x_i \to 0}}}_{\substack{\text{use thinner \& thinner rectangles} \\ \Rightarrow \text{must keep increasing the \# of rectangles} \\ \Rightarrow \text{better approximation}}} \underbrace{\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i}_{\substack{\text{"Riemann Sums"} \\ \text{approximate area}}} = \text{Exact Area under the curve}$$

Later on, there will be more examples of this moral in action such as

$$\underbrace{\text{Infinite Sums}}_{\text{"Series"}} = \underbrace{\lim_{n\to\infty}}_{\substack{\text{by adding up more and more} \\ \text{terms the value gets closer and} \\ \text{closer to the precise answer}}} \Big(\underbrace{\text{Partial Sums}}_{\substack{\text{sum up the first } n \text{ terms of the infinite} \\ \text{sum (approximate the full infinite sum} \\ \text{using finitely many of the numbers involved)}}} \Big)$$

$$\underbrace{\text{Functions}}_{\text{Taylor Series}} = \underbrace{\lim_{n\to\infty}}_{\substack{\text{approximation} \\ \text{keeps getting} \\ \text{better}}} \Big(\underbrace{\text{Taylor Polynomials}}_{\substack{\text{approximations of a function taking only its} \\ \text{first } n \text{ derivatives into consideration}}} \Big)$$

Summary

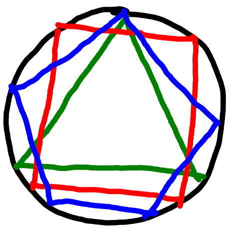
$\boxed{\text{Limits}} + \boxed{\text{Approximation}} = \boxed{\text{Exact Answers}}$

Example 1: Finding the slope of the tangent line to a curve

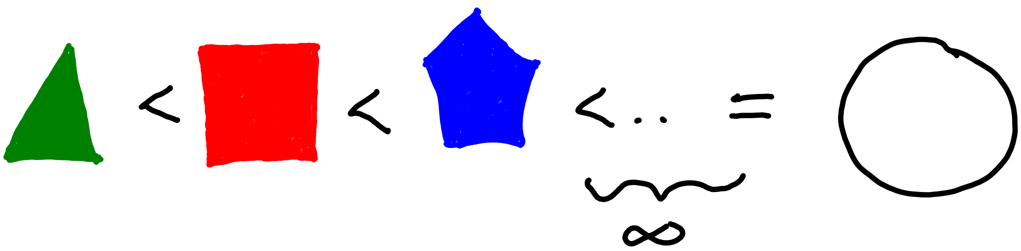
Example 2: Finding area under a curve

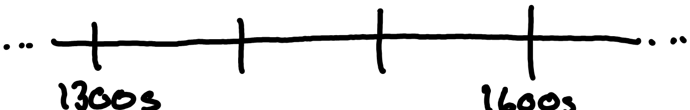
The solutions to these problems are very old...

Greece $> 2{,}000$ years ago



Area of circle
=
limit of areas of
approximating polygons
(inscribed regular n-gons)





people begin using the idea of the derivative to solve problems

The Fundamental Theorem of calculus

↳ The operations of taking the derivative & integration of functions are actually related

People did not realize that the two problems were related until about 400 years ago

Q: So what exactly is the relationship between

- the slope of the tangent line &
- the area under the curve

A: They're opposites

looking back @ our examples from before

$$f(x) = 2x$$

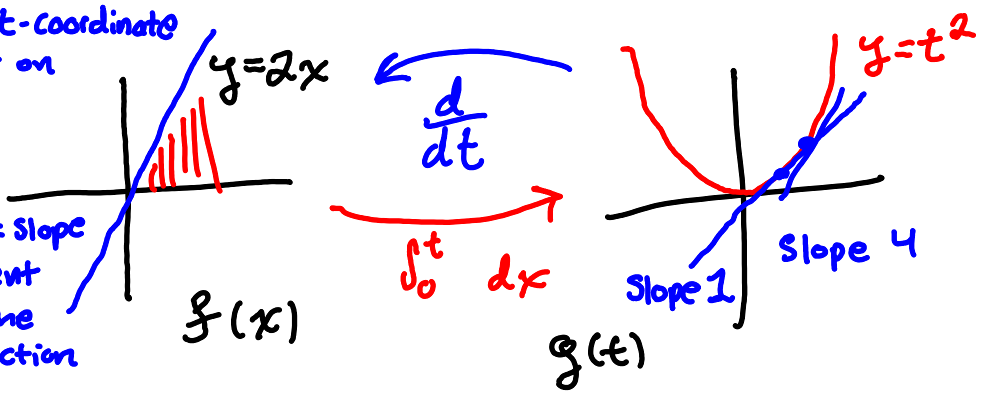
$$g(t) = \int_0^t f(x)\,dx = \int_0^t 2x\,dx = t^2$$

the "area function" of $f(x)$

$$\frac{d}{dt} g(t) = g'(t) = \lim_{h \to 0} \frac{g(t+h) - g(t)}{h} = 2t$$

x-axis
inputs = t-coordinate of point on $y = t^2$

y-axis
outputs = slope of tangent line to the area function



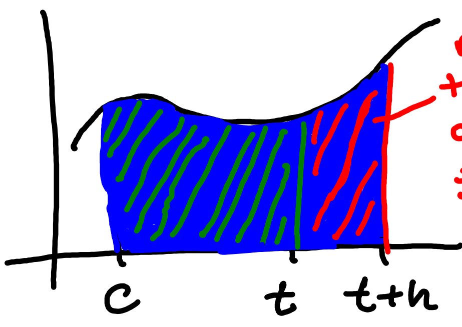
t-axis
inputs = length of base of Right △

y-axis = area of the △ if the height is $2 \times$ base

"The derivative of the area function is the starting function"

The operations cancel with one another

$$\underbrace{\frac{d}{dt}\int_0}_{\text{erase this}}^{t} f(\underset{\uparrow \text{ replace with } t}{x}) \overbrace{dx}^{\text{erase this}} = f(t)$$



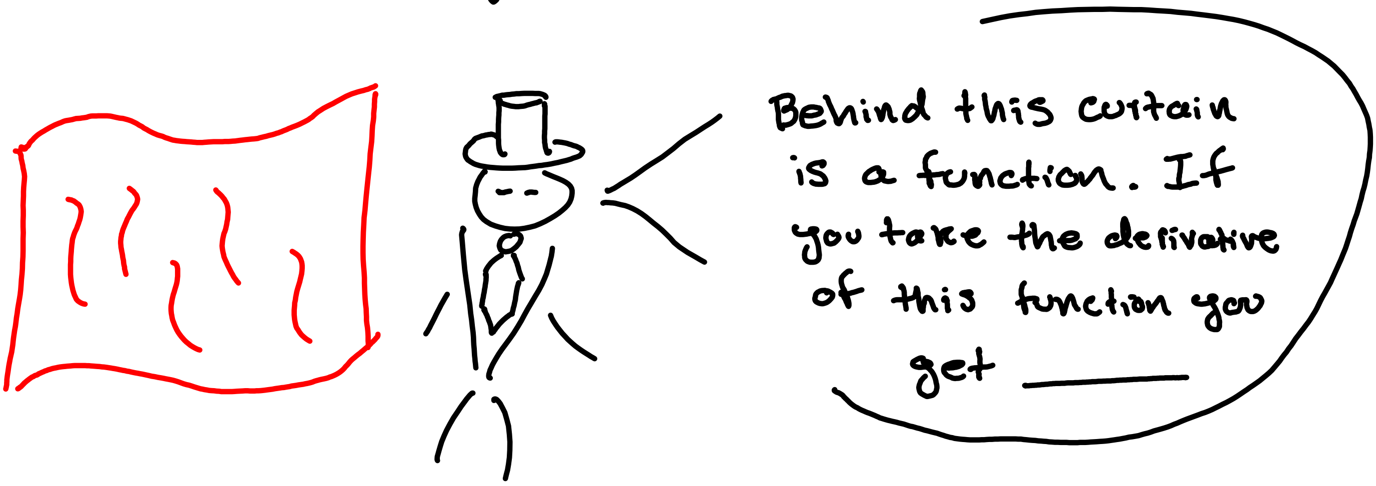
roughly the size of an $f(t) \times h$ rectangle

Here is why this is reasonable to expect

$$\frac{\int_c^{t+h} f(x)\,dx - \int_c^{t} f(x)\,dx}{h} = \frac{\int_t^{t+h} f(x)\,dx}{h} \approx \frac{f(t)\cdot h}{h} = f(t)$$

look @ picture on previous page

Here is another way to think about it



Q: What is behind the curtain?

take the derivative $\frac{d}{dx}$

Out pops

$$\frac{d}{dx}F(x) = F'(x) = 2x$$

Since we know $\frac{d}{dx}x^2 = 2x$ we see that a correct answer to the question would be $\boxed{F(x) = x^2}$ but there are, of course, other perfectly valid answers

Definition: We say $F(x)$ is an antiderivative of $f(x)$ if its derivative equals $f(x)$, i.e.

$$\boxed{F'(x) = f(x)}$$

Idea: FToC Says $\boxed{\text{Antiderivatives}} \approx$ Area Functions

why would this be true?

$$\frac{d}{dt}\int_0^t f(x)\,dx \approx \frac{d}{dt}\boxed{F(t)} = F'(t) = f(t)$$

$$\frac{d}{dt}\int_0^t f(x)\,dx \underset{\text{(FToC)}}{=} f(t) = F'(t) = \frac{d}{dt}\boxed{F(t)}$$

assuming f has an antiderivative

So

$$\int_0^t f(x)\,dx \quad \& \quad F(t)$$

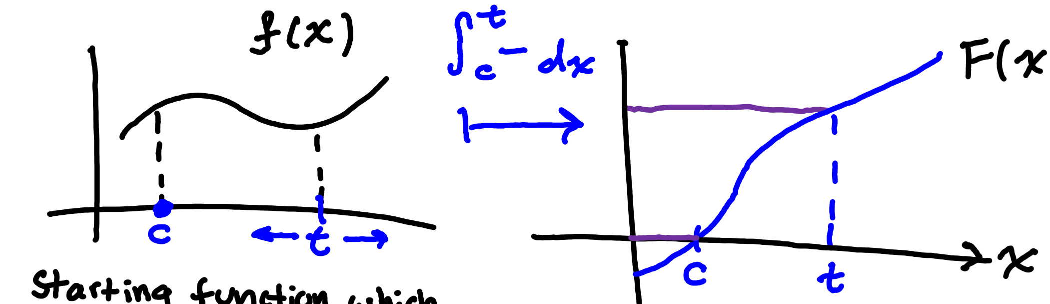
have the same derivative

⚠ this does not mean $\int_0^t f(x)\,dx = F(t)$

however, it will be the case that these two functions differ by a constant at worst

In Fact:

$$\underbrace{F'(t) = f(t)}_{\text{If } f \text{ has an antiderivative}} \implies \underbrace{F(b) - F(a) = \int_a^b f(x)\,dx}_{\text{then it controls the area function}}$$



Starting function which has an antiderivative

area function

Consider

$$\underbrace{F(t) - F(c)}_{\text{how much more area has accumulated since } c \text{ by the time } t \text{ has been reached}}$$

c = fixed but unknown constant (arbitrarily picked location from which to begin recording how much area is accumulating)

t = variable endpoint of integration

choosing this differently is what causes different antiderivatives to differ by a constant

What happens if we pick a different starting point?

$$\frac{d}{dt}\int_c^t f(x)\,dx = \frac{d}{dt}\Big(F(t) - \underbrace{F(c)}_{\substack{\text{derivative} \\ \text{of constant} \\ = 0}}\Big) = F'(t) = f(t)$$

So we don't change the derivative by picking a different start

the constant we picked goes away in the formula because it can only change the antiderivative by adding a constant. Antiderivatives are area functions ⊕ a starting locatin!

$$f(x) \xrightarrow{\text{area function}} \int_c^t f(x)\,dx$$

$$\int_c^t f(x)\,dx \xrightarrow{\text{derivative}} f(x)$$

$$f(x) = F'(x) = \frac{d}{dx}(F(x)+C) \xrightarrow{\text{antiderivative}} F(t) + C$$

$$\int_c^t f(x)\,dx \overset{\text{FToC}}{=} F(t) + \underbrace{C}_{\text{arbitrary constant}}$$

(in this case $C = -F(c)$)

starting point

Calculus I

- limits
- derivatives
  - ↦ constants (get zero)
  - ↦ monomials (power rule)
  - ↦ product rule ⇓ quotient rule
  - ↦ chain rule
  - ↦ trig functions
  - ↦ exponential functions
  - ↦ logarithmic functions
- FToC

Calculus II

Compute integrals (areas of weird shapes) by using FToC and what we understand about derivatives in reverse

- Reverse Power Rule
- Integration by parts (reverse product rule)
- U-Substitution (reverse chain rule)

⋮

$$\frac{d}{dx} a = 0 \quad \text{means} \quad \xrightarrow{\text{FToC}} \quad \int f(x)\,dx = \underbrace{F(x)}_{\text{antiderivative}} + \underbrace{C}_{\text{arbitrary constant}}$$

fixed but unspecified constant

$$\frac{d}{dx} ax = a \quad \text{means} \quad \int a\,dx = ax + C$$

$$\frac{d}{dx} ax^2 = 2ax \quad \text{means} \quad \int 2ax\,dx = x^2 + C$$

Linearity of the Derivative ⟹ Linearity of the Integral

$$\frac{d}{dx}\left(af(x)+bg(x)\right) = af'(x)+bg'(x) \qquad \int (af(x)+bg(x))\,dx = a\int f(x)\,dx + b\int g(x)\,dx$$

⋮

Checking Your Answers

The FToC allows us to check our answers without checking our work directly (i.e. it provides a different means of knowing that a function is the antiderivative of another without needing to integrate from scratch).

If you want to know you've computed an integral correctly, all you must do is take the derivative.

Easy (Calc I)

Hard (Calc II)

taking the derivative of functions

Finding the indefinite integral of a function

this answer can be checked by computing & making sure you get back the function you started with (the one that was to be integrated in the first place)

> So you might as well always check your answers by taking a derivative.

Checking your work (as opposed to checking your answer by the above strategy) means going back through the solution you wrote down to scan for any mistakes, but this is more dangerous due to the power of suggestion - if you made the mistake to begin with, you're less likely to catch it or even know that it as an error. It is always better to have (at least) two different ways of solving a problem than it is to do the same solution twice over.

"Educated Guess" & Check (a last resort/discovery focused integration technique)

A second useful consequence of FToC is that it affords us the ability to discover the answer to problems even when we don't necessarily understand what is going on.

<u>Observation</u> $\xrightarrow{\text{leads to}}$ <u>Educated guess</u> $\xrightarrow{\text{which we then}}$ <u>Check</u> $\frac{d}{dx}(x^3+C)$

derivatives tend to decrease the degree of polynomials by 1

$$\int x^2 = x^3 + C$$

increase exponent by 1

$$= 3x^2$$

oops, off by a constant that's an easy fix though

& modify

Check it →

$$\boxed{\int x^2 = x^3/3 + C}$$