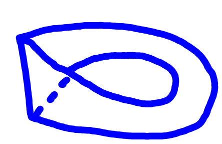
Given a map $f: X \longrightarrow Y$ of topological spaces, let $\mathrm{Cone}(f)$ & $\mathrm{Cyl}(f)$ denote the mapping cone & mapping cylinder of f respectively.

Q1: What (familiar) space is $M := \mathrm{Cyl}(S^1 \xrightarrow{\times 2} S^1)$, i.e., What is a more common name for the mapping cylinder of the two-fold cover of the circle?

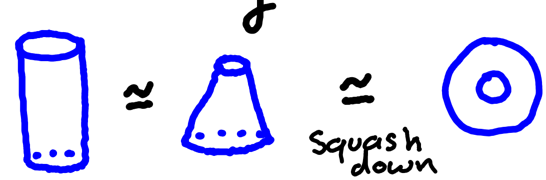
Recall: There is a canonical inclusion of X into $\mathrm{Cyl}(f: X \longrightarrow Y)$. Lets denote it by i_X.

Q2: What (familiar) space is $R := \mathrm{Cone}(S^1 \overset{i_{S^1}}{\hookrightarrow} M)$, i.e, what is a more common name for the mapping cone of the (canonical) inclusion of S^1 (as the domain) into the mapping cylinder of the two-fold cover of S^1?

A1: $M =$ the möbius strip 

Idea - Here are two simpler cases:

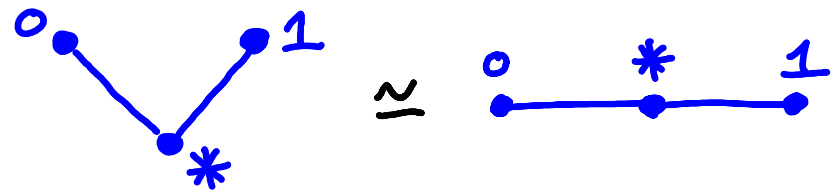
i) The mapping cylinder of the identity on S^1 is obviously just a cylinder/annulus.



Intuitively, the double cover adds a twist.

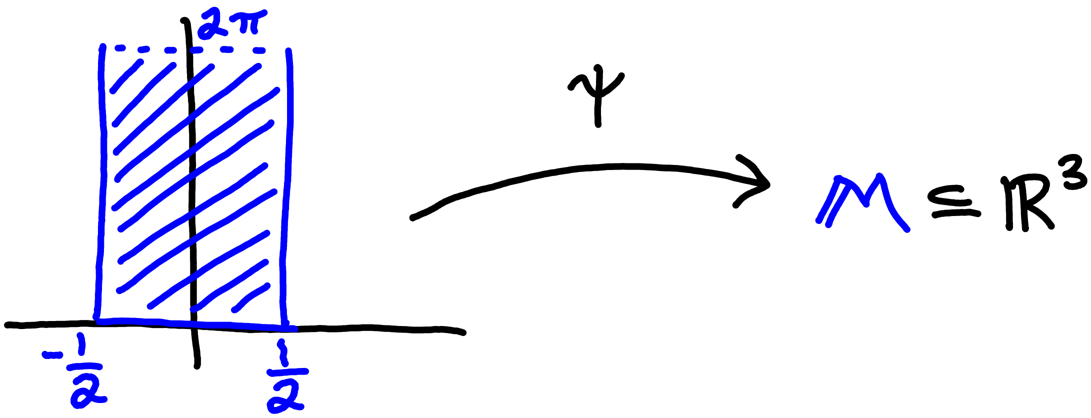
ii) Given discrete spaces $\mathbb{2} := \{0,1\}$ and $\mathbb{1} := \{*\}$ there is a (unique) continuous map $\mathbb{2} \xrightarrow{!} \mathbb{1}$. Of course, the mapping cylinder of $!$ is an interval.

$\mathrm{Cyl}(!) \simeq [0,1]$



Now, the goal is to understand how to combine these two examples with a twist, in some sense.

First, we give an explicit parameterization of the möbius strip $\mathcal{M}$ in $\mathbb{R}^3$.



$$\Psi(x,y) := \vec{\alpha}(y) + x\vec{\beta}(y)$$

$$\vec{\alpha}(\theta) := \langle \cos(\theta), \sin(\theta), 0 \rangle$$

$$\vec{\beta}(\theta) := \langle \sin(\theta/2)\cos(\theta), \sin(\theta/2)\sin(\theta), \cos(\theta/2) \rangle$$

From this description of M it is clear that there is a deformation retraction from M onto S^1 which is essentially obtained by contracting the domain of this parameterization horizontally until all that is left over is the part on the y-axis. In other words, this is the deformation retraction of M onto its "core circle" i.e., $\{(x,y,0) \in \mathbb{R}^3 \mid x^2+y^2=1\}$. To be precise, this homotopy is a continuous map $H : [0,1] \times ([-\tfrac{1}{2}, \tfrac{1}{2}] \times [0, 2\pi)) \longrightarrow \mathbb{R}^3$ which is given by the formula $H(t,x,y) = \psi((1-t)x, y)$.

<u>Recall:</u> There exists a deformation retract of $Cyl(f)$ onto the codomain of f.

<u>Claim:</u> The deformation retract of M guaranteed by the fact above is H.

To verify the claim one needs to check the following:

(i) $S^1 \cong \partial M \cong$ The image of $\{\pm \tfrac{1}{2}\} \times [0, 2\pi)$ under ψ

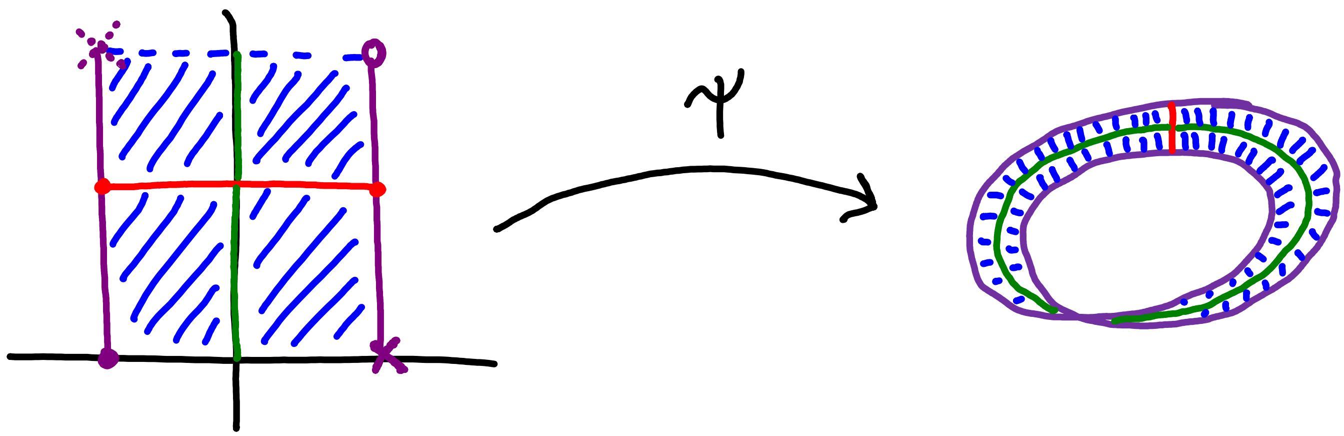
(ii) The image of $\psi|_{\{0\} \times [0,2\pi)}$ is homeomorphic to S^1

(iii) $\forall y \in [0,2\pi)$ the pre-image of the map

$H(1,-,-) : [-1/2, 1/2] \times [0,2\pi) \longrightarrow \mathbb{R}^3$ at $(\cos(y), \sin(y), 0)$

is the interval $[-1/2, 1/2] \times \{y\}$.

Here is a heuristic picture



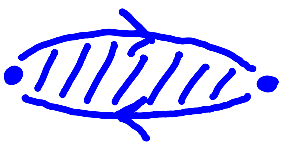
Since each of the three properties follows easily from the formulas for Ψ and H we now have that $\mathcal{M} \cong M$.

<u>A2:</u> R = the real projective plane $\mathbb{RP}^2$

Idea - The inclusion $i_{S^1} : S^1 \hookrightarrow M$ is the inclusion of ∂M into M (which we now know is the möbius strip).

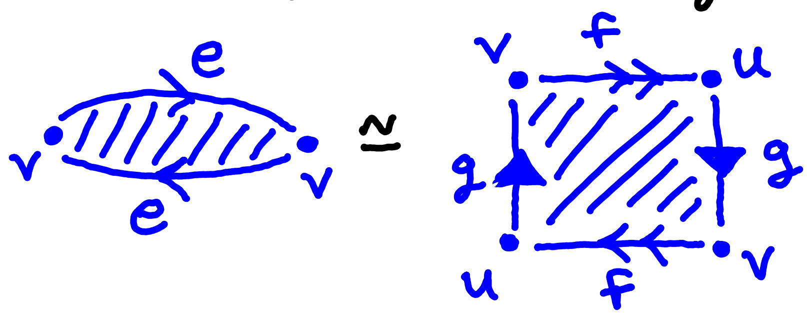
Since the cone on a circle is a disk, this question is asking what space is obtained from gluing a disk to the boundary of a möbius strip, or equivalently what is the quotient space $M/\partial M$. In any case, we expect a (non-orientable) 2-manifold w/out boundary based on the descriptions above. This limits the possibilities. Let's now prove that $R \cong \mathbb{R}P^2$.

Take, as our definition of $\mathbb{R}P^2$, the cell complex

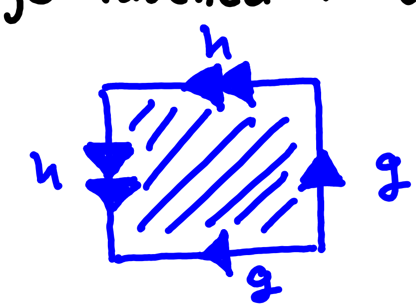


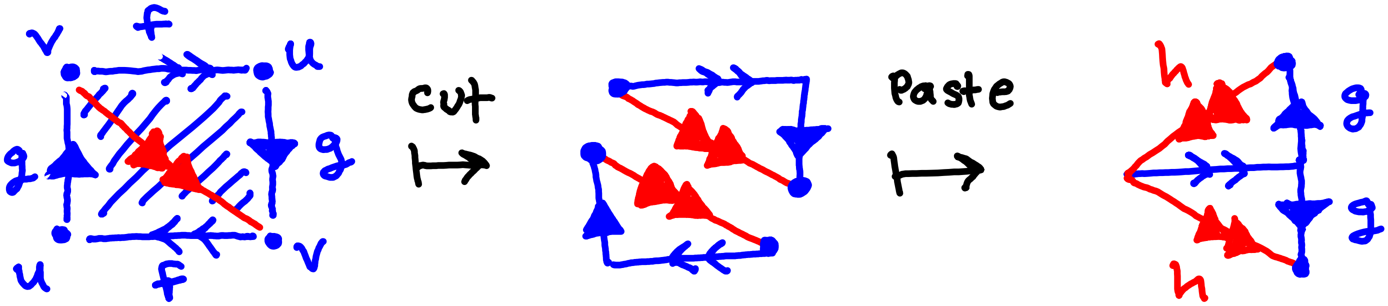
$D^2/\sim$ where $\sim$ identifies antipodes on ∂D^2. The quotient here is another way of describing the attaching map of the 2-cell to the 1-skeleton .

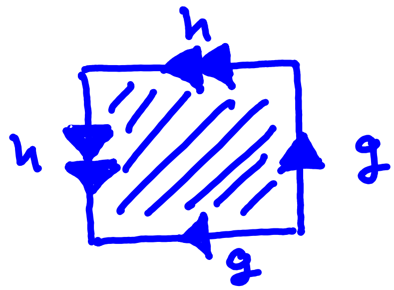
By relabelling each edge & deforming the disk a bit we see that



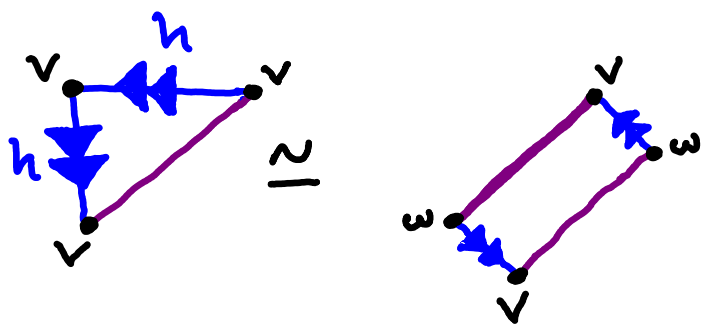
Next, we cut along the main diagonal of the square & paste along the edge labelled f which yields



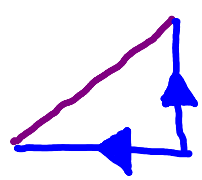
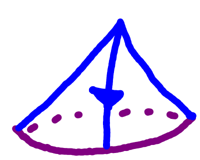


it remains to show that 

is a möbius strip with a disk glued along its boundary. So, where is the möbius strip in this picture? It's the upper left triangular part.



Note: it may be easier to think about this homotopy equivalence in the right-to-left direction first.

On the other hand,  is a disk because if we glue the corresponding sides we get  which can be flattened down to a disk.