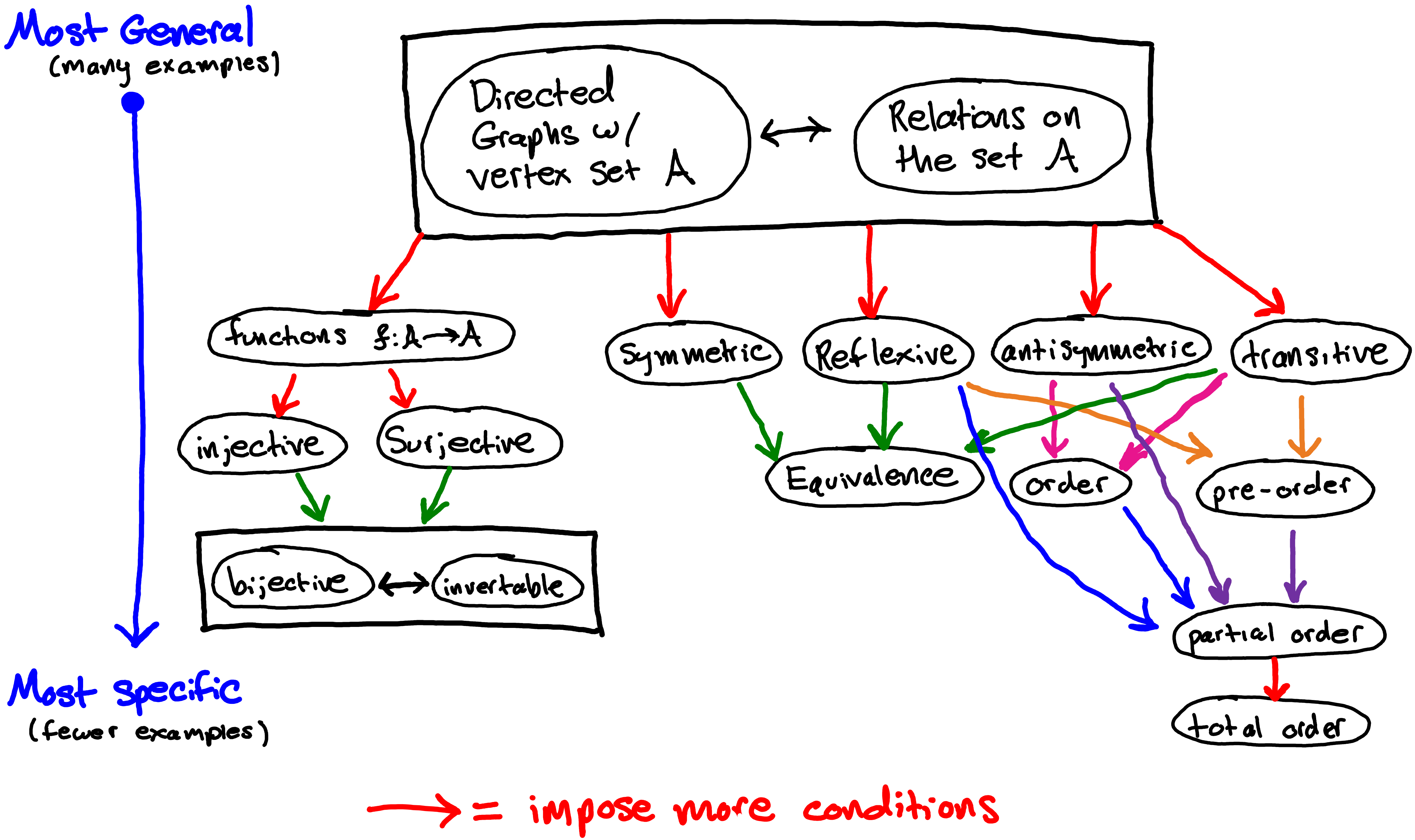
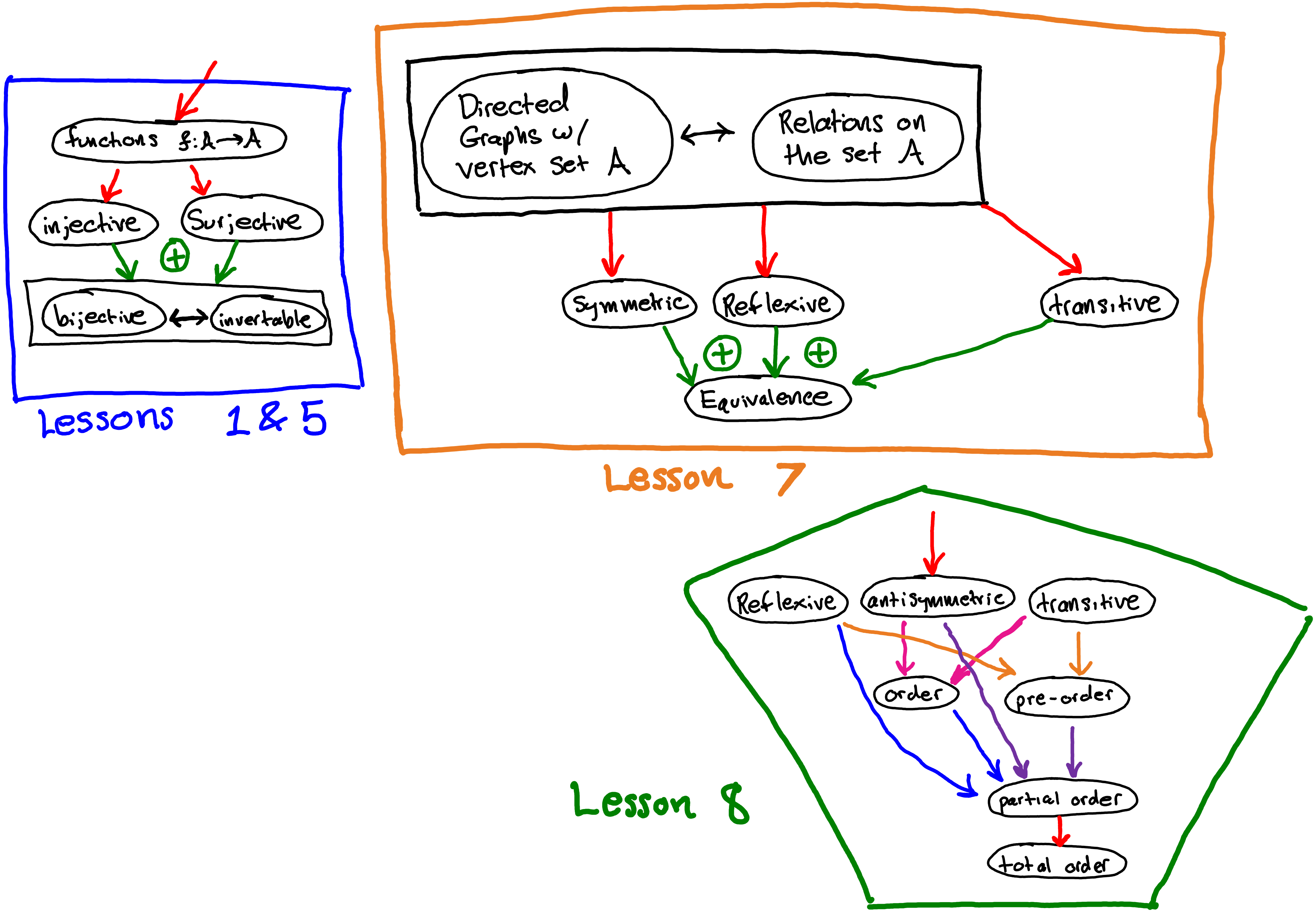


(more conditions means we are talking about fewer objects)


functions f:A→A
injective
Surjective
bijective
invertable
Lessons 1 & 5
Directed Graphs w/ vertex set A
Relations on the set A
Symmetric
Reflexive
transitive
Equivalence
Lesson 7
Reflexive
antisymmetric
transitive
order
pre-order
partial order
total order
Lesson 8

Relations on A

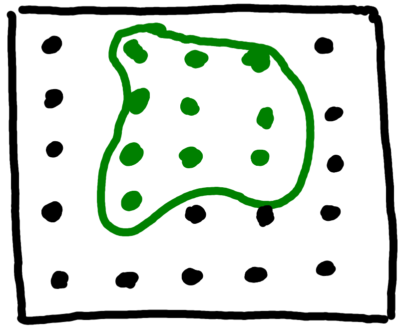
- A subset $R \subseteq A \times A = A^2$ is a relation on A
- R relation on $A \iff R \in \mathcal{P}(A^2)$ (power set)

<u>Notation</u>: write $aRb \iff (a,b) \in R$

$\equiv$

Picture / Table

- Cartesian products of a finite set w/ itself makes a square



relations on A are subsets of this square

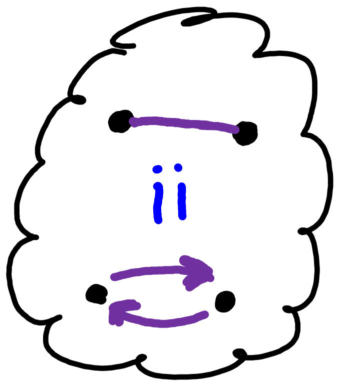
$\equiv$

As a Directed Graph

- Vertex set is A
- Arrows come from R

$a,b \in A \wedge (a,b) \in R$

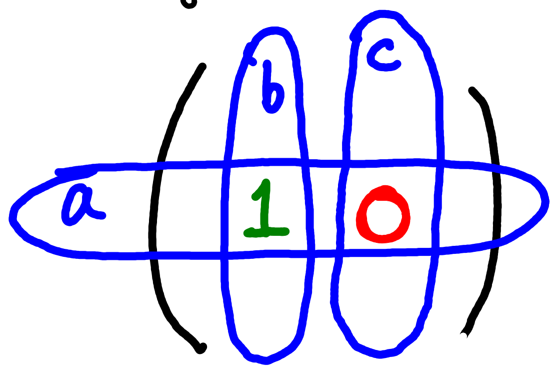
$\Updownarrow$



$\equiv$

As a Matrix

- $m_{ij} = 1$ if $(i,j) \in R$
- $m_{ij} = 0$ if $(i,j) \notin R$



$(a,b) \in R$

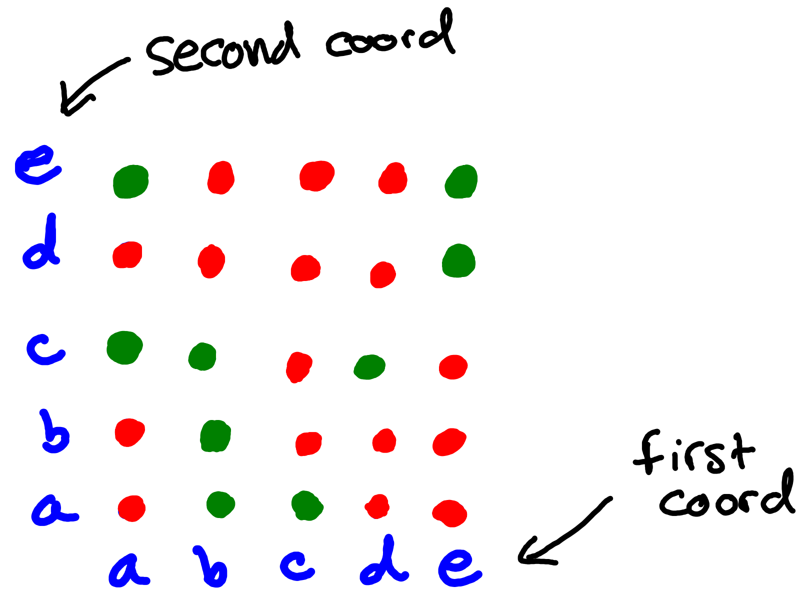
$(a,c) \notin R$

Example Relation

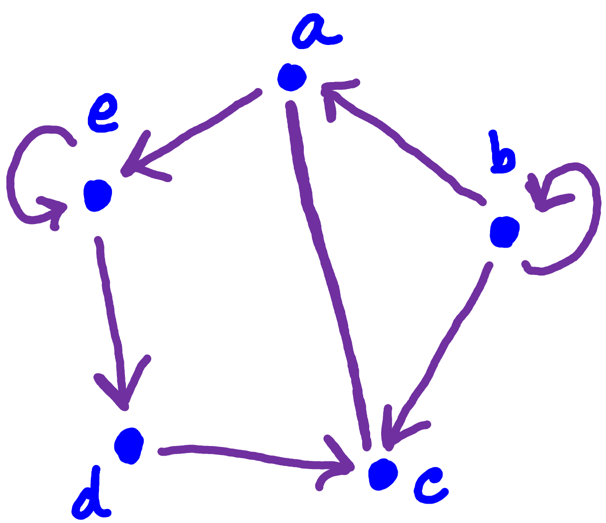
$A := \{a, b, c, d, e\}$

$R := \{(a,e), (e,e), (e,d), (d,c), (c,a), (b,a), (b,c), (b,b), (a,c)\}$

Picture / Table



As a Directed Graph



Adjacency Matrix

As a Matrix

- rows are first coord
- columns are second coord

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

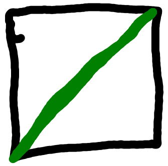
Reflexive

A relation $R \subseteq A^2$ is **reflexive** if every element is related to itself

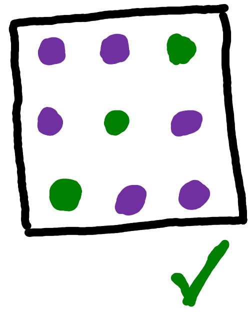
$\forall a \in A \ (a,a) \in R \iff$ R is reflexive

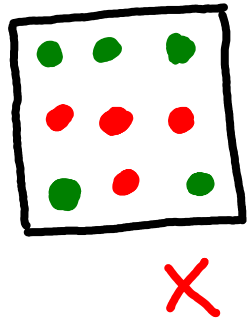
● = allowed to be ● or ●
● = must be red
● = must be green

$\Delta = \{(a,a) \mid a \in A\}$

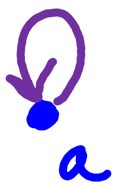


diagonal



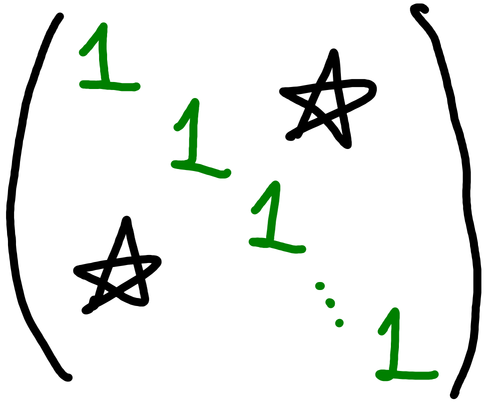


$(a,a) \in R \iff$



Reflexive $\iff$ there is a loop at every vertex

there are 1s along the "main diagonal" m_{ii} for all i $\iff$ reflexive



Symmetric

$\forall a \forall b \; (a,b) \in R \rightarrow (b,a) \in R$

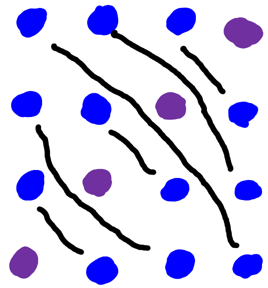
A relation $R \subseteq A^2$ is **Symmetric** if $(a,b) \in R$ implies $(b,a) \in R$

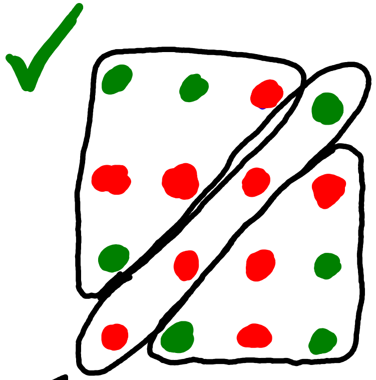
e.g. $(a,b) \in R \Leftrightarrow$ b is the child of a

(Not symmetric) (symmetric)

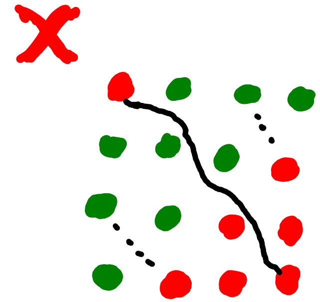
$(a,b) \in \tilde{R} \Leftrightarrow$ b is a relative of a

= both red xor both green

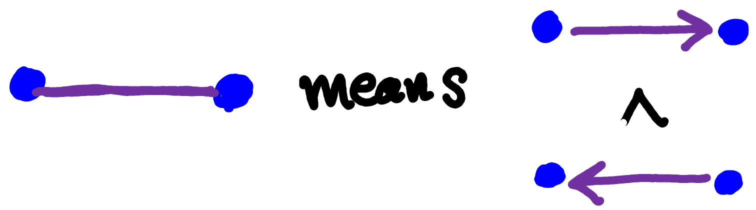




triangles agree when flipped over this diagonal



By definition

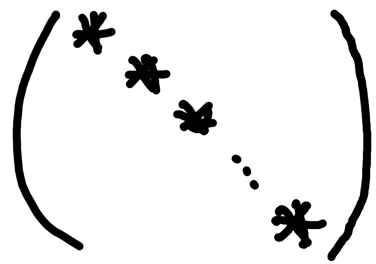


Hence

Symmetric $\Leftrightarrow$ Every connection is an edge (No arrows)

A matrix (m_{ij}) is Symmetric $\Leftrightarrow m_{ij} = m_{ji}$ for all i & j.

exchange rows & columns



– reflect over main diagonal & matrix does not change

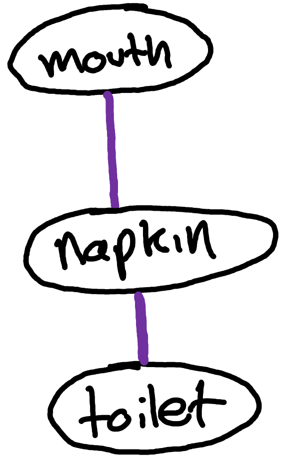
The Transitive Property (Transitivity)

$R \subseteq A^2$ is <u>transitive</u> if $\forall a \forall b \forall c \ (a,b) \in R \wedge (b,c) \in R \rightarrow (a,c) \in R$

Non-example: "Handshake relation"

touch is not transitive

1) Clean mouth w/ napkin
2) clean toilet w/ napkin
3) Relax (your mouth did not touch the toilet)

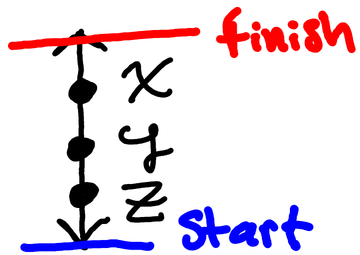


Notice that edges are not arrows because touch is symmetric (A shakes hands w/ B $\Rightarrow$ B shakes hands w/ A)

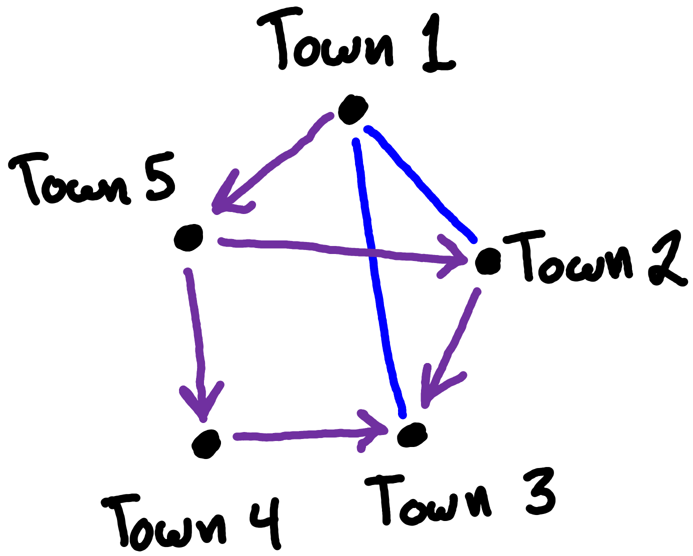
Example: $X = \{x \mid x \text{ is running in today's race}\}$

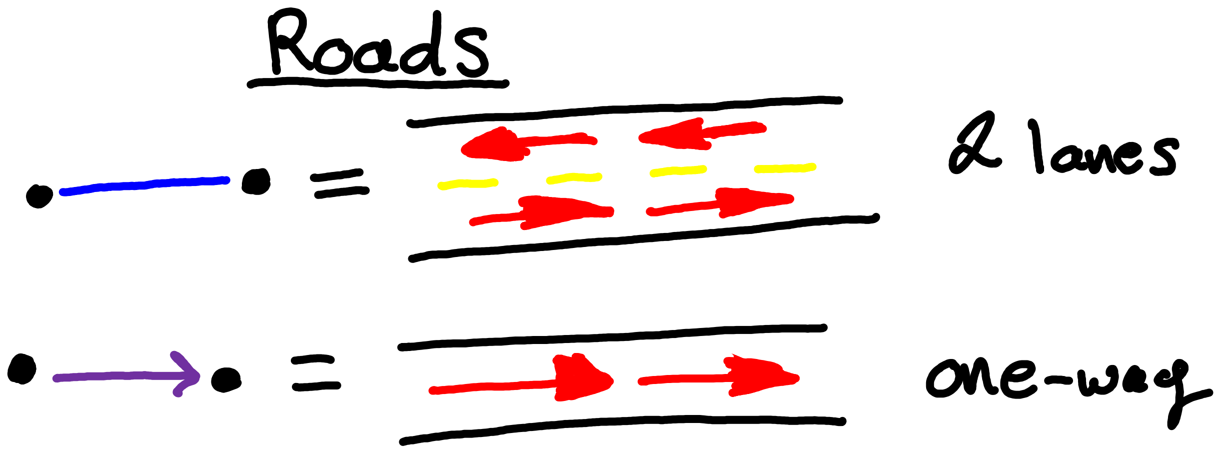
$R := \{(x,y) \mid x \text{ finishes before } y\}$

$(x,y),(y,z) \in R \Rightarrow$



we see x finishing before $z \Leftrightarrow (x,z) \in R$



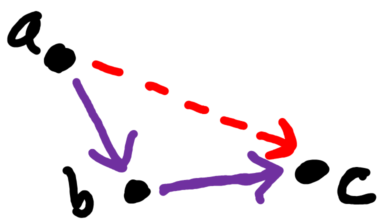


The directed graph above represents a relation $R \subseteq \{\text{Town } i, i=1,\dots,5\}^2$

Is R reflexive? No  $\Leftrightarrow$ (Town i, Town i) $\in R$

Is R symmetric? No  $\Longrightarrow$ $\neg$Symmetric

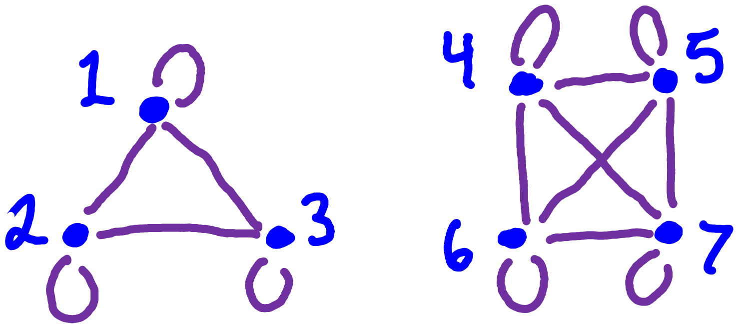
Is R transitive? No (Town 5, Town 4), (Town 4, Town 3) $\in R$ but (Town 5, Town 3) $\notin R$



-Need paths of length 2 (two arrows) to live inside a completed triangle

Equivalence Relations

Reflexive ⊕ Symmetric ⊕ Transitive



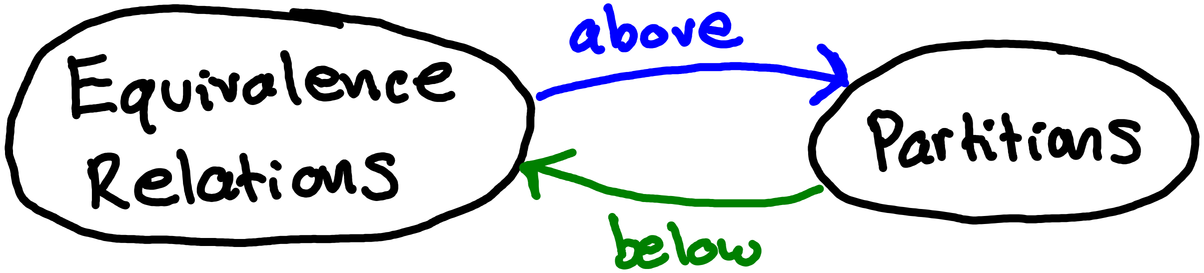
"~" is an equivalence relation on $\mathbb{Z}$

$1 \sim 2 \sim 3, \; 4 \sim 5 \sim 6 \sim 7$

$$\mathbb{Z} = \mathbb{B} \sqcup (\mathbb{Z} - \mathbb{B}) = \{1,2,3\} \sqcup \{4,5,6,7\}$$

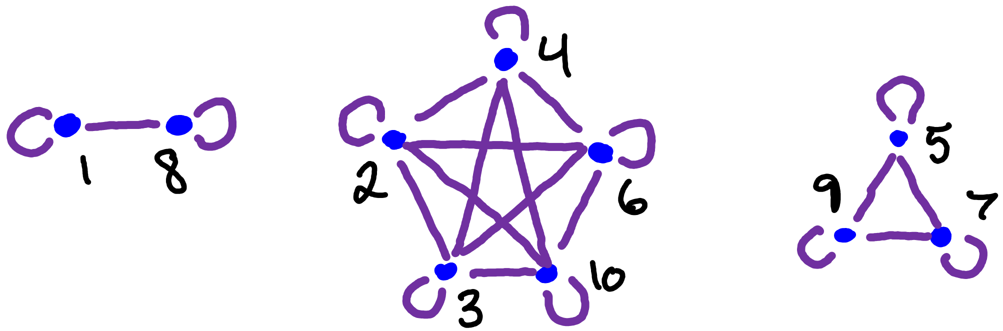
Equivalence Relations **Partition** the underlying set.

Def: A *partition* of a set X is a collection of mutually disjoint (& nonempty) subsets of X whose union is X.



$$[10] = \{1,8\} \uplus \{2,4,6,10,3\} \uplus \{5,7,9\}$$

- 3 "Equivalence Classes"
- 2, 5 & 3 "Representatives" (respectively)



- 3 "connected components"
- "complete Graphs on 2, 5 & 3 vertices" respectively

Easy to write down the relation now

$$R = \{(a,b) \mid a \sim b\}$$

↑ equivalence relation

Application: The Mod 2 equivalence relation

$R := \{(a,b) \in \mathbb{Z}^2 \mid a-b \text{ is even}\}$

Notation: If $(a,b) \in R$ we will write $a \equiv b \pmod 2$

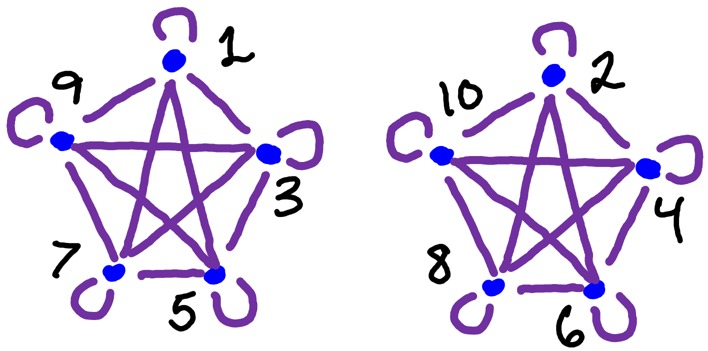
Claim: $a \equiv b \pmod 2 \iff$ both are even xor both are odd

proof:

$$\text{Even} - \text{Odd} = 2n - (2k+1) = 2(n-k) - 1 = \underbrace{\text{Odd}}_{\notin R}$$

$$\left.\begin{aligned} \text{Even} - \text{Even} &= 2n - 2k = 2(n-k) = \text{Even} \\ \text{Odd} - \text{Odd} &= 2n+1 - (2k+1) = 2(n-k) = \text{Even} \end{aligned}\right\} \in R$$

$\square$



- All odd numbers represent the same equivalence class
- All even numbers represent the same equivalence class
- No equivalence class has both an even number & an odd number in it

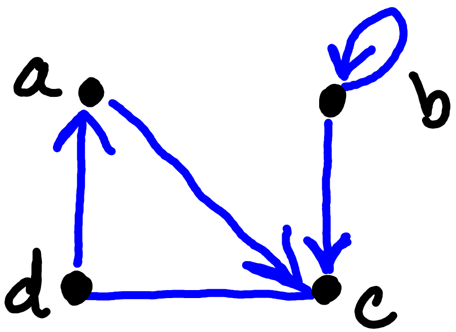
Relations	0,1-Matrices	Directed Graphs
$\cup$ - Set union	$\vee$ - Join (treat 0 as F & 1 as T then do $\vee/\wedge$ (or/and) entry by entry)	Set operations w/ arrows rather than elements
$\cap$ - set intersection	$\wedge$ - Meet (treat 0 as F & 1 as T then do $\vee/\wedge$ (or/and) entry by entry)	
$-$ - set difference		
$(-)^{-1}$ - Inverse	$(-)^{T}$ - transpose	reverse directions of arrows
$\circ$ - Composite	$\odot$ - boolean product	Directed Paths
$(-)^{*}$ - Connectivity	$M_R^{*} = M_R \vee M_R^{[2]} \vee \dots \vee M_R^{[n]}$	Directed Paths
Closure	**Closure**	**Closure**
• $R \cup \Delta$ - Reflexive	• Make diagonals all 1s	• Loops @ all vertices
• $R \cup R^{-1}$ - Symmetric	• Add 1s on off-diagonal until matrix is symmetric	• Arrows become Edges
• $\overline{R}$ - transitive	• Marshall's Algorithm	• Directed Paths

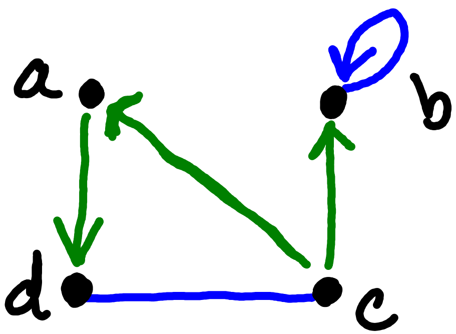
Inverse Relation: $(x,y) \in R^{-1} \Leftrightarrow (y,x) \in R$

$$R := \{(a,c),(b,b),(b,c),(c,d),(d,c),(d,a)\}$$

$$R^{-1} = \{(c,a),(b,b),(c,b),(d,c),(c,d),(a,d)\}$$

Dual Graph: Reverse direction of arrows





Matrix Transpose: $(m_{ij})^T = (m_{ji})$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Rows Here

Become Columns Here

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Closure Operations

Let R be a relation on A and P be a property (Reflexivity/symmetry/transitivity) that R **fails to have**.

We can ask for the smallest relation containing R (as a set) which **Does have** property P.

Q: How can we describe the closure of R w.r.t. P in another way?

A: Let $X = \{ S \subseteq A^2 \mid R \subseteq S \wedge S \text{ has property } P \}$

then the closure we seek is always $\bigcap_{S \in X} S$

Q: How do we compute closures in practice?

A.

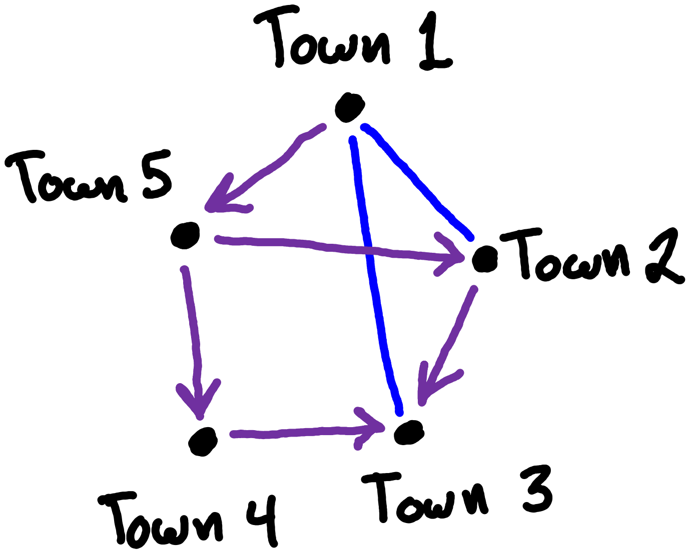
Reflexive

$\Delta := \{ (a,a) \in A^2 \}$ "the diagonal"

$R \cup \Delta$ = "Reflexive closure"

Symmetric

$R \cup R^{-1}$ = "Symmetric closure"



The transitive Closure

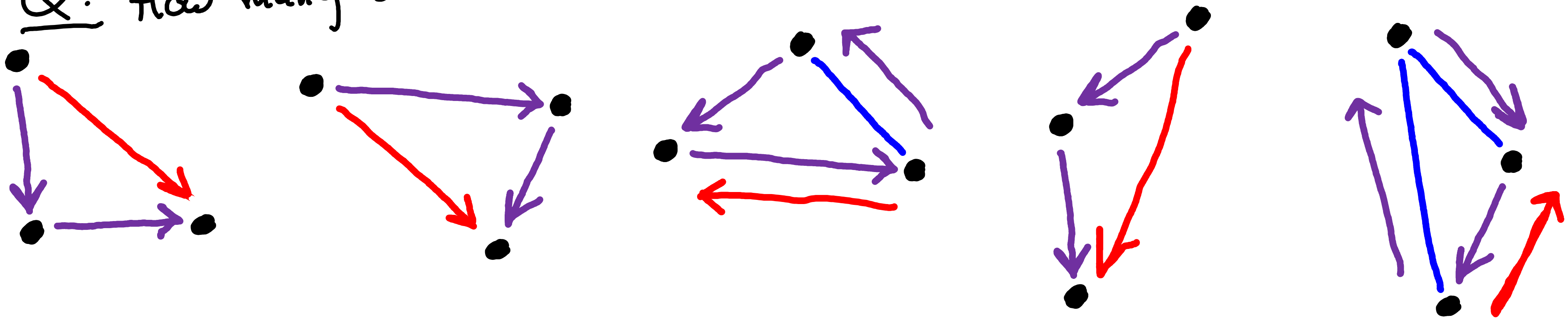
Recall: this "relation" is not transitive

$\Rightarrow \neg \forall a \forall b \forall c \ (a,b),(b,c) \in R \to (a,c) \in R$

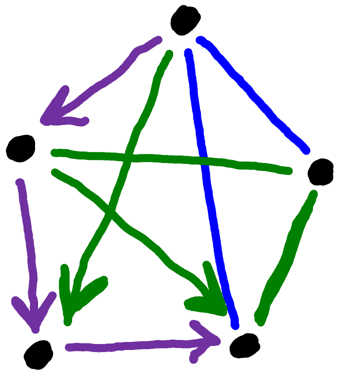
$\equiv \exists a \exists b \exists c \ \neg (a,b),(b,c) \in R \to (a,c) \in R$

(i.e. some triangles need to be filled in)

Q: How many such counterexamples can we find?

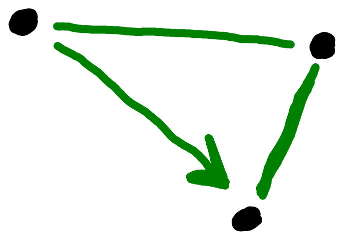


Filling in all of the missing triangles we end up with

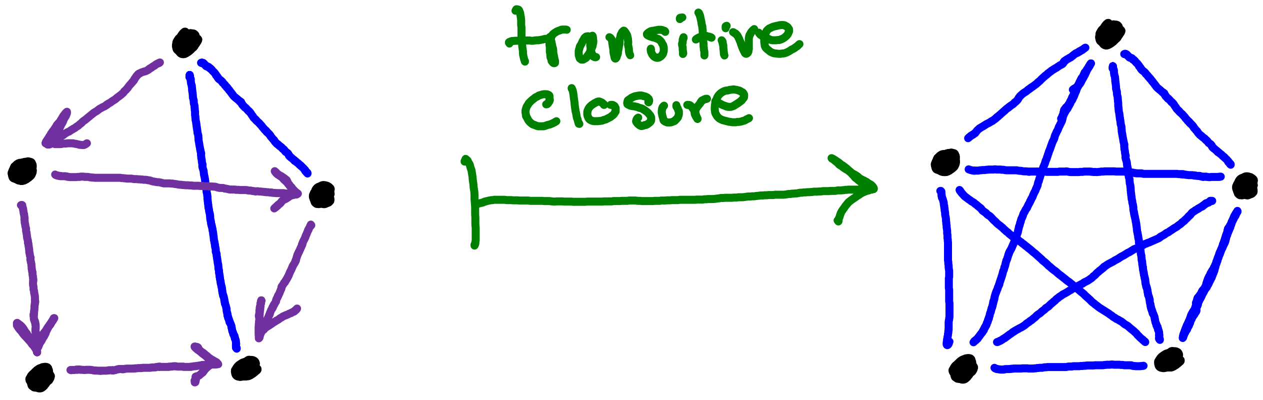


Q: Is this new "relation" transitive?

A: No



Finding the transitive closure means iterating the Question above & filling in missing triangles ($R \circ R$ = composition)



— 2 lane road

→ 1 way road

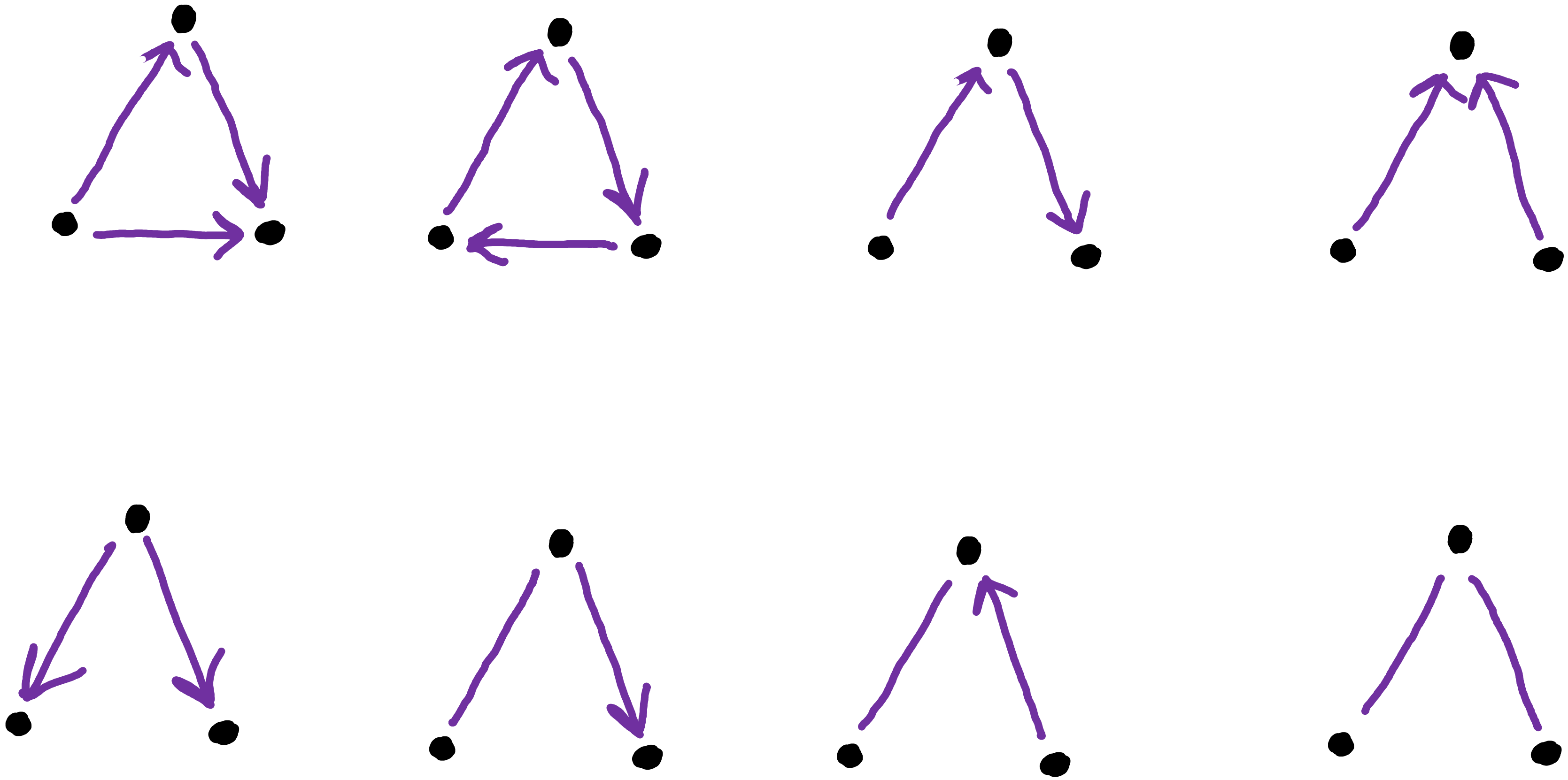
• → • ∃ sequence of roads (directed path) from • to •

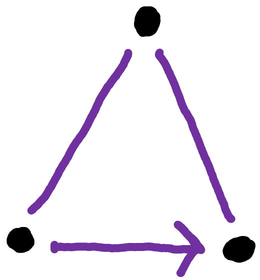
Edges ~ "you can drive DIRECTLY from tail to tip"

(R = original relation)

Edges ~ "you can eventually drive from tail to tip using roads"

(R^* = Connectivity Relation)





Q: Which are transitive when thought of as a relation on $\mathcal{B}$?

What is the transitive closure of the nontransitive ones?

Can you describe a condition on an adjacency matrix for it to describe a transitive relation?