Chapter 6 Fundamental Groupoids of Topological Spaces

$\mathcal{T}$:= Category of topological spaces & continuous maps

$\mathcal{T}'$:= Category of pairs of spaces & maps of pairs

$\mathcal{T}_1$ = Category of pointed topological spaces & continuous maps

$$\begin{tikzcd}
\mathcal{T}_1 \arrow[hook, dr] & \\
& \mathcal{T}' \\
\mathcal{T} \arrow[hook, ur] &
\end{tikzcd}
\qquad T \mapsto (T,T)$$

Functors

$P : \mathcal{T}' \longrightarrow \mathcal{C}$ category of paths

$\pi : \mathcal{T}' \longrightarrow \mathcal{G}$ fundamental groupoid

note: pre-composition w/ inclusions induce functors w/ codomains

$\mathcal{T}_1 \rightsquigarrow \mathcal{C}_1$ & $\mathcal{G}_1$

$\mathcal{T} \rightsquigarrow \mathcal{C}$ & $\mathcal{G}$

notation:

$\mathcal{T}'$: $\pi(T, I)$ fundamental groupoid of pair

$\mathcal{T}$: $\pi(T)$ fundamental groupoid of T

$\mathcal{T}_1$: $\pi(T,*)$ fundamental group of T

Main Results

(i) $\left\{\begin{matrix}\text{Retracts}\\ A \longrightarrow B\end{matrix}\right\} \xleftrightarrow{\cong} \left\{\begin{matrix}\text{full subgroupoids}\\ B \subset A \text{ meeting all}\\ \text{components of } A\end{matrix}\right\}$

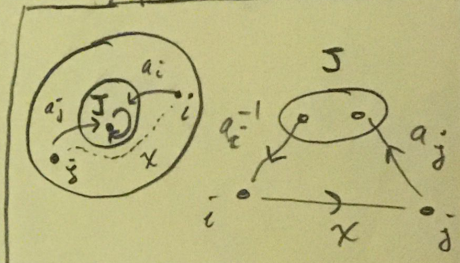
Special case

(ii) $\left\{\begin{matrix}\text{Connected}\\ \text{Groupoids}\end{matrix}\right\} \xleftrightarrow{\cong} \{\text{Groups}\}$

lemma

(iii) $\left\{\begin{matrix}\text{pairs of}\\ \text{equivalent}\\ \text{Groupoids}\end{matrix}\right\} \xleftrightarrow{\cong} \left\{\begin{matrix}\text{disjoint unions}\\ \text{of pairwise}\\ \text{isomorphic groups}\end{matrix}\right\}$

picture for (i)



$x \mapsto a_i^{-1} x a_j$

Idea Set up an analogy

Homotopies $\overset{\text{Prop 13}}{\rightsquigarrow}$ Natural Equivalences

$\mathcal{T}'$

$H \mapsto$ the choice-free map "follow the vertex i"

$(x\theta)\, b_j = b_i (x\phi)$ (Natural transformation)

picture (uses inverses)

$i \bullet \overset{x}{\text{———}} \bullet j \quad A_{ij}$

$i\phi \bullet \overset{x\phi}{\text{———}} \bullet j\phi$

$\|$

$i \bullet \overset{b_i^{-1}}{\text{——}} \bullet_{i\theta} \overset{x\theta}{\text{——}} \bullet_{j\theta} \overset{b_j}{\text{——}} \bullet j \quad B_{ij}$

$$\begin{tikzcd} A \arrow[d, shift right, "\phi"'] \arrow[d, shift left, "\theta"] \arrow[d, phantom, "\simeq"] \\ B \end{tikzcd}$$

Special case

g&h induce same inner automorphism $\triangle\!!\ \sim$

Self-homotopies $\mathcal{T}'$ $\rightsquigarrow$ $\left\{\begin{matrix}\text{deformations}\\ \alpha \simeq id_A\end{matrix}\right\} \xleftrightarrow{\cong} \left\{\begin{matrix}\text{All maps of the form}\\ x \mapsto a_i^{-1} x a_j\\ \overset{\cap}{A_{ij}} \text{ where } a_i \in A_{ii}\ \forall i\end{matrix}\right\}$

in $\mathcal{G}_1$ $\{\text{deformations}\} \xleftrightarrow{\cong} \left\{\begin{matrix}\text{inner}\\ \text{automorphisms}\end{matrix}\right\}$

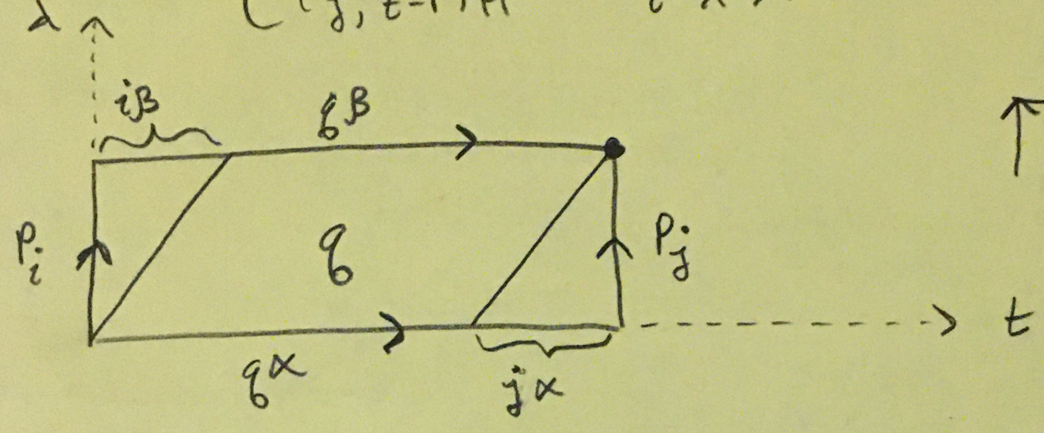
Special case deformation-retractions $\mathcal{T}' \rightsquigarrow \left\{\begin{matrix}\text{retracts}\\ \alpha^2 = \alpha\end{matrix}\right\} \xleftrightarrow{\cong} \{\alpha|_{A\alpha} = id_{A\alpha}\} \xleftrightarrow{\cong} \left\{\begin{matrix}\mu p = 1_B\\ p\mu \simeq 1_A\end{matrix}\right\}$

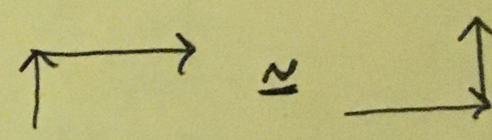
Homotopy (proof of natural equivalence)

$h : [0, r+1] \times [0,1] \longrightarrow Y$

homotopy $H : \alpha \Rightarrow \beta$

$$(t,\lambda)h = \begin{cases} (i,t)H & t-\lambda \leq 0 \\ ((t-\lambda)q, \lambda)H & 0 \leq t-\lambda \leq r \\ (j, t-r)H & t-\lambda \geq r \end{cases}$$





$$\langle q \rangle \pi(\alpha) \cdot \langle p_j \rangle = \langle p_i \rangle (\langle q \rangle \pi(\beta))$$

$$\begin{tikzcd} i\alpha \arrow[r, "\langle q \rangle \pi(\alpha)"] \arrow[d, "p_i"'] & j\alpha \arrow[d, "p_j"] \\ i\beta \arrow[r, "\langle q \rangle \pi(\beta)"'] & j\beta \end{tikzcd}$$

$\langle p_i \rangle$ are components of natural equivalence