Arrays & Sequences

Arrays Lists that can store objects

Value	1	2	4	8	16	32	64	128	256	?
index	0	1	2	3	4	5	6	7	8	9

Finite Sequences: Ordered lists of (real) numbers

terms $a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9$

position

$$a_i := 2^i$$

Functions from finite sets

$$f : \{0\} \cup [9] \longrightarrow \mathbb{N}, \quad f(i) = 2^i$$

input values

output values

Notation: $\{a_i\}_{i=m}^{n}$ $\quad a_m, a_{m+1}, \ldots, a_{n-1}, a_n$

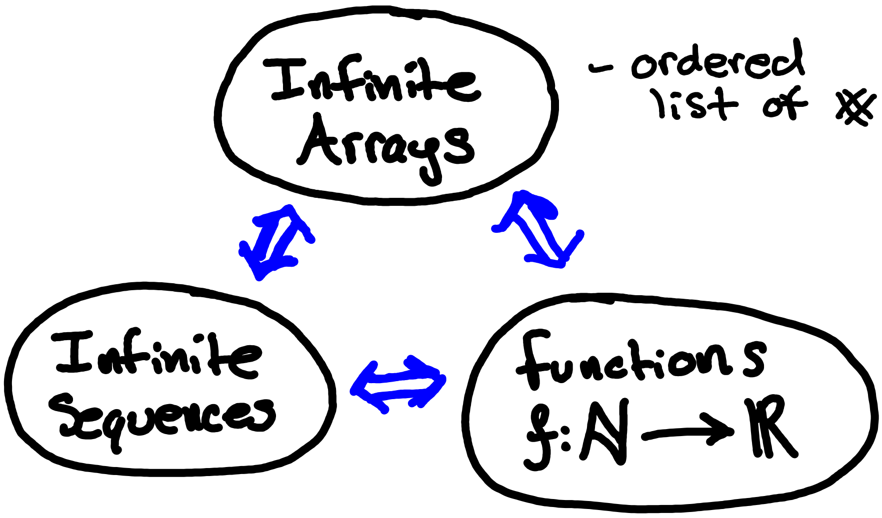
finite sequence whose indices are between m & n inclusive.

$\{a_i\}_{i \in \{0,2,3,9\}}$ $\quad a_0, a_2, a_3, a_9$

indices can come from a set called an "indexing set"

Infinite Sequences

$$\{a_i\}_{i=1}^{\infty} := \{a_i\}_{i \in \mathbb{N}} \qquad a_1, a_2, \ldots, a_n, \ldots$$



Q: How do we specify a sequence?

- Examples of terms w/ small indices (need enough to establish a pattern) — Show part of an array & claim there is a pattern

(a) $1, 8, 15, 22, 29, \ldots$

$+7 \quad +7 \quad +7 \quad +7 \quad \ldots$

(b) $1, 4, 16, 64, 256, \ldots$

$\times 4 \quad \times 4 \quad \times 4 \quad \times 4 \quad \ldots$

(c) $1, 2, 4, 7, 11, 16, 21, \ldots$

$+1 \quad +2 \quad +3 \quad +4 \quad +5 \quad +6 \quad \ldots$

(d) $a, ar, ar^2, ar^3, \ldots$

$\times r \quad \times r \quad \times r \quad \ldots$

- provide an expression for the n^{th} term in terms of a general input] view as a function

(a) $a_n = 1 + n \cdot 7 \longleftrightarrow f(n) = 1 + n \cdot 7$

(b) $b_n = 4^n \longleftrightarrow f(n) = 4^n$

(c) $c_n = 1 + (0 + 1 + 2 + \ldots + i) = 1 + \sum_{i=0}^{n} i$

later

(d) $d_n = ar^n \longleftrightarrow f(n) = ar^n$

(e) $e_n = \frac{1}{n} \longleftrightarrow f(n) = 1/n$

What is the rule for getting to the next term?

$$1/1, \xrightarrow{\times 1/2} 1/2, \xrightarrow{\times 2/3} 1/3, \xrightarrow{\times 3/4} 1/4, \ldots$$

- Sequences can be described **recursively**: n^{th} term is described using any terms coming before it & some initial conditions are provided (first few terms)

(a) $$1, \xrightarrow{+7} 8, \xrightarrow{+7} 15, \xrightarrow{+7} 22, \xrightarrow{+7} 29, \ldots$$

$a_0 = 1, \quad a_n = a_{n-1} + 7 \quad n > 0$

(b) $1,\ 4,\ 16,\ 64,\ 256,\ \ldots$

$\times 4 \quad \times 4 \quad \times 4 \quad \times 4 \quad \ldots$

$a_0 = 1, \quad a_n = 4a_{n-1} \qquad n > 0$

(c) $1,\ 2,\ 4,\ 7,\ 11,\ 16,\ 21,\ \ldots$

$+1 \quad +2 \quad +3 \quad +4 \quad +5 \quad +6 \quad \ldots$

$a_0 = 1, \quad a_n = a_{n-1} + n \qquad n > 0$

(d) $a,\ ar,\ ar^2,\ ar^3,\ \ldots$

$\times r \quad \times r \quad \times r \quad \ldots$

$a_0 = a, \quad a_n = r a_{n-1} \qquad n > 0$

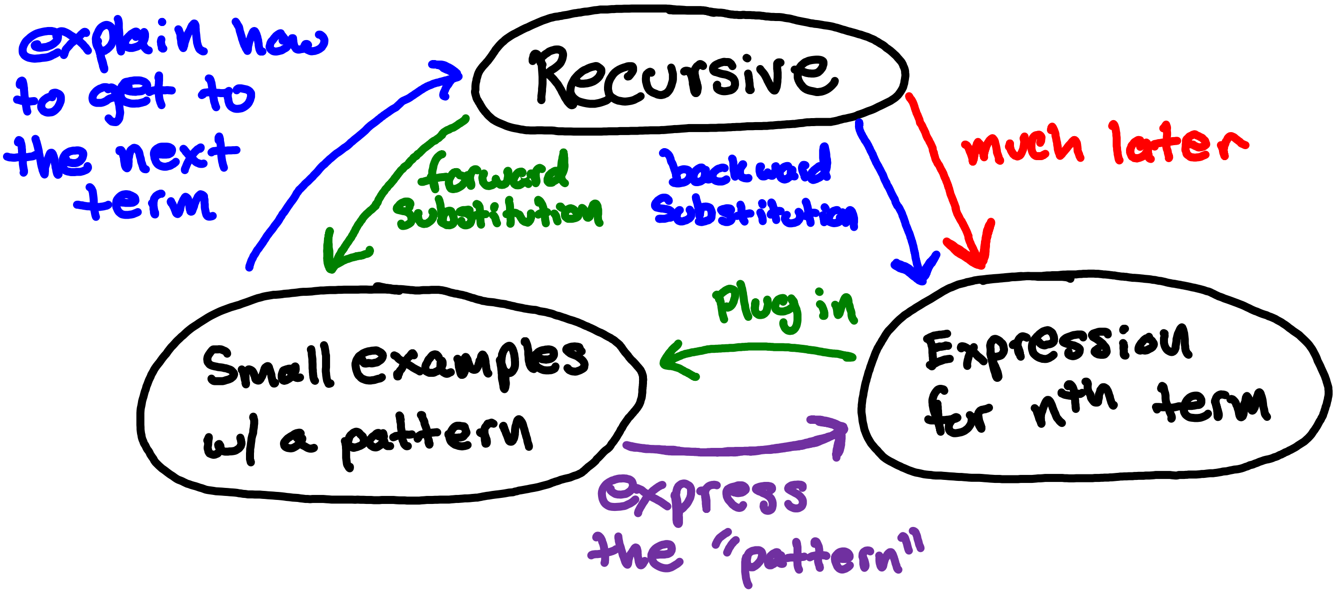
(e) $1,\ 1/2,\ 1/3,\ 1/4,\ \ldots$

$\times 1/2 \quad \times 2/3 \quad \times 3/4 \quad \ldots$

$a_1 = 1, \quad a_n = \frac{n-1}{n} a_{n-1} \qquad n > 1$

Q: What kinds of things do we do with sequences?

0) Basic: Translate between the three descriptions above
1) Math: Add up all the terms (summation)
2) CS: Iterate through them (loops, filling arrays, sorting searching)



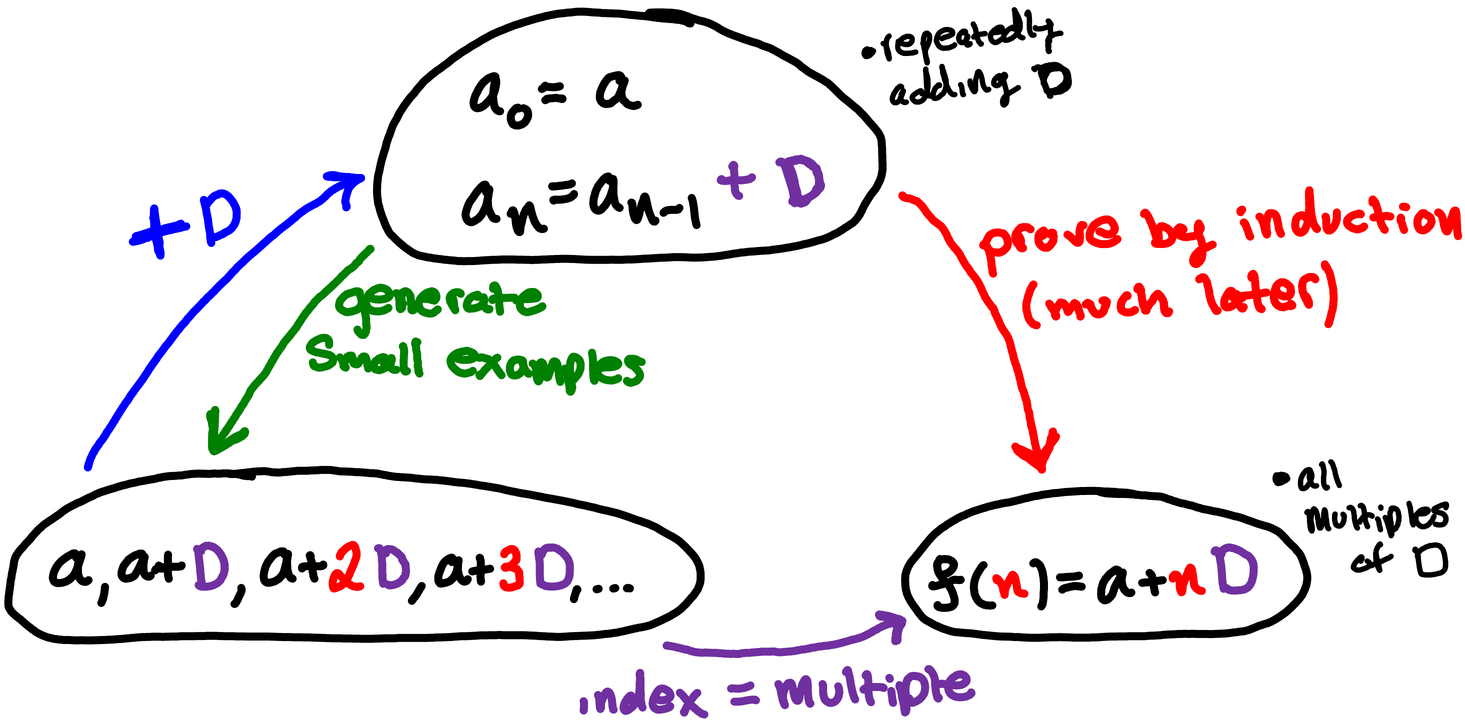
■ ⇒ easy

■ ⇒ harder

■, ■ = medium difficulty

Two Categories

(I) Arithmetic Progression (multiplication is repeated addition)



$$\underbrace{\longrightarrow}_{\text{can do directly but not yet}} \overset{\text{for now}}{\Longleftarrow} \underbrace{\longrightarrow + \longrightarrow}_{\text{generate small terms \& find the pattern}}$$

finding them in the wild

Note: D is the "common difference"

$a_{n+1} - a_n = D$

(II) Geometric Progression (exponents are repeated multiplication)

- Example (d) from earlier ($r =$ "common ratio")

Remark: treating $+$ like $\times$ & $\times$ like exponents takes you from example (I) to example (II)

Special Examples

(III) Sum of the first n integers (will be investigated further below)

- Example (c) above. The next example

Remark: the relationship between (I) & (II) is shared by (III) & (IV) resp.

(IV) (Factorials)

$$\begin{array}{l} a_0 = 1 \\ a_n = n a_{n-1} \end{array} \xrightarrow{\text{induction}} n! := n \cdot (n-1) \cdots 2 \cdot 1$$

$a_n = n! = n \cdot (n-1)! = n \cdot a_{n-1}$

(V) (fibonacci numbers)

let the colors explain

$$\underbrace{\overbrace{0, 1}^{\oplus}}, \overbrace{1, 2}^{\oplus}, 3, 5, 8, 21, 29, \ldots$$

$a_0 = 0$

$a_1 = 1$

$a_n = a_{n-1} + a_{n-2}$

solve a recurrence relation

$$a_n = \frac{\eta^n - (1-\eta)^n}{\sqrt{5}}$$

$$\eta = \frac{1+\sqrt{5}}{2}$$

induct!

Summation

$\sum$ Array ↔ $\sum$ Sequence ↔ $\sum$ Function

(I) Finite

sequence	Sum of terms
$\{a_i\}_{i=m}^{n}$	$\sum_{i=m}^{n} a_i = \sum_{k=m}^{n} a_k$
$\{a_i\}_{i \in I}$	$\sum_{i \in I} a_i = \sum_{i \in I} f(i)$

Dummy indexing variable

↳ iterates through all terms like a loop.

e.g. Begin w/ an ∞ sequence

$\{a_n\}_{n=1}^{\infty}$ $\quad a_n = n \quad$ 1, 2, 3, 4, 5, ...

Partial Sum makes a new sequence (finite)

$$\{b_i\}_{i=1}^{N} \qquad b_i = \sum_{n=1}^{i} a_n = \sum_{n=1}^{i} n = 1+2+3+\dots+i$$

(note: (c) from earlier $c_i = b_i + 1$)

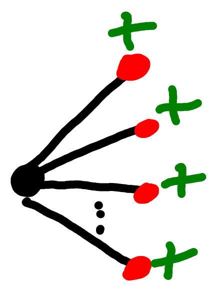
If we store the first 10 terms of a_n in an array, we can use a loop to find the partial sums. ($\{a_n\}_{n=1}^{10}$ as an array)

Counting := (int array)

1	2	3	4	5	6	7	8	9	10
0	1	2	3	4	5	6	7	8	9

$$b_K = \sum_{i=0}^{K-1} \text{Counting}(i)$$

```
i = 0
b_K = 0
For i ≤ K-1
    b_K += Counting(i)
    i += 1
```



Filling an empty array w/ b_Ks

Sums :=

0	1	2	3	4
1	3	6	10	15

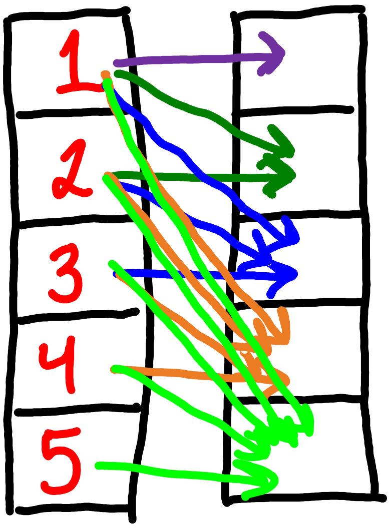
Nested Loops

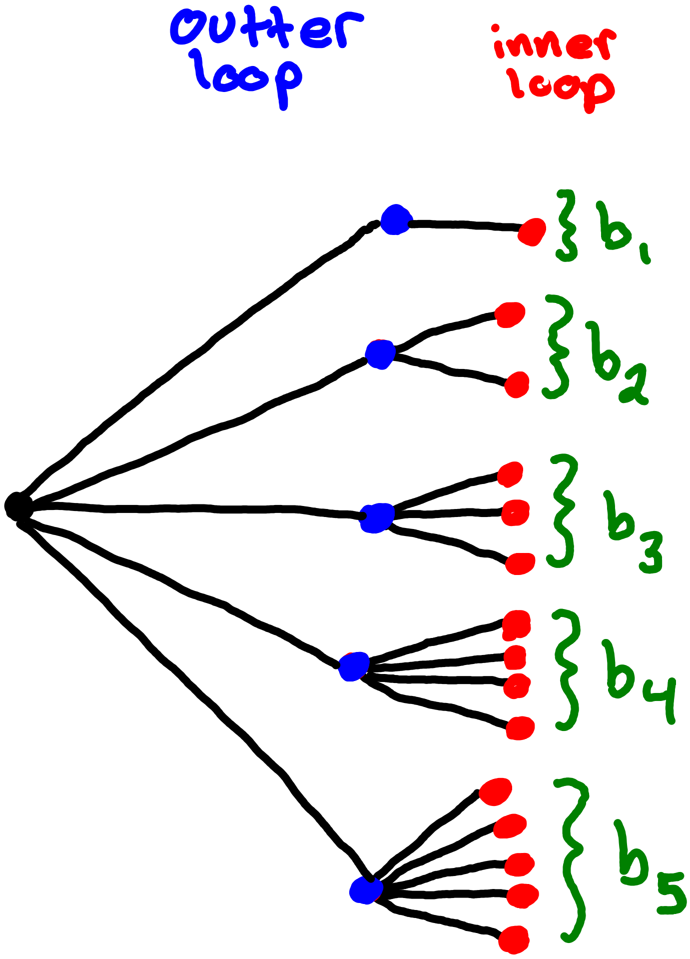
Computing b_K uses one loop

traversing the array takes another

```
i=0
j=0
K=0
For i ≤ 9
    b=0
    For j ≤ i
        b += Counting(j)
        j += 1
    Sums(i) = b
    i += 1
```

Computing $\sum_{n=1}^{i} n$ as above





Note: *Red dots in above picture is

$$1+2+3+4+5=\sum_{n=1}^{5} n = b_5$$

Double Sums ≤ Nested Loops

$\{a_n\}_{n=1}^{\infty}$	$a_n = n$	$1, 2, 3, 4, 5, \ldots$
$\{b_i\}_{i=1}^{N}$	$b_i = \sum_{n=1}^{i} a_n$	partial sums of $\{a_n\}$
$\{c_j\}_{j=1}^{M}$	$c_j = \sum_{i=1}^{j} b_i$	partial sums of $\{b_i\}$

$$c_j = \sum_{i=1}^{j} b_i = \sum_{i=1}^{j}\left(\sum_{n=1}^{i} a_n\right)$$

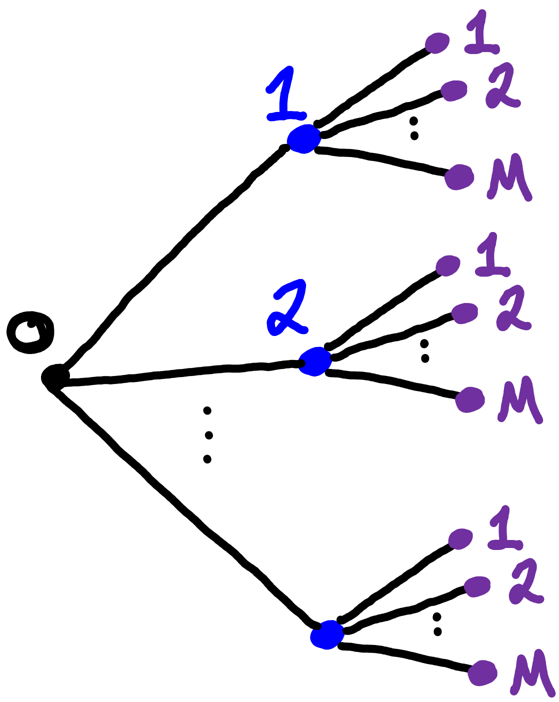
Exercise: Write an algorithm for filling an empty array of size 3 with the first 3 terms of $\{c_j\}$

Multiplication, $\sum\sum$, Nested Loops & Trees

$$N \cdot M = \sum_{i=1}^{N}\sum_{j=1}^{M} 1$$

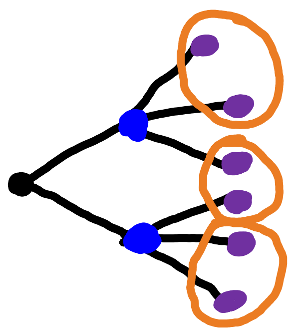
N groups of M

```
For i ≤ N
    For j ≤ M
        total += 1
```



e.g.

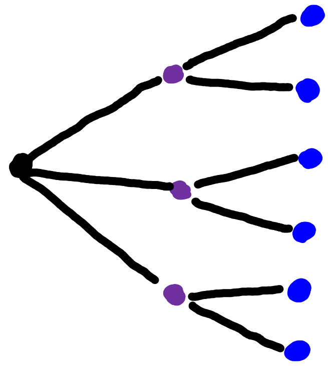
$$2 \cdot 3 = \sum_{i=1}^{2}\sum_{j=1}^{3} 1$$



(how do we make this look line 3 groups of 2?)

$$3\cdot 2 = \sum_{i=1}^{3}\sum_{j=1}^{2} 1$$

(how do we make this look like 2 groups of 3?)

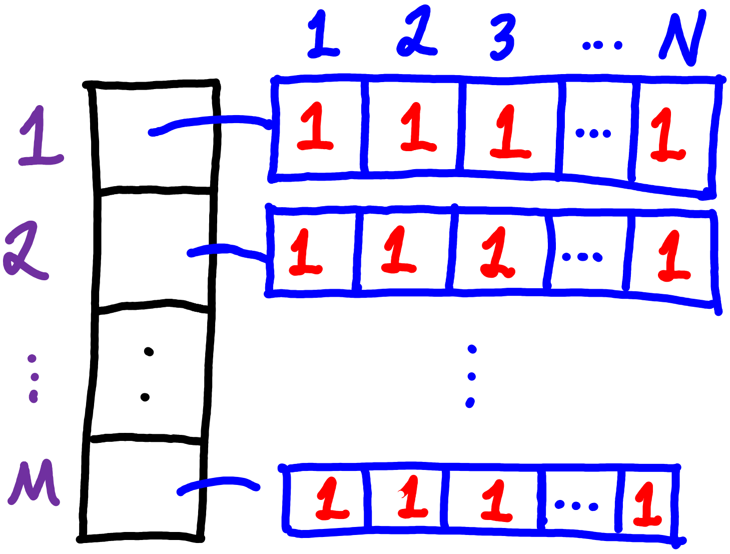


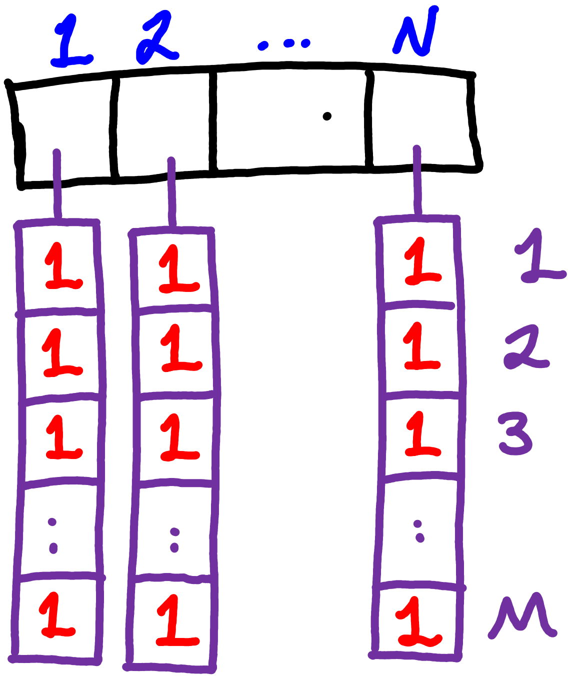
With this definition of multiplication the fact that $N\cdot M = M\cdot N$ is not obvious & requires a proof

proof: Stuck $\Rightarrow$ Change Perspective

(we had this analogy "$\sum \longleftrightarrow$ Loops" so, lets think of these sums as filling arrays)

We arrange the 1s we need to sum in a double array (an array of arrays)





Outter Loop

Inner Loop

$$\begin{array}{cc} 4 & \begin{bmatrix}1&1&1\\1&1&1\\1&1&1\\1&1&1\end{bmatrix} \\ & 3 \end{array}$$

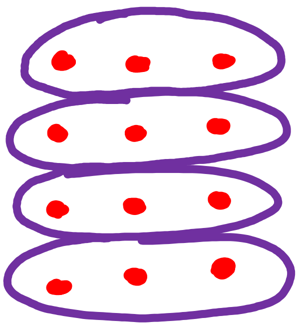
Outter Loop

Inner Loop

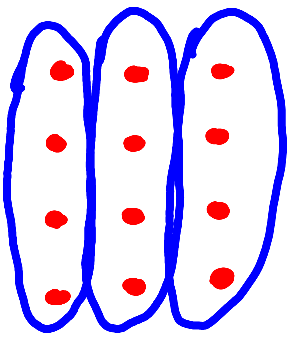
$$\begin{array}{cc} 3 & \begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix} \\ & 4 \end{array}$$

it makes the same rectangular grid of numbers (Matrix) except they differ in orientation

We counted the same thing in two different ways



First by counting rows



Then by counting columns

□

Matrices (sequences of sequences)

Rectangular Arrays of numbers

(a_{ij}) Has i rows & j columns

e.g.

The 3 by 2 matrix

$$(i+j) = \begin{pmatrix} 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{pmatrix}$$

A general 3×3 matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The 3×3 matrix

$$\left((-1)^{i+j}\right) = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

Actually Computing Sums

(I) $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

(II) $$\sum_{i=0}^{n} ar^i = \begin{cases} \dfrac{ar^{n+1} - a}{r-1} & r \neq 1 \\ (n+1)a & r = 1 \end{cases}$$

proof of (I): (Famously due to Gauss)

$$\begin{aligned}
&\sum_{i=1}^{n} i = 1 + 2 + \dots + (n-1) + n \\
+\ &\sum_{i=1}^{n} (n-i+1) = n + (n-1) + \dots + 2 + 1 \\
\hline
&\sum_{i=1}^{n} (n+1) = \underbrace{(n+1) + (n+1) + \dots + (n+1) + (n+1)}_{n \text{ times (repeated addition is multiplication)}}
\end{aligned}$$

$$\underbrace{2\left(\sum_{i=1}^{n} i\right)}_{\substack{\text{order of summation} \\ \text{does not matter if} \\ \text{you only add finitely} \\ \text{many terms}}} = n\cdot(n+1) \iff \sum_{i=1}^{n} i = \frac{n\cdot(n+1)}{2} \quad \square$$

proof of (II): $n \neq 1$ case

$$S_n := \sum_{i=0}^{n} ar^i$$

$$rS_n = r\sum_{i=0}^{n} ar^i = \sum_{i=0}^{n} ar^{i+1}$$

Like u-sub from CALC I change variable, bounds & function

Change of variables

$K := i+1$

$i=0 \Rightarrow K=1$

$i=n \Rightarrow K=n+1$

$$= \sum_{K=1}^{n+1} ar^K$$

$$= ar^1 + ar^2 + \dots + ar^{n+1}$$

add 0 in the form $(a-a)$

$$= (a-a) + ar^1 + ar^2 + \dots + ar^{n+1}$$

move terms around

$$= ar^{n+1} - a + \sum_{K=0}^{n} ar^K$$

$$= ar^{n+1} - a + S_n$$

Conclude: $rS_n = S_n + ar^{n+1} - a$

$$\Longleftrightarrow \quad (r-1)S_n = ar^{n+1} - a$$

uses the assumption $r \neq 1$

$$\Longleftrightarrow \quad \boxed{S_n = \frac{ar^{n+1} - a}{r-1}}$$

Remark: Another reason you may want to change variables in a sum is so you can combine two sums that start @ different indices.