Categories & Groupoids - Higgins 1968

Chapter 1

<u>Convention</u>: Functions written on the Right
Functors written on the Left

$$\theta_x f = \theta[(x)f] = (x)[\theta(f)]$$

Functors ~ Canonical Constructions

<u>Setting</u> We mainly consider the following categories

$$\mathcal{S}\text{ets} \supseteq \mathcal{D}\text{irected graphs} \supseteq \text{small } \mathcal{C}\text{ategories} \supseteq \mathcal{G}\text{roupoids}$$

When we only consider objects from a fixed set I...

<u>I-Categories</u>: $\mathcal{S}_I \supseteq \mathcal{D}_I \supseteq \mathcal{C}_I \supseteq \mathcal{G}_I$

- $ob(K_I) = I$
- $\forall \varphi \in mor(K_I)$ φ fixes I

Special case

$I = \{\star\}$ singleton set

$\Rightarrow$ $\mathcal{D}_I = \mathcal{S}$, $\mathcal{C}_I$ = Category of Monoids

& $\mathcal{G}_I$ = Category of Groups.

Functors (6/7)

Forgetful Functors:

$$\mathcal{G} \xrightarrow{V} \mathcal{C} \xrightarrow{V} \mathcal{D} \underset{E}{\overset{V}{\rightrightarrows}} \mathcal{S}$$

V - underlying vertex set

E - underlying edge set

Incidence map defines a natural transformation

$$\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "E(-)"{name=U}] \arrow[r, bend right, "V^2(-)"'{name=D}] & \mathcal{S} \arrow[from=U, to=D, Rightarrow, "\delta"'] \end{tikzcd}$$

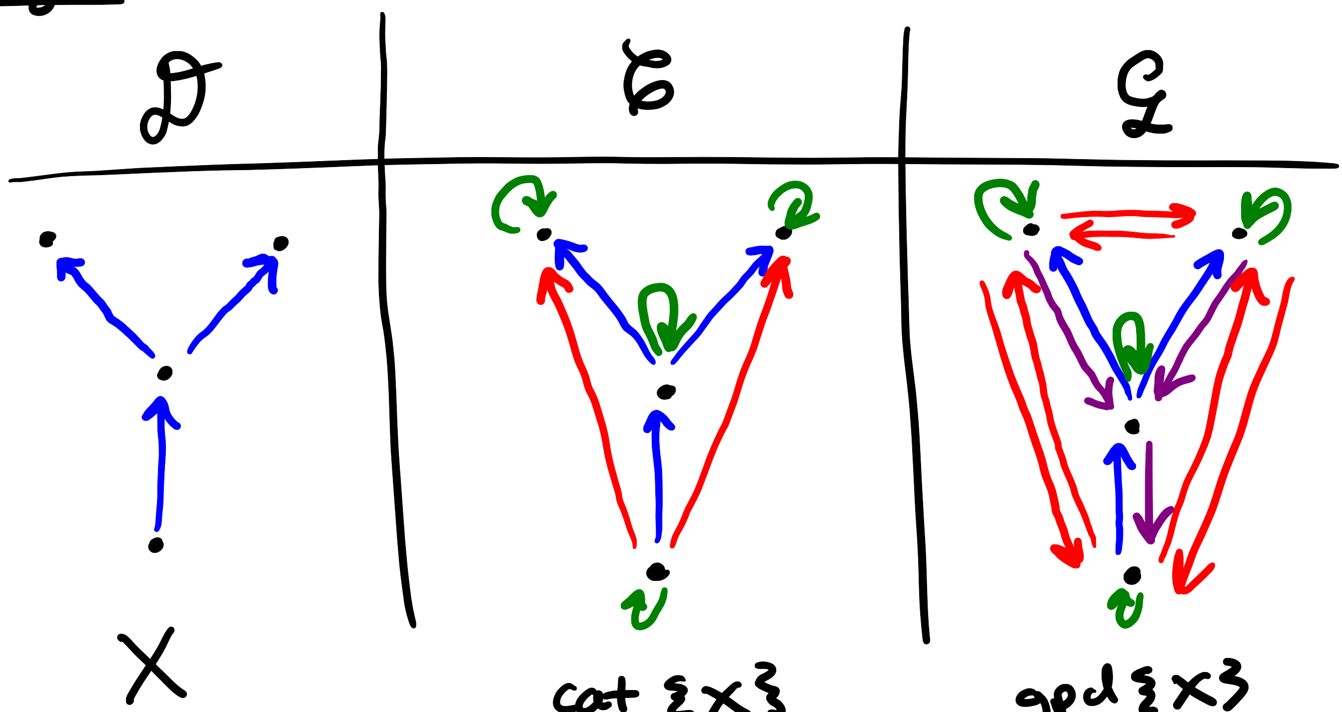
Generating more structure:

$\mathrm{cat}\{-\}: \mathcal{D} \longrightarrow \mathcal{C}$ category generated by a graph

$\mathrm{gpd}\{-\}: \mathcal{D} \longrightarrow \mathcal{G}$ groupoid generated by a graph

$$\mathrm{Cat}\{X\} := \bigcap_{\substack{X \subseteq Y \\ Y \in \mathrm{ob}(\mathcal{C})}} Y \qquad \mathrm{gpd}\{X\} := \bigcap_{\substack{X \subseteq Z \\ Z \in \mathrm{ob}(\mathcal{G})}} Z$$

e.g.



■ = directed edges

■ = identity loops

■ = Composite edge

■ = inverses

Functors (7/7)

Simplicial Groupoids: $\Delta(-) : \mathcal{S} \longrightarrow \mathcal{G}$

$$X \longmapsto (X, X^2)$$

i.e. $V(\Delta(X)) := X$

$E(\Delta(X)) := X^2$

note: $\exists!$ category structure on $\Delta(X)$ since all hom sets have 1 element.

e.g.

$\Delta^0 := \Delta(\{0\}) =$ 

, $\Delta^1 := \Delta(\{0,1\}) =$ 

$\Delta^2 := \Delta(\{0,1,2\}) =$ 

...

Set Theory

$\left\{ \begin{matrix} \text{binary relations on} \\ \text{subsets of } X \end{matrix} \right\} \xleftrightarrow{\sim} \left\{ \begin{matrix} \text{Subgraphs of} \\ \Delta(X) \end{matrix} \right\}$

$\left\{ \begin{matrix} \text{pre-orders on subsets} \\ \text{of } X \end{matrix} \right\} \xleftrightarrow{\sim} \left\{ \begin{matrix} \text{Subcategories} \\ \text{of } \Delta(X) \end{matrix} \right\}$

$\left\{ \begin{matrix} \text{equivalence relations on} \\ \text{subsets of } X \end{matrix} \right\} \xleftrightarrow{\sim} \left\{ \begin{matrix} \text{subgroupoids} \\ \text{of } \Delta(X) \end{matrix} \right\}$

$\left\{ \begin{matrix} \text{groupoids } \Delta(I) \text{ for} \\ I \subseteq X \end{matrix} \right\} \xleftrightarrow{\sim} \left\{ \begin{matrix} \text{full subgroupoids} \\ \text{of } \Delta(X) \end{matrix} \right\}$

$\left\{ \text{unicursal } I\text{-categories} \right\} \xleftrightarrow{\sim} \left\{ \begin{matrix} \text{Subcategories} \\ \text{of } \Delta(I) \end{matrix} \right\}$

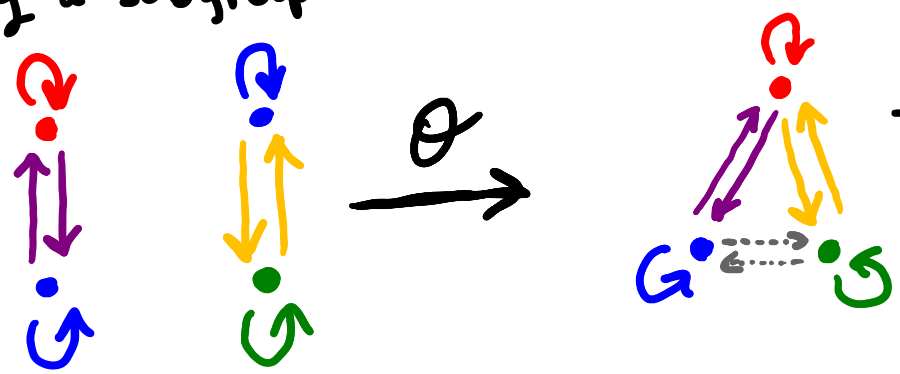
Note: $\Delta(I)$ is final in $\mathcal{C}_I$ & $\mathcal{G}_I$

Notation: δ^* canonical map to $\Delta(I)$

↳ $\mathrm{dom}(\delta^*)$ unicursal $\Rightarrow \delta^*$ injective and the image is a subobject

CH 1 Ex*1: The image of a groupoid map is not neccessarily a subgroupoid of the target.

e.g.



– not even a subcategory

Proposition 1: $K \in \{\mathcal{C}, \mathcal{G}\}$

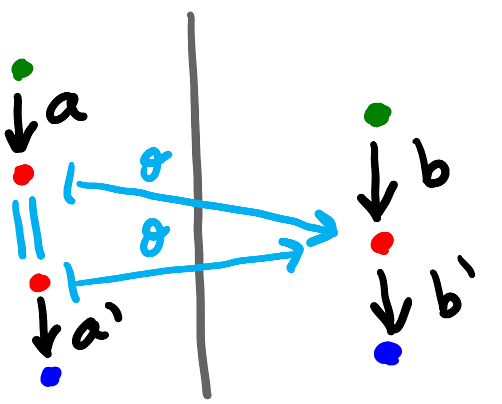
(i) $\theta : A \to B \in \mathrm{mor}(K)$

(ii) $V(\theta)$ injective

$\Rightarrow A\theta \in \mathrm{ob}(K)$

proof: $A \xrightarrow{\theta} B$

(ii) $\Rightarrow$

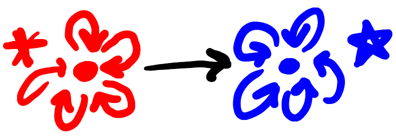


hence (i) $\Rightarrow bb' = (aa')\theta$ □

Corollary: $\theta : G \to H$ group homomorphism $\Rightarrow G\theta \leq H$

subgroup

proof:



$\exists !\, V(\theta) : \{*\} \leftrightarrow \{*\}$ □

Remark: If $K = \mathcal{G}$ in prop 1 we may drop (ii)

<u>proposition 2</u> (cat{–} & gpd{–} are functorial)

(i) $A = \mathrm{cat}\{X\}$ $\quad X \in \mathrm{ob}(\mathcal{D})$

(ii) $\forall \theta : X \longrightarrow \underbrace{B}_{\text{category}}$ $\quad \theta \in \mathrm{mor}(\mathcal{D})$ maps inherit structure of the more general object

$\Rightarrow \exists!\, \theta_* : A \longrightarrow B$ s.t.

(I) $\theta_* \in \mathrm{mor}(\mathcal{C})$

(II) $\theta_*|_X = \theta$

$\boxed{\text{again } \mathcal{K} \in \{\mathcal{C}, \mathcal{G}\}}$

<u>proof:</u> The equalizer, E, of any $\mathcal{K}$-maps is a subobject of the domain so

$$X \subseteq E \Rightarrow \mathrm{cat}\{X\} \subseteq E \quad \square$$

An alternative proof is an immediate consequence of the following

<u>CH 1 Ex *2</u> (inside v.s. outside approach to cat{–}/gpd{–})

If $X \subseteq A$ subgraph of a category

Then $a \in \mathrm{Cat}\{X\} \iff$ either

(i) $\exists i \in V(A)$ s.t. $a = e_i$

(ii) $\exists n \in \mathbb{N} \setminus \{0\}$ s.t. $a = x_1 \dots x_n$, $x_i \in E(X)$ $\forall i \leq n$

& $\delta_2(x_i) = \delta_1(x_{i+1})$ $\forall i < n$

Moreover, $a \in \mathrm{gpd}\{X\} \iff$ either (i) or

(ii)' $\exists n \in \mathbb{N} \setminus \{0\}$ s.t. $a = x_1 \dots x_n$, $x_i \in E(X) \amalg E(X^{op})$ $\forall i \leq n$

& $\delta_2(x_i) = \delta_1(x_{i+1})$ $\forall i < n$

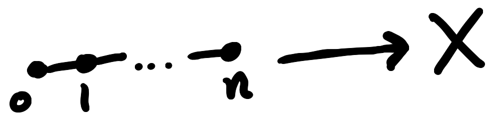
proof: Essentially the same as the group theoretic proof.

(I) Show these elements form a category/groupoid

(II) Show all such elements are in $\mathrm{cat}\{X\}$ or $\mathrm{gpd}\{X\}$

(III) By definition of $\mathrm{cat}\{X\}$ & $\mathrm{gpd}\{X\}$ we are done $\square$

"Outside approach"	"Inside approach"
$\bigcap\{\text{subobjects containing } X\}$	$\{\text{words on } X\} \updownarrow \cong \{\text{paths in } X\}$



Chapters 3 & 4 provide a third proof of this fact using adjunctions.

CH 3 constructs $\vec{P}: \mathcal{D} \longrightarrow \mathcal{C}$ $(\mathrm{cat}\{-\})$

CH 4 constructs $\pi: \mathcal{D} \longrightarrow \mathcal{G}$ $(\mathrm{gpd}\{-\})$

Corollary: $S \subseteq G$ subset of a group

$H := \langle S \rangle \leq G$

$\Rightarrow$ maps $\phi: H \longrightarrow G$ determined by $S\phi$ $\square$

CH 1 Ex *3 (Inverses take care of themselves)

$$\vartheta : A \longrightarrow B \in \operatorname{Mor}(\mathcal{G}) \Rightarrow \operatorname{cat}\{A\vartheta\} \in \operatorname{ob}(\mathcal{G})$$

proof: $x \in A\vartheta \Rightarrow \exists\, a \in A$ s.t. $a\vartheta = x$

$a \in A \in \operatorname{ob}(\mathcal{G}) \Rightarrow \exists\, a^{-1} \in A$

$\vartheta \in \operatorname{mor}(\mathcal{G}) \Rightarrow a\vartheta \cdot a^{-1}\vartheta = (aa^{-1})\vartheta = e_i\vartheta$

Similarly $a^{-1}\vartheta a\vartheta = e_j\vartheta$

hence $(ii) \Leftrightarrow (ii)'$ Since $E(A\vartheta)^{op} = E(A\vartheta)$ $\square$

CH 1 Ex *4 $G \in \operatorname{ob}(\mathcal{G})$

The lattice of subgroupoids of G is a sublattice of the lattice of subcategories of G.

proof: $H \leq G$ subgroupoid $\Rightarrow$ $H \leq G$ subcategory

Thus, need to show (N.T.S.) smallest subcategory of G containing H_1 & H_2 (in $\mathcal{G}$) is a groupoid.

$$\exists \text{ canonical } H_1 \coprod_{H_1 \cap H_2} H_2 \xrightarrow{\varphi} \operatorname{gpd}\{H_1 \coprod_{H_1 \cap H_2} H_2\} \in \operatorname{mor}(\mathcal{G})$$

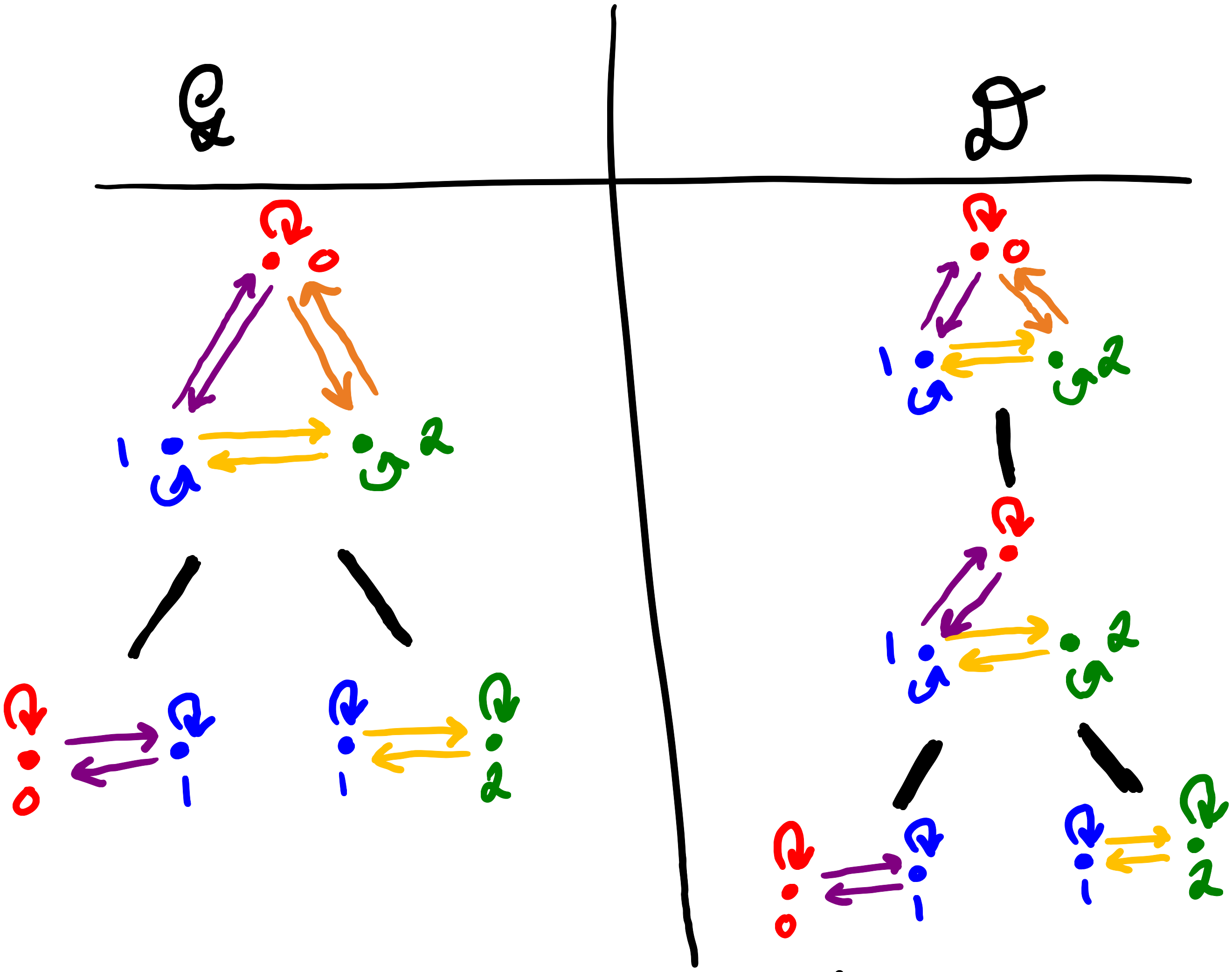
$$\& \operatorname{cat}\{(H_1 \coprod_{H_1 \cap H_2} H_2)\varphi\} \in \operatorname{ob}(\mathcal{G}) \quad (\text{prop } 2)$$

$$\Rightarrow \operatorname{cat}\{(H_1 \coprod_{H_1 \cap H_2} H_2)\varphi\} \supseteq \operatorname{gpd}\{(H_1 \coprod_{H_1 \cap H_2} H_2)\varphi\}$$

& the reverse inclusion is obvious $\square$

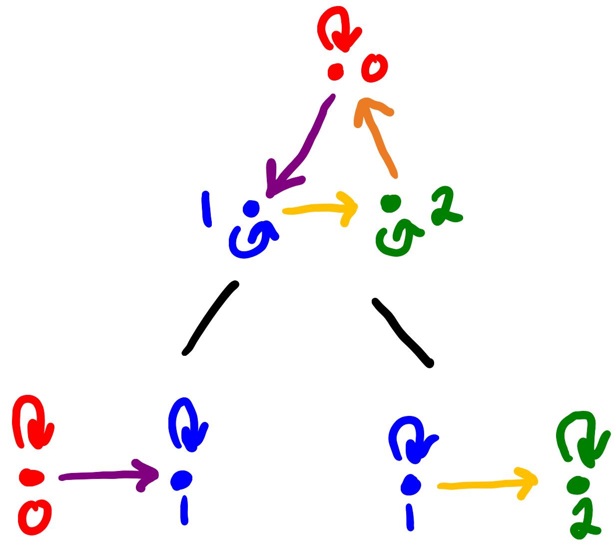
note: the lattice of subgroupoids of G need not be a sublattice of the lattice of subgraphs of G

Moral: being in $\mathcal{G}$ or $\mathcal{C}$ sometimes forces you to toss extra stuff in.



the joins don't coincide.

example in $\mathcal{C} \setminus \mathcal{G}$



CH 1 EX *5

(I) $\{\text{monomorphisms in } \mathcal{C} \text{ or } \mathcal{G}\} \longleftrightarrow \left\{\begin{matrix}\text{injections in} \\ \mathcal{C} \text{ or } \mathcal{G}\end{matrix}\right\}$

(II) $A \xrightarrow{\alpha} B \in \text{mor}(\mathcal{C})$, $\text{cat}\{A\alpha\} = B$

$\Rightarrow \alpha$ epimorphism

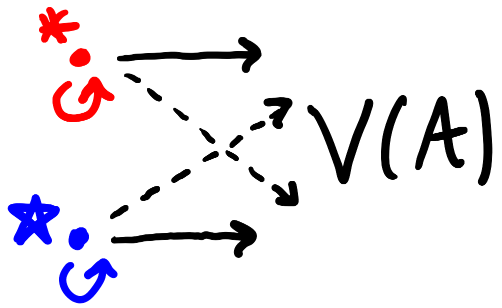
(III) The converse of (II) fails in $\mathcal{C}$

(IV) converse of (II) holds in $\mathcal{G}$ but is hard

proof of (I):

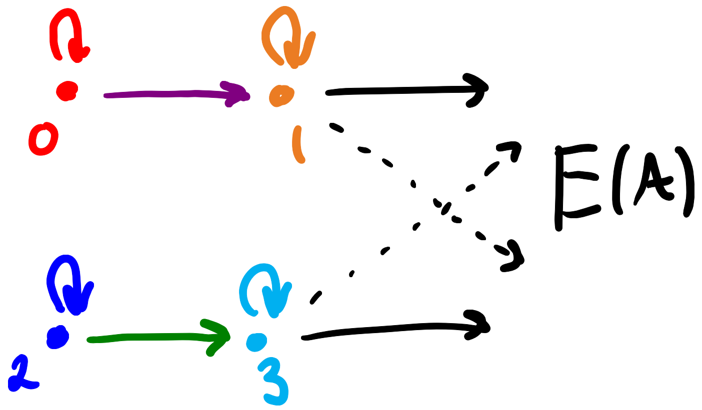
$\alpha :=$ ⟶

$\beta :=$ - - - ->



this is essentially the argument of the same fact in $\mathcal{S}$

$\Rightarrow V(\theta)$ injective



$\Rightarrow E(\theta)$ injective in $\mathcal{C}$

(use copies of $\Delta(\{0,1\})$ in $\mathcal{G}$)

Recall: (i) maps $\theta \in \text{mor}(K)$ for $K \in \{\mathcal{C}, \mathcal{G}\}$ are injective $\Leftrightarrow V(\theta)$ & $E(\theta)$ injective.

(ii) $\alpha : A \to B$ monomorphism iff $\forall \beta, \gamma$ $C \overset{\beta}{\underset{\gamma}{\rightrightarrows}} A \xrightarrow{\alpha} B$ commutes $\Leftrightarrow \beta = \gamma$ $\square$

proof of (II):

$\mathrm{Cat}\{A\alpha\} = B \Rightarrow V(\alpha)$ surjective (otherwise we miss an identity element of B)

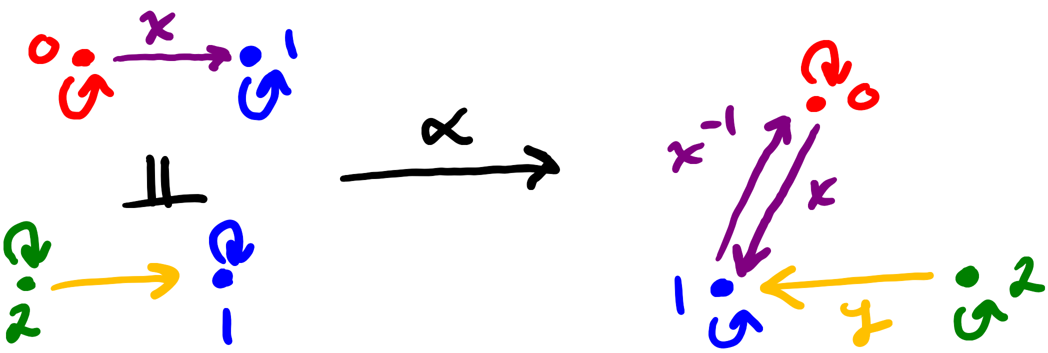
Suppose $A \xrightarrow{\alpha} B \underset{\gamma}{\overset{\beta}{\rightrightarrows}} C$ commutes for arbitrary β, γ

then $\alpha\beta = \alpha\gamma \Leftrightarrow \beta|_{A\alpha} = \gamma|_{A\alpha} \underset{(\text{prop } 2)}{\Leftrightarrow} \beta = \gamma$

& the same proof works in $\mathcal{G}$ by Ex*3.

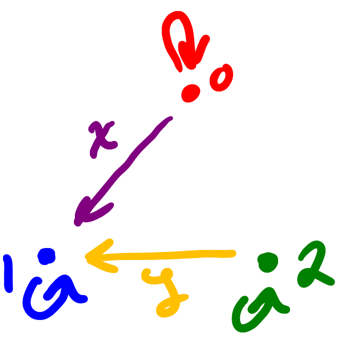
take $\alpha \in \mathrm{mor}(\mathcal{G})$ but only look @ $\mathrm{cat}\{A\alpha\}$ $\square$

Example for (III):



maps out of this are determined by the image of x & y

However, the image does not generate the target as a category since

$\mathrm{im}(\alpha) =$  $\in \mathrm{ob}(\mathcal{G})$ $\square$

Remark: the proof of the corresponding fact in $\mathcal{S}$ fails in this case because the image of edges may be determined by another line χ & χ^{-1} above.

Left

monomorphism

$\Big\downarrow$ think

(injective $\Downarrow$ left-exact)

$$A \underset{\gamma}{\overset{\beta}{\rightrightarrows}} B \xrightarrow{\alpha} C$$

$\forall \beta, \gamma$

$$\alpha \circ \beta = \alpha \circ \gamma \Rightarrow \beta = \gamma$$

i.e. α Left cancellable

Right

epimorphism

$\Big\downarrow$ think

(surjective $\Downarrow$ right-exact)

$$A \xrightarrow{\alpha} B \underset{\gamma}{\overset{\beta}{\rightrightarrows}} C$$

$\forall \beta, \gamma$

$$\beta \circ \alpha = \gamma \circ \alpha \Rightarrow \beta = \gamma$$

i.e. α Right cancellable

the '$\circ$' here are to emphasize we are breaking the convention of applying functions on the right for this to be true.