FToC Review

Multiplication v.s. division

$\times$ opposites $/$

easier harder

Cancel out

$$\frac{A \times B}{B} = A$$

reciprocals = inverses

$$A \times \frac{1}{A} = 1$$

Fractions create new stuff

$$\frac{\text{whole \#}}{\text{whole \#}} \neq \text{whole \#}$$

differentiation v.s. Integration

$\frac{d}{dt}$ opposites $\int_c^t - \, dx$

easier harder

Cancel out

$$\frac{d}{dt}\int_c^t f(x)\,dx = f(t)$$

area functions = antiderivatives

$$\int f(x)\,dx = F(x) + C$$

where $F'(x) = f(x)$

Integrals make new functions

$\int e^{x^2}\,dx$ cannot be expressed in terms of the functions & rules of math we are used to

Analogy: The relationship described by the Fundamental theorem of Calculus is like the relationship between division & multiplication

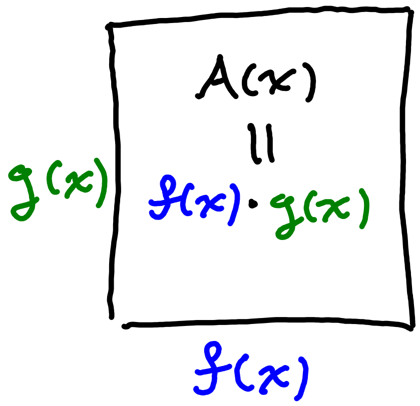
$$\frac{d}{dx} = \lim(\text{Slopes}) = \text{limit}\ \frac{\text{rise}}{\text{run}} \leftarrow \text{Division}$$

Multiplication

$$\int = \lim \sum \begin{pmatrix}\text{Areas of}\\ \text{Rectangles}\end{pmatrix} = \text{limit} \sum (\text{height}) \times (\text{width})$$

The Product Rule: How do we compute the derivative of a product of two functions?

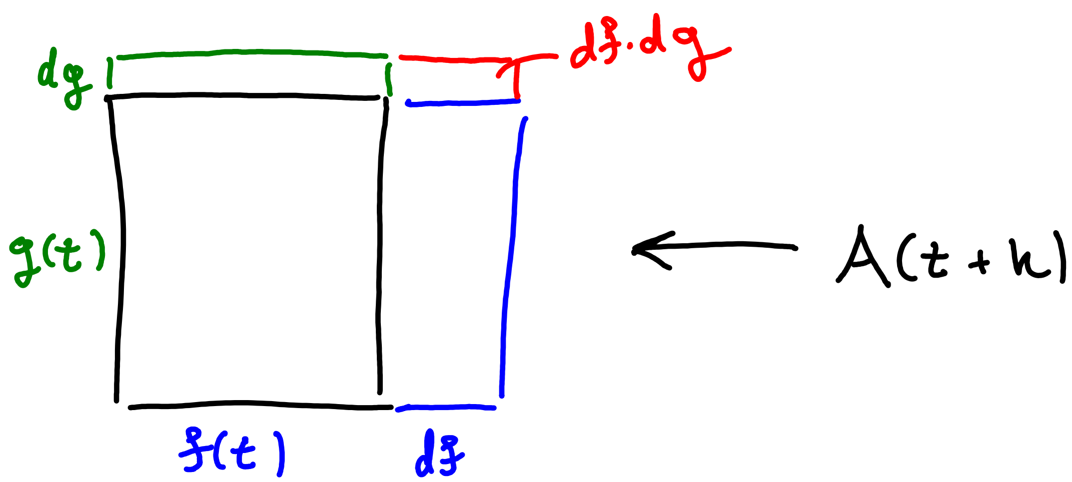
$$\frac{d}{dx}\underbrace{A(x)}_{\text{area of a rectangle whose side lengths change as functions of } x} = \frac{d}{dx}\Big[\underbrace{f(x)}_{\text{view as length}}\underbrace{g(x)}_{\text{view as height}}\Big] = \ ?$$



We have two functions (fixed but unknown) each controlling the side lengths of a rectangle.

<u>Recall</u>: $\frac{d}{dx}$ = "the instantaneous rate of change of the function with respect to x"

So we can approximate $\frac{d}{dx}[f(x)g(x)]$ by looking at the change in the rectangle's area when x changes slightly



$f(t)$ = length @ time t $\qquad$ $f(t)+df$ = length a short time later

$g(t)$ = height @ time t $\qquad$ $g(t)+dg$ = height a short time later

$$A(t+h) = (g(t)+dg)(f(t)+df)$$
$$= f(t)g(t) + df\,g(t) + f(t)\,dg + df\,dg$$

So $$\underbrace{A(t+h) - A(t)}_{\text{change in area after } h \text{ units of time}} = df\,g(t) + f(t)\,dg + \underbrace{df\,dg}_{\text{small as compared with the other two terms}}$$

& we get $$\approx f(t)g'(t) + f'(t)g(t)$$

product rule:

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)g'(x) + f'(x)g(x)$$

problems: Compute $\frac{d}{dx}\left[f(x)g(x)\right]$ using the above formula & the facts below

(i) $\frac{d}{dx}(\text{constant}) = 0$

(ii) $\frac{d}{dx}x = 1$

1) $f(x) = 7$, $g(x) = x$

2) $f(x) = x$, $g(x) = x$

3) $f(x) = x^2$, $g(x) = x$

4) $f(x) = x^3$, $g(x) = x$

5) $f(x) = x^2$, $g(x) = x^2$

6) $f(x) = x$, $g(x) = x^4$

$\underline{Q}$: What is $\frac{d}{dx}x^n = ?$

Solutions:

1) $\frac{d}{dx}7\cdot x = 7\cdot\frac{d}{dx}x + \left(\frac{d}{dx}7\right)\cdot x \overset{(i)\&(ii)}{=} 7\cdot 1 + 0\cdot x$ $\boxed{=7}$

product Rule

2) $\frac{d}{dx}x^2 = \frac{d}{dx}x\cdot x \underset{(ii)}{=} x\cdot 1 + 1\cdot x = x + x$ $\boxed{=2x}$

3) $\frac{d}{dx}x^3 = \frac{d}{dx}x^2\cdot x = x^2\frac{d}{dx}x + \left(\frac{d}{dx}x^2\right)\cdot x$

$= x^2\cdot 1 + 2x\cdot x = x^2 + 2x^2$ $\boxed{=3x^2}$

problem 2)

4) $\frac{d}{dx}x^4 = \frac{d}{dx}x^3\cdot x = x^3\cdot\frac{d}{dx}x + \left(\frac{d}{dx}x^3\right)\cdot x$

$= x^3 + 3x^2\cdot x = x^3 + 3x^3$ $\boxed{=4x^3}$

problem 3)

5) $\frac{d}{dx}x^4 = \frac{d}{dx}x^2x^2 = x^2\cdot\frac{d}{dx}x^2 + \left(\frac{d}{dx}x^2\right)\cdot x^2$

$= 2\left(x^2\cdot\frac{d}{dx}x^2\right) = 2\cdot(x^2\cdot 2x)$

problem 2)

$\boxed{=4x^3}$

6) $\frac{d}{dx}x^5 = \frac{d}{dx}x \cdot x^4 = x\frac{d}{dx}x^4 + \left(\frac{d}{dx}x\right)\cdot x^4$

problem 5) $\nearrow$ $= x \cdot 4x^3 + 1 \cdot x^4 = 4x^4 + x^4$

$$\boxed{= 5x^4}$$

The Power Rule:

$$\frac{d}{dx}x^{n} = n x^{n-1}$$

I) Bring the exponent down (multiply by exponent)

II) Subtract 1 from the exponent

(Why? $\frac{d}{dx}x^n = \frac{d}{dx}x^{n-1}\cdot x = x^{n-1}\frac{d}{dx}x + \left(\frac{d}{dx}x^{n-1}\right)\cdot x$

$$= x^{n-1} + x \cdot \underbrace{\frac{d}{dx}x^{n-1}}_{\text{we already know what this is}} = x^{n-1} + (n-1)x^{n-2}\cdot x$$

$$= x^{n-1} + (n-1)x^{n-1} = x^{n-1}(1+(n-1)) = nx^{n-1})$$

$\uparrow$ factor out x^{n-1}

Note: this argument finds a general formula for derivatives of monomials by finding a pattern in the result of iteratively factoring out a single x & applying the product rule. This concept is used often to compute all integrals/derivatives of some general form.

Reverse Power Rule:

Morning

I) put on your socks

II) put on your shoes

Night

I) take off your shoes

II) take off your socks

to reverse a series of instructions you do the opposite of each instruction in reverse order

Power Rule

I) ~~Bring~~ exponent ~~down~~ multiply by

II) Subtract 1 from the exponent

opposites

I) Divide by the exponent

II) Add 1 to the exponent

Reverse order

Reverse Power Rule

I) Add 1 to the exponent

II) Divide by the exponent

$$\int x^2 dx = \frac{x^3}{3} + C$$

Step 1

Step 2

Checking our answer:

$$\frac{d}{dx}\left(\frac{x^3}{3}+C\right)=\frac{1}{3}\frac{d}{dx}x^3+\frac{d}{dx}C$$

$$=\frac{1}{3}\cdot 3x^2+0=\underline{\underline{x^2}}\ \checkmark$$

Power Rule $\xrightarrow{\text{opposite}}$ Reverse Power Rule

$$\boxed{\frac{d}{dx}x^n=n\cdot x^{n-1}}$$

$$\int x^n\,dx=\frac{x^{n+1}}{n+1}+C$$

except when $n=-1$ because then the $n+1$ in the denominator $=0$. Don't divide by zero!

Next,

Chain Rule $\xrightarrow{\text{opposite}}$ U-Substitution

$$\boxed{\frac{d}{dx}f\circ g(x)=f'(g(x))g'(x)}$$

"Reverse Chain Rule"

$$\frac{d}{dx}(x^2+x)^4$$

use chain rule

$u=x^2+x$

$\frac{d}{du}u^4=4u^3$

$\frac{d}{dx}u=2x+1$

$\frac{d}{dx}(u(x))^4=\frac{d}{du}\frac{du}{dx}=4(x^2+x)^3\cdot(2x+1)$

expand out & use power rule

$(x^2+x)^4=(x^2+x)(x^2+x)(x^2+x)(x^2+x)$

$\vdots$

not enough room or time to finish this calculation & it wouldn't be worth all of the work in any case ... try something else...

$$\int 4(x^2+x)^3\cdot(2x+1)\,dx$$

<u>use u-substitution</u>

expand completely & use reverse power rule

$$4(x^2+x)^3\cdot(2x+1)$$
$$=4(x^2+x)(x^2+x)(x^2+x)(2x+1)$$

notice that the function looks like it came from chain rule.

$$\begin{tikzcd} x^2+x \arrow[r, "\text{derivative}"] \arrow[d, "\text{gets plugged in to}"'] & 2x+1 \arrow[d, "\text{multiply by}"] \\ 4\cdot u^3 & dx \end{tikzcd}$$

Set $u = x^2+x$

$du = (2x+1)dx$

$$\int 4(x^2+x)^3(2x+1)dx = \int 4u^3\cdot du$$

$$= \frac{4u^4}{4} + C = u^4 + C = (x^2+x)^4 + C$$

<u>Slogan</u>: Look for an expression whose derivative appears multiplied by dx

↳ this even works if the derivative is off by a constant multiple

$$\int \underline{x}e^{x^2}\underline{dx}$$ is $u = x^2$ a good choice?

<u>Check</u>: $du = 2x\,dx$

dont care about constants

need to see $x\cdot dx$ in integral

Since the derivative of u appears multiplied by dx in the integral, we can completely eliminate all occurrences of x to get a new integral in terms of u.

$$du = 2x\,dx \quad\Downarrow\quad \frac{du}{2} = x\,dx$$

$$\int e^{x^2} x\,dx = \int e^{u} \frac{du}{2}$$

$$= \frac{1}{2}\int e^{u}\,du$$

$$= \frac{1}{2}e^{u} + C = \frac{1}{2}e^{x^2} + C$$

How else can we see that $u = x^2$ is a good choice?

try taking the derivative of $\frac{1}{2}e^{x^2} + C$

$$\frac{d}{dx}\left(\frac{1}{2}e^{x^2} + C\right) = \frac{1}{2}\underbrace{\frac{d}{dx}e^{x^2}}_{\text{need to use chain rule}} + \frac{d}{dx}C$$

$$f(t) = e^{t}$$

$$\boxed{u(x) = x^2} \qquad e^{x^2} = f(x^2) = f(u(x))$$

$$\frac{d}{dx}e^{x^2} = \frac{d}{dx}f(u(x)) = f'(u(x))\,u'(x)$$

$$\left.\begin{aligned} f'(t) &= e^{t} \\ u'(t) &= 2x \end{aligned}\right\} \longrightarrow = e^{x^2}\cdot 2x$$

$$\frac{d}{dx}\left(\frac{1}{2}e^{x^2} + C\right) = \frac{1}{2}(2x)e^{x^2} = xe^{x^2}$$

$$\frac{d}{dx} f(u(x)) = f'(u(x))u'(x)$$

$$\int f'(\underbrace{u(x)}_{\text{expression}})\underbrace{u'(x)dx}_{\substack{\text{derivative}\\ \text{of the expression}\\ \text{multiplied by } dx}} = \int f'(u)du = f(u(x))$$

Be Creative! – It takes practice to be able to solve lots of different kinds of integrals (like the ones you will be asked on exams) & you will find yourself trying lots of things that do not end up working out. Doing so now will save time on the exam since you will have a better feeling for what will not work. Additionally, it is good to keep in mind the fact that there are many ways to solve calculus problems so there is a lot of room for creativity in your solutions. This also means you will want some way to answer questions confidently since the answer you wrote down, while correct, can look wildly different from the answer given by a different approach. Before adapting the belif that your answer must not be correct, try to relate the two expressions to prove or disprove their equality. If all else fails, you can always check integrals by taking a derivative.

Example:

$\int \sqrt{2x+1}\,dx$ can be solved with at least two different substitutions

$$u = 2x+1$$

OR $$u = \sqrt{2x+1}$$

Here is how the second substitution works

$$\int \sqrt{2x+1}\,dx = \int \sqrt{2x+1} \cdot \frac{\sqrt{2x+1}}{\sqrt{2x+1}}\,dx$$

$$= \int \frac{2x+1}{\sqrt{2x+1}}\,dx$$

If $u = \sqrt{2x+1}$

$$du = \frac{1}{2}\frac{1}{\sqrt{2x+1}} \cdot 2\,dx = \frac{dx}{\sqrt{2x+1}}$$

& $u^2 = 2x+1$

$$=$$

$$\int u^2\,du$$

$$= \frac{u^3}{3} + C \quad \boxed{= \frac{1}{3}\sqrt{2x+1}^{\,3} + C}$$

U-Sub is like putting together a puzzle

The picture on the box is like your original integral

u & du are the only 2 pieces of the puzzle, the only problem is, it's your job to tell me what the puzzle pieces actually are

For Example

$$\int x\sqrt{x+2}\,dx$$

$u = x+2$
$du = dx$

don't have a puzzle piece to represent x yet

but notice

$u = x+2$

by rearranging the equation defining u so as to solve for x we get $x = u - 2$

So, $\underbrace{\int x\sqrt{x+2}\,dx}_{\text{in terms of } x}$ $=$ $\underbrace{\int (u-2)\sqrt{u}\,du}_{\text{in terms of } u}$ $\qquad \underbrace{u = x+2}_{\text{the relationship we picked for } x \,\&\, u}$

"the picture on the box" — "the completed puzzle" — "puzzle piece #1"

$\underbrace{u-2 = x}$

a clever use of the u-sub allowed us to express x itself in terms of u.

"puzzle piece #3"

$\underbrace{du = dx}$

"puzzle piece #2"

this is always determined by the choice of puzzle piece #1 and no choices are involved

it must be the derivative of the equation defining u

to finish the problem

$$\int (u-2)\sqrt{u}\,du = \int (u\cdot u^{1/2} - 2u^{1/2})\,du = \int u^{3/2}\,du - 2\int u^{1/2}\,du$$

$$= \frac{2}{5}u^{5/2} - 2\cdot\frac{2}{3}u^{3/2} + C = \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + C$$

More Examples

a) $\displaystyle\int \sqrt{1+x^2}\,x^5\,dx$

b) $\displaystyle\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx$ (Careful w/ the bounds)

c) $\displaystyle\int e^x\sqrt{1+e^x}\,dx$ (Hint: $\frac{d}{dx}e^x = e^x$)

a) $\int \sqrt{1+x^2}\, x^5 dx = \int \sqrt{1+x^2}\, x^4 \cdot \underbrace{x dx}$

Split one x off from x^5

$u = 1+x^2$

$du = 2x dx$

Q: How do we replace the x^4 part?

$u = 1+x^2 \underset{\text{Subtract 1 from both sides}}{\Longrightarrow} u-1 = x^2 \underset{\text{Square both sides}}{\Longrightarrow} (u-1)^2 = (x^2)^2 = x^4$

Q. What's up with the 2?

"the picture on the box" only has $x dx$

but we have $du = 2x dx$ So we just solve

for the part we need to replace $\frac{du}{2} = x dx$

$$\int \sqrt{1+x^2}\, x^5 dx = \int \sqrt{u}\,(u-1)^2 \frac{du}{2}$$

$$= \int (u-1)(u-1)u^{1/2} \frac{du}{2} = \frac{1}{2}\int (u^2 - 2u + 1)u^{1/2}\, du$$

$$= \frac{1}{2}\int (u^{5/2} - 2u^{3/2} + u^{1/2})du = \frac{1}{2}\left(\frac{2}{7}u^{7/2} - 2\,\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right) + C$$

$$\boxed{= \frac{1}{7}(1+x^2)^{7/2} - \frac{2}{5}(1+x^2)^{5/2} + \frac{1}{3}(1+x^2)^{3/2} + C}$$

b) $\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx$

Note: the dx inside the integral is really just there to tell you that the bounds on this integral (0 & 4) are x-values

lower bound

upper bound

For the moment, let's forget the bounds

$$\int \frac{x}{\sqrt{1+2x}}\,dx = \underbrace{\int \frac{x}{\sqrt{u}}\,\frac{du}{2}}_{\text{still need to replace } x} = \int \frac{u-1}{2}\cdot\frac{1}{\sqrt{u}}\cdot\frac{du}{2}$$

$u = 1+2x$

$du = 2dx$

$u-1 = 2x$

$\Rightarrow x = \frac{u-1}{2}$

(FYI the integral in the middle doesn't actually make sense to write)

$$\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx = \underbrace{\int_?^? \frac{1}{4}\frac{(u-1)}{\sqrt{u}}\,du}_{\text{the bounds here are supposed to be values of } u}$$

how do we find these new bounds?

well, we know the bounds in terms of x & the relationship between u & x

$u = 1+2x$

	x	u
Lower Bound	0	$1+2\cdot 0 = 1$
Upper Bound	4	$1+2\cdot 4 = 9$

So $\int_0^4 \frac{x}{\sqrt{1+2x}}dx = \frac{1}{4}\int_1^9 \frac{u-1}{\sqrt{u}}du$

Finishing the problem

$$\frac{1}{4}\int_1^9 (u^{1/2} - u^{-1/2})du = \frac{1}{4}\left[\frac{2}{3}u^{3/2} - 2u^{1/2}\right]_1^9$$

$$= \left(\frac{1}{6}(9)^{3/2} - \frac{1}{2}(9)^{1/2}\right) - \left(\frac{1}{6}1^{3/2} - \frac{1}{2}1^{1/2}\right)$$

$$= \frac{27}{6} - \frac{3}{2} - \frac{1}{6} + \frac{1}{2} = \frac{26}{6} - 1 = \boxed{\frac{20}{6}}$$

c)

$$\int e^x\sqrt{1+e^x}\,dx = \int \sqrt{1+e^x}\,e^x dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C$$

$u = 1+e^x$

$du = e^x dx$

using the hint

$$\boxed{= \frac{2}{3}(1+e^x)^{3/2} + C}$$

Some Notes Regarding Expectations

Definite Integrals $\int_a^b f(x)dx =$ Always output a number

- If you solve using a u-substitution all integrals in the work must involve a single variable (typically x or u). Don't use both x & u in the same integral like I did in problem b) above
- If you solve using a u-substitution make sure you put the correct bounds on the integrals at all times. Remember dx means bounds are x values

  du means bounds are u values

* Some people prefer to wait until the end of a calculation to mess with the bounds. If you plug the expression in terms of x back in once you've integrated in terms of u, you don't need to change the bounds, but you will need to write *something* as the bounds of the integrals in terms of u. If you choose to do this you may write something like

not recommended but you may do this if you insist

$$\int_a^b f(x)dx = \int_*^* g(u)du = G(u) + C \Big|_*^*$$

$u = g(x)$

$$= G(g(x)) + C \Big|_a^b$$

Indefinite Integrals

$$\int f(x)dx = \text{Always Equal a function}$$

- when solving by substitution all integrals must involve only a single variable

- Don't Forget $+C$