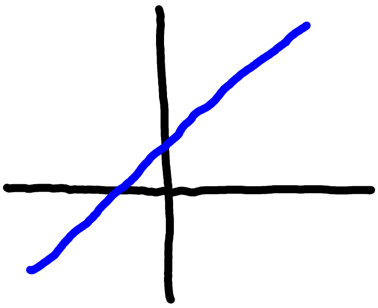
Calculus

$f : \mathbb{R} \to \mathbb{R}$

$f(x) = x + 1$

<u>Graph of f</u>

$\{(x, f(x)) \in \mathbb{R} \times \mathbb{R}\}$



passes vertical line test

Injective

$\Updownarrow$

passes horizontal line test

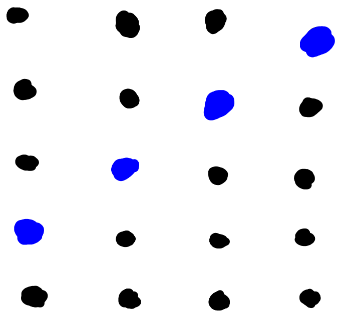
Discrete

$f : 4 \to 5$

$x \mapsto x + 1$

<u>f as a Relation</u>

$\{(x, f(x)) \in 4 \times 5\}$

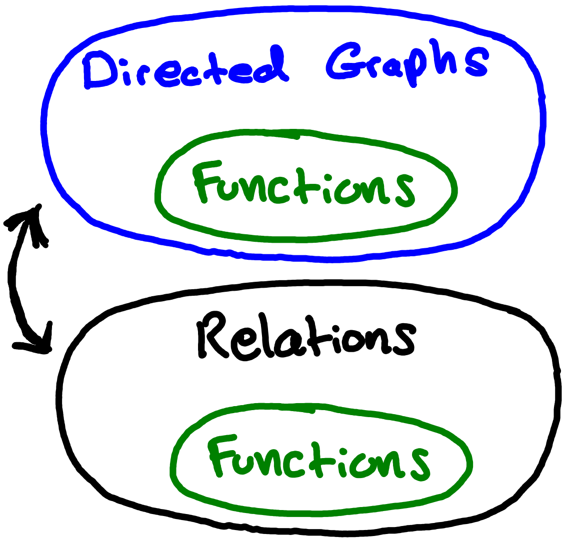


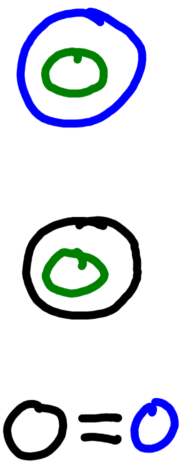
passes vertical line test

Injective

$\Updownarrow$

passes horizontal line test





- Lecture 1
- The task at hand
- Later

Sets (the details)

(i) $\{\text{element}, \text{element}, \ldots, \text{element}\}$

(ii) $A = B \iff$ they have the same elements

↳ Order of elements does not matter

↳ Repeat elements are ignored

e.g. $\emptyset$ No elements

(i) $\{\emptyset\}$ 1 element

$\{\emptyset, \{\emptyset\}\}$ 2 elements

$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ 3 elements

$\{\{\{\emptyset\}\}, \emptyset, \{\emptyset, 0, \{a,b\}\}, c\}$ How many elements?

Answer: $|\mathrm{Domain}(f)|$

↑ the function on the "Discrete" side of the previous page

(ii) $\{a,a,a,b,b,c\} = \{c,a,b\}$

proof: $\{a,a,a,b,b,c\} = \{a,b,b,c\} = \{a,b,c\}$

$\{a,b,c\} = \{a,c,b\} = \{c,a,b\}$

Q: What is $\mathbb{3} \cup \mathbb{5}$?

A: $\mathbb{3} \cup \mathbb{5} = \{1,2,3\} \cup \{1,2,3,4,5\}$

$= \{1,2,3,1,2,3,4,5\}$

$= \{1,1,2,2,3,3,4,5\}$

$= \{1,2,3,4,5\} = \mathbb{5}$

Q: What is $(\mathbb{4} \times \mathbb{5}) \cap ((\mathbb{3} - \mathbb{1}) \times \mathbb{2})$?

A: List the elements

$\mathbb{3} - \mathbb{1} = \{1,2,3\} - \{1\} = \{2,3\}$

$(\mathbb{3} - \mathbb{1}) \times \mathbb{2} = \{(2,1),(2,2),(3,1),(3,2)\} \subseteq \mathbb{4} \times \mathbb{5}$

$\Rightarrow (\mathbb{4} \times \mathbb{5}) \cap ((\mathbb{3} - \mathbb{1}) \times \mathbb{2}) = (\mathbb{3} - \mathbb{1}) \times \mathbb{2}$ □

Claim: $A \subseteq B \Rightarrow A \cap B = A$

(remains to prove this claim in order to finish the justification of the "Answer" above)

proof: Direct Proof

Assume $A \subseteq B$ (*)

We prove $A \cap B = A$ by showing

$A \cap B \subseteq A$ & $A \cap B \supseteq A$

$\subseteq$ Let $x \in A \cap B$. By definition of the intersection we have $x \in A \wedge x \in B$ is true. Since

$$\begin{array}{l} P \wedge Q \\ \hline \therefore\ P \end{array}$$

is a valid argument, we have that $x \in A$. This proves $A \cap B \subseteq A$.

$\supseteq$ Let $x \in A$. We use the assumption (*) & the definition of "subset" to see

$$x \in A \Rightarrow x \in B$$

Since

$$\begin{array}{l} P \\ P \Rightarrow Q \\ \hline \therefore\ Q \end{array}$$

is a valid argument we have $x \in B$. Finally, the valid argument

$$\begin{array}{l} P \\ Q \\ \hline \therefore\ P \wedge Q \end{array}$$

allows us to conclude

$x \in A \wedge x \in B$ which, by definition of intersection, means $x \in A \cap B$. This proves $A \subseteq A \cap B$.

Since $X \subseteq Y$ & $Y \subseteq X \Leftrightarrow X = Y$ we are done

$\square$

Suppose $A \subseteq B$. What is $A \cup B = ?$

prove your answer.

<u>Claim:</u> $A \subseteq B \Rightarrow A \cup B = B$

<u>proof:</u> $B \subseteq A \cup B$ always

<u>Suppose</u> $A \subseteq B$ and let $x \in A \cup B$

either $x \in B$ or

$x \in A \Rightarrow x \in B$ (by assumption)

thus $A \subseteq B \Rightarrow A \cup B \subseteq B$

which proves the claim $\square$

- when proving $=$ by $\subseteq$ & $\supseteq$

  OR $\Leftrightarrow$ by $\Rightarrow$ & $\Leftarrow$

there is usually an "easy direction" and a "hard direction."
(you may want to use a hypothesis to prove "hard directions")

this is exactly what happened in the above proofs.

Functions (the details)

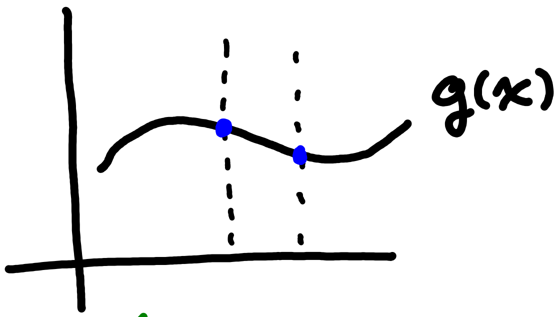
Q: What does it take to specify a function?

$$\underset{\text{name}}{f} : \underbrace{4}_{\text{domain}} \longrightarrow \underbrace{5}_{\text{codomain}}$$

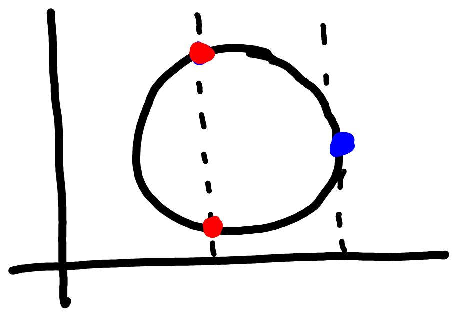
(An ordered pair of sets)

A rule for assigning to each element of the domain, exactly 1 element of the codomain.

Calculus: "the vertical line test"

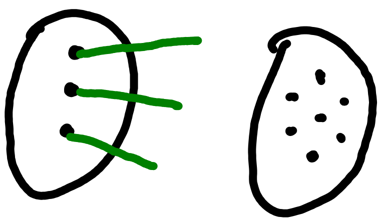


✓ Function

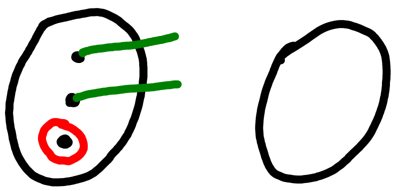


X Not a Function

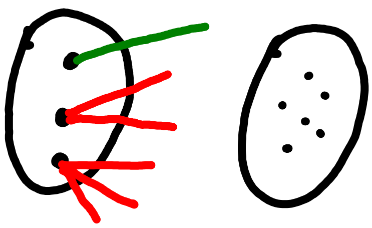
Discrete: $\forall x \, \exists! y \;\; f(x) = y$



✓ Function



X Not a Function



X Not a function

$f: A \longrightarrow B$ can be thought of as a subset of $A \times B$ (i.e. a relation)

$$F := \{(a, f(a)) \in A \times B\}$$

<u>Exercise</u>: A relation $R \subseteq A \times B$ satisfies

(i) $\forall a \in A \ \exists b \in B$ s.t. $(a,b) \in R$

(ii) $(a,b) = (a,c) \Rightarrow b = c$

iff the rule of assignment

$$f(a) = b \Leftrightarrow (a,b) \in R$$

defines a function $f: A \longrightarrow B$.

Equality of Functions

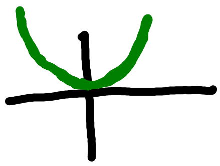
$f: A \longrightarrow B$
$g: C \longrightarrow D$

$$f = g \iff \begin{array}{l} A = C \\ B = D \\ f(a) = g(a) \ \forall a \in A \end{array}$$

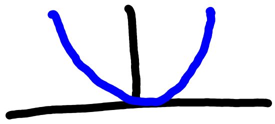
<u>e.g.</u> $f: \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto |x|$ $\qquad g: \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto \sqrt{x^2}$

$$f = g$$

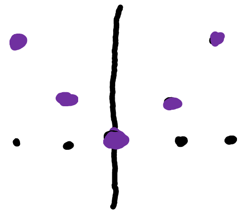
$$\tilde{f}: \mathbb{R} \longrightarrow \mathbb{R}, \quad x \mapsto x^2$$



$$\tilde{g}: \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}, \quad x \mapsto x^2$$



$$\tilde{h}: \mathbb{Z} \longrightarrow \mathbb{R}, \quad x \mapsto x^2$$



$$\tilde{h} \neq \tilde{f} \neq \tilde{g} \neq \tilde{h}$$

(different domain/codomain)

Injective Functions

$f: A \hookrightarrow B$ s.t.

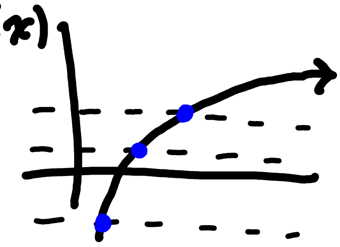
$$f(a_1) = f(a_2) \Rightarrow a_1 = a_2$$

OR equivalently (contrapositive)

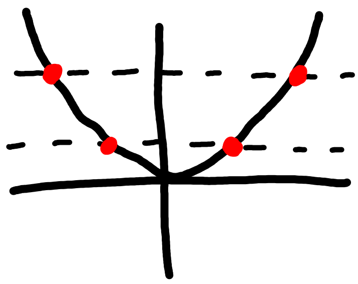
$$a_1 \neq a_2 \Longrightarrow f(a_1) \neq f(a_2)$$

Calculus: "the Horizontal Line test"

$f(x) = \ln(x)$



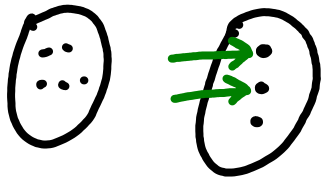
✓ Injective



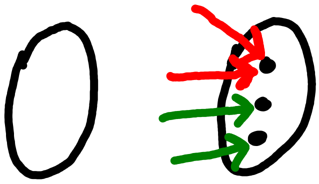
$f(x) = x^2$

✗ Not injective

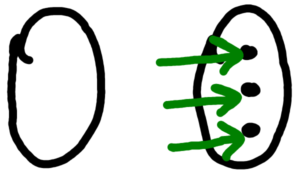
Discrete: $\forall x \forall y\, (f(x)=f(y) \Rightarrow x=y)$



✓ Injective



✗ Not Injective

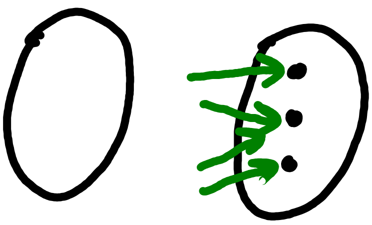


✓ Injective

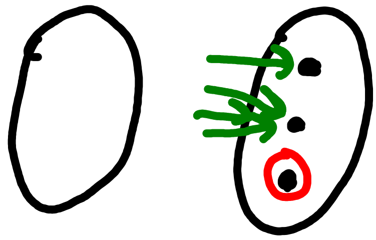
Surjective Functions (everything gets pointed to)

$f: A \twoheadrightarrow B$ s.t.

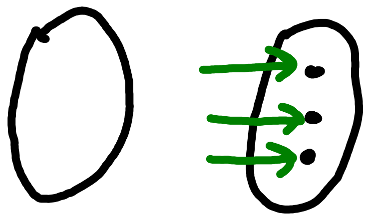
$\forall b \in B \;\; \exists a \in A$ with $f(a)=b$



✓ Surjective



✗ Not Surjective



✓ Surjective

Range

$$\text{Range}(f) := \{ b \in B \mid \exists a \in A \;\; f(a)=b \} \subseteq B$$

(everything that is pointed to)

the codomain of f

$\boxed{f: A \twoheadrightarrow B \text{ Surjective}} \Leftrightarrow \boxed{\text{Range}(f) = B}$

Bijections

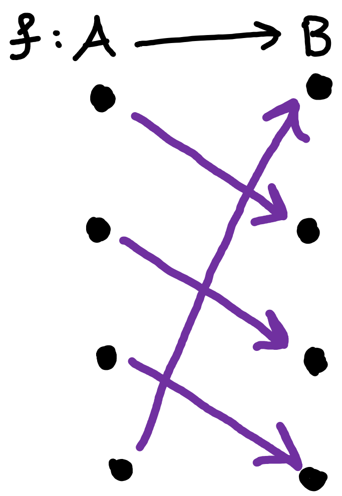
Notation: $f : A \hookrightarrow\!\!\!\!\twoheadrightarrow B$, $f . A \xrightarrow{\sim} B$

injective ⊕ surjective

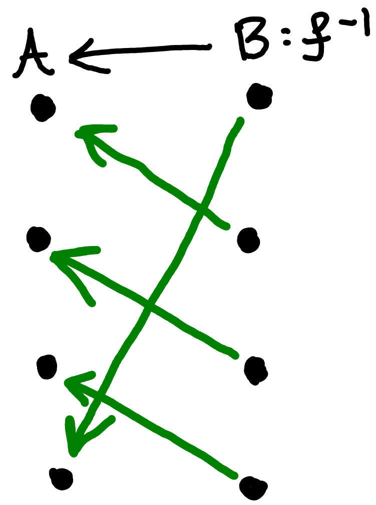
The sets A & B are in one-to-one correspondence.

Inverse Functions

Beginning w/ a bijection $f : A \hookrightarrow\!\!\!\!\twoheadrightarrow B$



If we reverse all the arrows

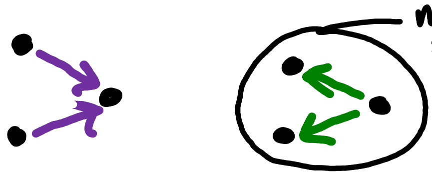


the result is again a function

Q: Is beginning with a bijection necessary? **YES**

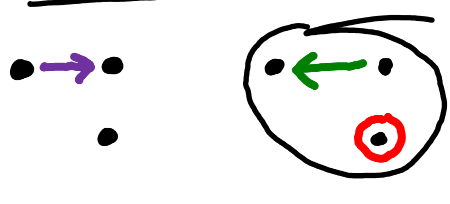
(i.e. If we reverse the arrows beginning w/ something other than a bijection, can we still get a function?) **No**

Fails Injectivity



not allowed in functions

Fails Surjectivity

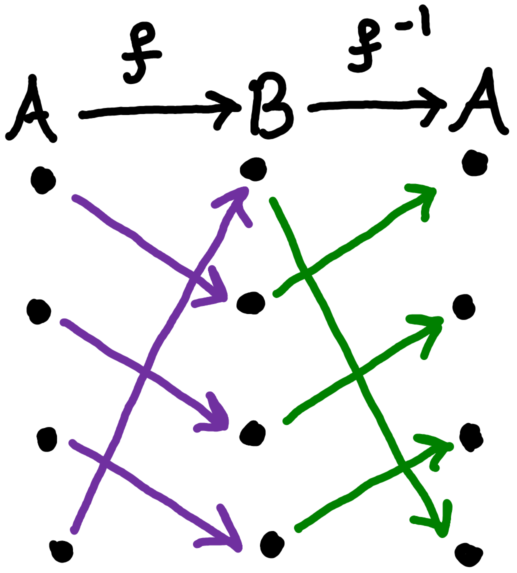


functions assign a value to Everything in the domain

Composition of Functions

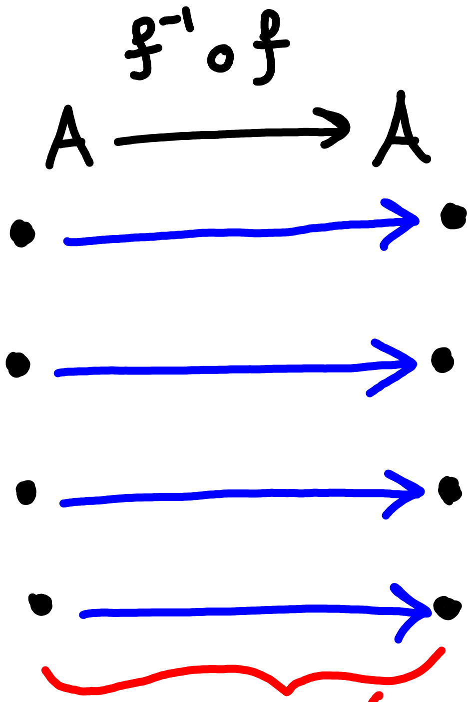
$f: A \to B$, $g: B \to C$

$g \circ f$



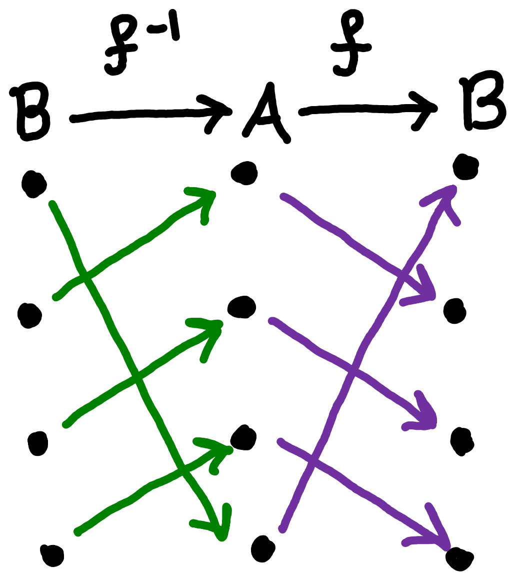
forget the middle

compose $\mapsto$

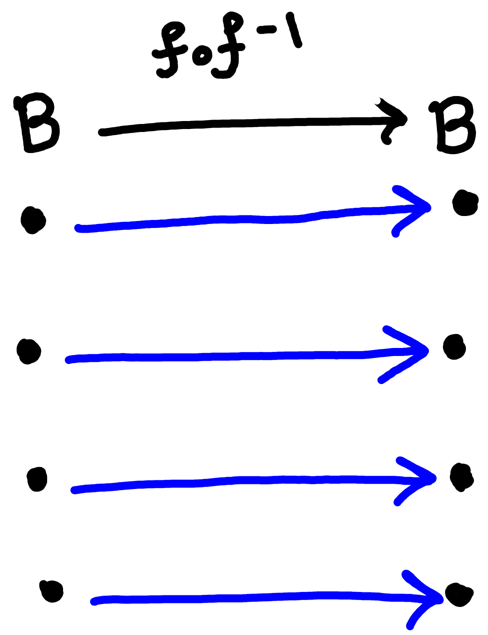


identity on A

connect if $\exists$ path of length 2 connecting them



compose $\mapsto$



identity on B

Fact: $\psi: A \to B$ is a bijection $\iff \exists\, \varphi: B \to A$ such that

$\varphi \circ \psi = id_A$ & $\psi \circ \varphi = id_B$

Injection

$f : A \hookrightarrow B$

Bijection

$f : A \hookrightarrow\!\!\!\!\twoheadrightarrow B$

reverse $\longmapsto$ arrows

Partial Function from $B \to A$ of

$g : \text{Range}(f) \to A$

is an honest function

Inverse

$f^{-1} : B \hookrightarrow\!\!\!\!\twoheadrightarrow A$

$f^{-1} \circ f = id_A$

$f \circ f^{-1} = id_B$

$\alpha : X \longrightarrow Y$

$\beta : Y \longrightarrow Z$

can be composed

$\beta \circ \alpha$

beta "after" alpha