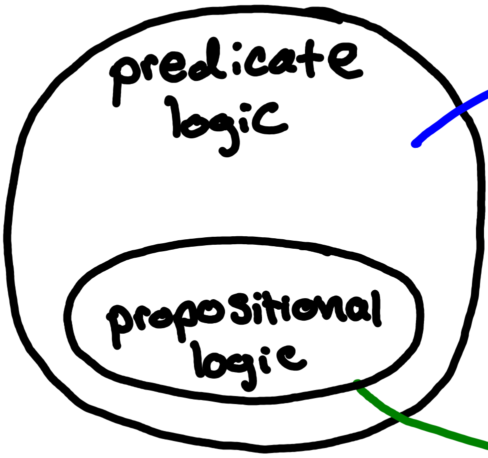


Infinite Domains

- ↳ $\forall$ becomes $\bigwedge_{i=1}^{\infty}$
- ↳ $\exists$ becomes $\bigvee_{i=1}^{\infty}$

Finite Domains

- ↳ $\forall$ becomes $\bigwedge_{i=1}^{n} = \wedge \dots \wedge$
- ↳ $\exists$ becomes $\bigvee_{i=1}^{n} = \vee \dots \vee$
- ↳ $\neg\forall$, $\neg\exists$ are understood using De Morgan's laws
- ↳ checking truth values is like iterating through a loop

$$\forall x \forall y = \forall y \forall x$$

$$\exists x \exists y = \exists y \exists x$$

$$\boxed{\forall x \exists y \neq \exists y \forall x}$$

↳ Careful with the order of quantifiers!

All facts about ∞ sets we will learn are captured by the 2 statements below

> "Countable unions of countable sets are countable"
>
> $$|X| < 2^{|X|}$$

- ↳ $|\mathbb{N}| = |\mathrm{Even}(\mathbb{N})| = |\mathrm{Odd}(\mathbb{N})| = |\mathbb{Z}| = |\mathbb{Q}|$
- ↳ Hilbert's Grand Hotel
- ↳ Diagonalization
- ↳ $\mathbb{R}$ is uncountable

English	Adjectives / Properties	+	Subject	=	Statement
Calculus	Function & Domain	+	independent variable	=	Real Number
Predicate Logic	Predicate & Domain $P(x)$ $\mathbb{R}$ $Q(x,y)$ $\mathbb{Z}^2$	+	variable	=	Proposition

<u>eg</u>

$H(x)$ "x has green hair"

Domain {all humans}

$H(\text{Alice})$

"Alice has green hair"

$P(x,y)$ "$x+y=0$" Domain $\mathbb{Z}$

$P(2,-2)$ "$2+(-2)=0$"

$P(3,1)$ "$3+1=0$"

$\cancel{P(2.2, 75)}$ Domain $2.2 \notin \mathbb{Z}$

$P(x,4)$ Not a proposition (yet)

Quantifiers

Universal: $\forall$ "For All"

Exestential: $\exists$ "There Exists"

$$\begin{tikzcd} \text{Predicate} \arrow[r, shift left, "\text{replace variables with elements of the domain}"] \arrow[r, shift right, swap, "\text{``bind'' all variables w/ quantifiers}"] & \text{Proposition} \end{tikzcd}$$

Domain $\mathcal{D} = \{1, 2, 3\}$

Predicate $\text{Even}(x)$ "x is even"

$$\forall x \ \text{Even}(x) \equiv \text{Even}(1) \land \text{Even}(2) \land \text{Even}(3)$$

$$F \Leftarrow F \land T \land F$$

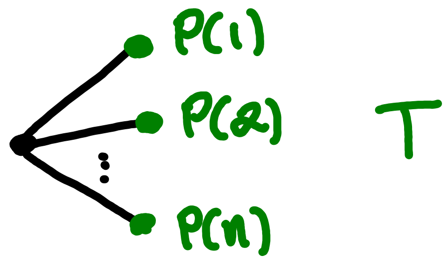
$$\exists x \ \text{Even}(x) \equiv \text{Even}(1) \lor \text{Even}(2) \lor \text{Even}(3)$$

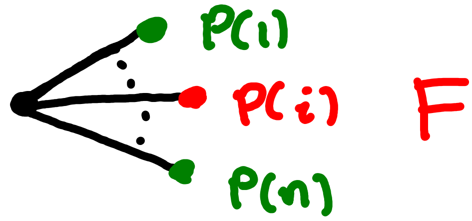
$$T \Longleftarrow F \lor T \lor F$$

Domain $\{1, 2, 3, \ldots, n\}$

Q: Is $\forall x\, P(x)$ true?

```
i=1
for i ≤ n
    if !P(i) == T
        return NO
    i = i+1
return YES
```

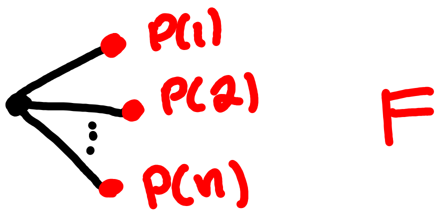


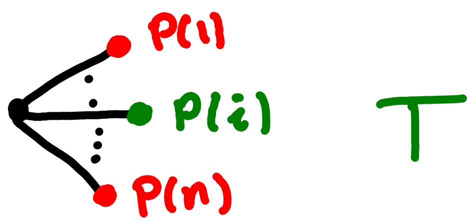


$$\bigwedge_{i=1}^{n} P(i) := P(1) \wedge P(2) \wedge \ldots \wedge P(n)$$

Q: Is $\exists x\, P(x)$ true?

```
i=1
for i ≤ n
    if P(i) == T
        return YES
    i=i+1
return NO
```

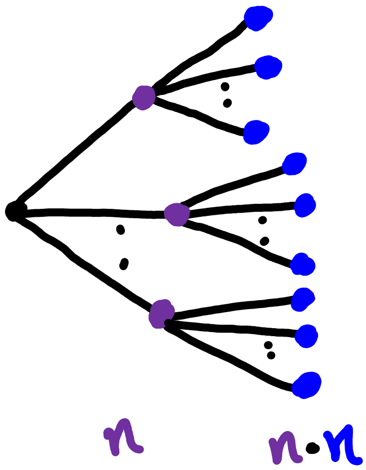




$$\bigvee_{i=1}^{n} P(i) := P(1) \vee P(2) \vee \ldots \vee P(n)$$

Q: Is $\forall x \forall y\, Q(x,y)$ true?

$$\bigwedge_{i=1}^{n} \bigwedge_{j=1}^{n} Q(i,j)$$



for $i \leq n$
  for $j \leq n$
    T or F ?
  $j = j+1$
$i = i+1$

Negation

"it is not the case that everyone will study"

$\equiv$

"Someone will not study"

$$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$$

proof:

Finite Domain

$$\neg \forall x_i P(x_i) \equiv \neg \bigwedge_{i=1}^{|D|} P(x_i)$$

(De Morgan)

$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$

$$\equiv \bigvee_{i=1}^{|D|} \neg P(x_i)$$

$$\equiv \exists x_i \left(\neg P(x_i)\right)$$

&

(similar proof)

$$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$$

$$\forall x \exists y \neq \exists y \forall x$$

$D := \mathbb{Z}$

$Q(x,y) :=$ "$x+y=0$"

$\forall x \exists y\, Q(x,y) =$ "Additive inverses of integers exist and are integers" **T**

$\exists y \forall x\, Q(x,y) =$ "There is some integer that always gives 0 when summed with any other integer" **F**

Order of operations

$$(\;),\ \forall/\exists,\ \neg,\ \underbrace{\wedge,\ \vee}_{\text{and/or}},\ \rightarrow,\ \leftrightarrow$$

Binding / Scope

Bound

Free

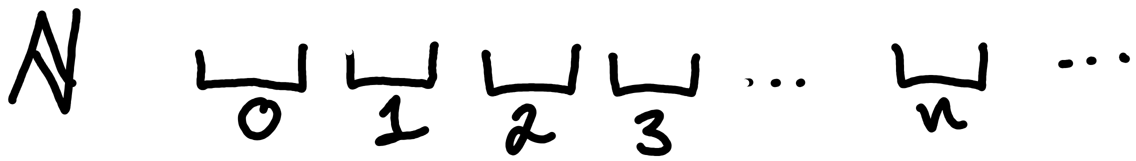
$$\exists x \underline{(P(x) \wedge Q(x))} \vee \forall x\, \underline{R(x)}$$

$$\exists x \underline{(x + y = 1)}$$

Scope of a variable in a loop

Counting w/ Infinity

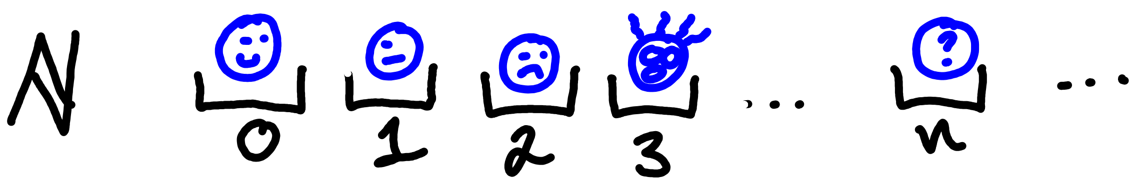
Recall: Counting $\longleftrightarrow$ Bijections



- think of having an infinite string of numbered bins

$\psi: X \xrightarrow{\sim} \mathbb{N}$

then a bijection from a set to $\mathbb{N}$ is a way of placing each element of the set in its own bin so that every bin is filled.



$\psi: \mathbb{N} \xrightarrow{\sim} X$

then a bijection to a set from $\mathbb{N}$ is a way of numbering or labelling the elements of the set using all of the natural numbers exactly once (i.e. a sequence)

The Smallest infinity (Countable ∞)

$$|\mathbb{N}| = |Even(\mathbb{N})| = |Odd(\mathbb{N})| = |\mathbb{Z}| = |\mathbb{Q}| =: \aleph_0$$

Aleph null

<u>Def</u>: $\aleph_0 = |X| \Leftrightarrow \exists \varphi : \mathbb{N} \xrightarrow{\sim} X$ or $\varphi : X \xrightarrow{\sim} \mathbb{N}$ (one implies the other)

"$\infty \pm 1 = \infty$"

1 new guest @ Hilbert's Grand Hotel (illustrated/explained below)

$$\psi_1 : \mathbb{N} \cup \{*\} \xrightarrow{\sim} \mathbb{N}$$

$$\begin{array}{ccccccc} * & 0 & 1 & 2 & \cdots & n-1 & \cdots \\ 0 & 1 & 2 & 3 & \cdots & n & \cdots \end{array}$$

$\psi_1(n) = n+1$

$\psi_1(*) = 0$

$$\psi_1^{-1} : \mathbb{N} \xrightarrow{\sim} \mathbb{N} \cup \{*\}$$

$\psi_1^{-1}(n) = n-1, \ n > 0$

$\psi_1^{-1}(0) = *$

$$\begin{array}{ccccccc} \#0 & \#1 & \#2 & \#3 & & \#n & \\ * & 0 & 1 & 2 & \cdots & n-1 & \cdots \end{array}$$

$(\psi_1 \circ \psi_1^{-1}) = id_{\mathbb{N}}$

$$\begin{array}{cccccc} 0 & 1 & 2 & \cdots & n & \cdots \\ 0 & 1 & 2 & \cdots & n & \cdots \end{array}$$

$id_{\mathbb{N}}(n) = n$

<u>Exercise:</u> Compute $\psi_1^{-1} \circ \psi_1$

$\psi_2 : \mathbb{N} \setminus \{0\} \xrightarrow{\sim} \mathbb{N}$ $\qquad$ $\mathbb{N}\setminus\{0\} =: \mathbb{N}^+$

$$\underbrace{1}_{0} \quad \underbrace{2}_{1} \quad \underbrace{3}_{2} \quad \underbrace{4}_{3} \quad \cdots \quad \underbrace{n+1}_{n} \quad \cdots$$

$\psi_2(n) = n-1$

$\therefore$ (therefore) $\qquad$ "$\infty = \infty + 1 = \infty + 1 + 1 = \ldots = \infty + n$"

Countable $\sqcup$ Finite $=$ Countable

"$\infty + \infty = \infty$ & $\infty - \infty = \infty$"

$$\mathrm{Even}(\mathbb{N}) := \{x \in \mathbb{N} \mid \exists n \in \mathbb{N}\ (x = 2n)\}$$

$$\mathrm{Odd}(\mathbb{N}) := \{x \in \mathbb{N} \mid \exists n \in \mathbb{N}\ (x = 2n+1)\}$$

$$\mathbb{N} = \mathrm{Even}(\mathbb{N}) \sqcup \mathrm{Odd}(\mathbb{N})$$

<u>Expect</u> $\quad |\mathrm{Even}(\mathbb{N})| = |\mathrm{Odd}(\mathbb{N})|$ &

($\star$) $\quad |\mathbb{N}| = |\mathrm{Even}(\mathbb{N})| + |\mathrm{Odd}(\mathbb{N})| = 2|\mathrm{Even}(\mathbb{N})|$

$\Rightarrow \quad |\mathrm{Even}(\mathbb{N})| = |\mathbb{N}|/2 \qquad$ ($\star\star$)

$\varphi_1 : \mathrm{Even}(\mathbb{N}) \xrightarrow{\sim} \mathbb{N}$, $\quad \varphi_1(2n) = n$, $\quad \boxed{\varphi_1^{-1}(n) = 2n}$

$$\underbrace{0}_{0} \quad \underbrace{2}_{1} \quad \underbrace{4}_{2} \quad \underbrace{6}_{3} \quad \cdots \quad \underbrace{2n}_{n} \quad \cdots$$

$\varphi_2 : Odd(\mathbb{N}) \xrightarrow{\sim} \mathbb{N}$, $\varphi_2(2n+1) = n$, $[\varphi_1^{-1}(n) = 2n+1]$

$$\underbrace{1}_{0} \quad \underbrace{3}_{1} \quad \underbrace{5}_{2} \quad \cdots \quad \underbrace{2n+1}_{n} \quad \cdots$$

$\therefore \quad |Even(\mathbb{N})| = |\mathbb{N}| = |Odd(\mathbb{N})|$

Thus "$\underset{(\star\star)}{\infty/2 = \infty}$ & $\underset{(\star)}{2 \cdot \infty = \infty}$ & $\underset{(\star)}{\infty + \infty = \infty}$)

Another Example

$$\mathbb{N}^- := \{-x \mid x \in \mathbb{N}^+\}$$

$\mathbb{Z} = \mathbb{N} \sqcup \mathbb{N}^-$ so we expect $|\mathbb{Z}| = |\mathbb{N}| + |\mathbb{N}^-|$

$= 2|\mathbb{N}|$

$$\underbrace{0}_{0} \quad \underbrace{1}_{1} \quad \underbrace{-1}_{2} \quad \underbrace{2}_{3} \quad \underbrace{-2}_{4} \quad \underbrace{3}_{5} \quad \cdots$$

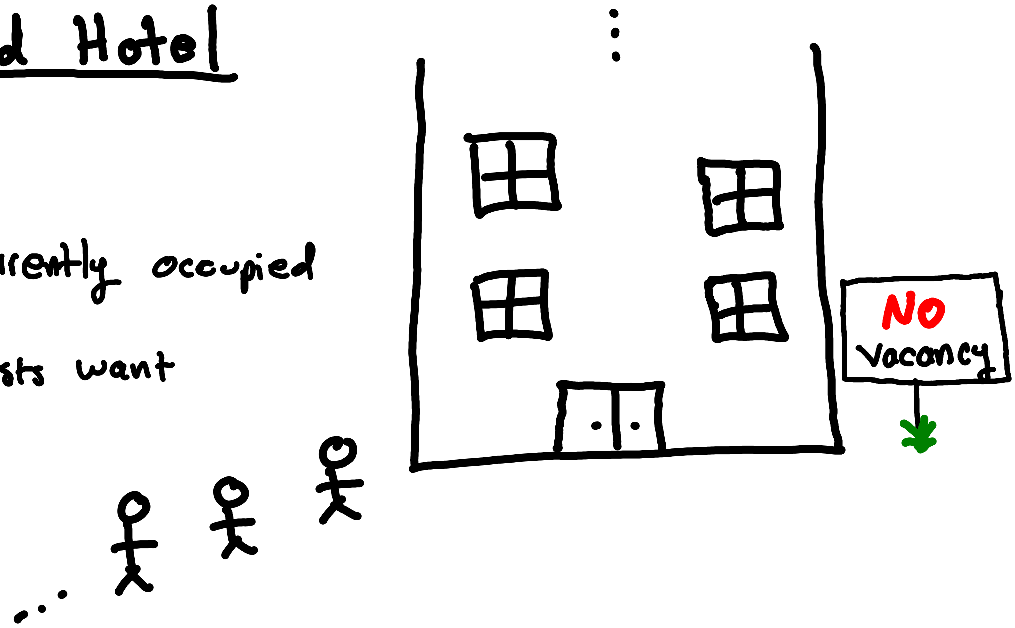
"$\infty = \infty + \infty = \infty + \infty + \infty = \ldots = n \cdot \infty$"

Sum rule for ∞ sets [Finite unions of countable sets are countable

$$\left| \bigsqcup_{i=1}^{n} \mathbb{N}_i \right| = \aleph_0$$

Hilbert's Grand Hotel

- ∞ hotel
- Every room is currently occupied
- ∞ many new guests want rooms of their own



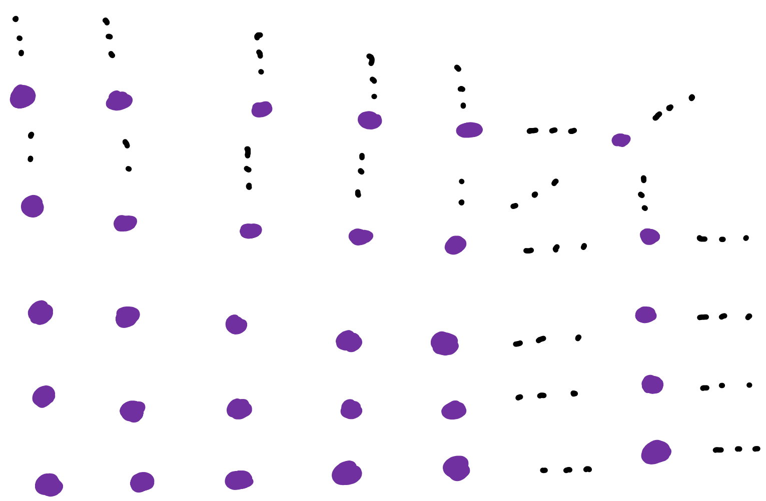
Q: Can we accommodate everyone?

A: Yes, have every guest move to the room # which is double their current room #. Then every odd # room is empty allowing for ∞ many new guests

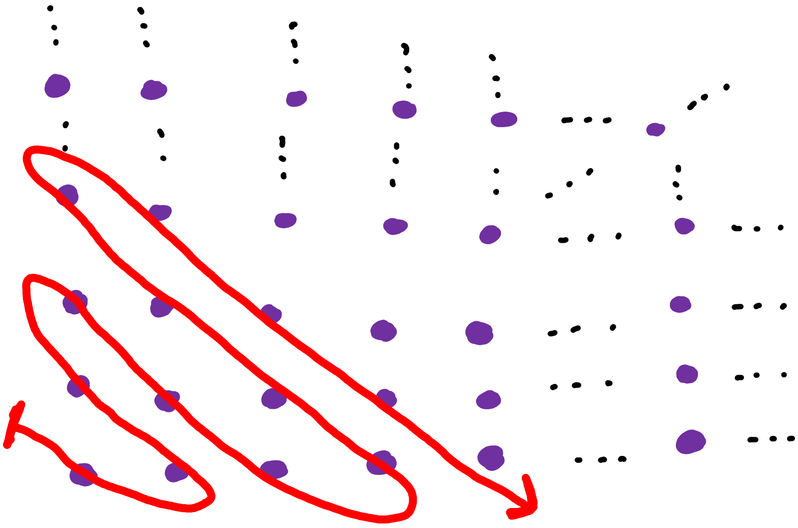
"$\infty \cdot \infty = \infty$" (Multiplication Rule w/ ∞)

$\mathbb{N} \times \mathbb{N} =$

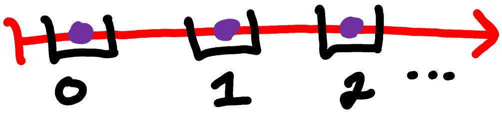
(Quadrant of dots)



For finite sets $|A \times B| = |A| \cdot |B|$

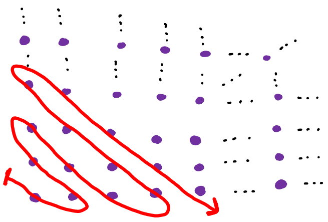


$$|\mathbb{N}| = |\mathbb{N}^2|$$



Application $|\mathbb{Q}|$

$$\mathbb{Q}^+ := \{ a/b \in \mathbb{Q} \mid 0 < a, b \in \mathbb{N} \}$$



Numerator ↑						
⋮						
5	5/1	5/2	5/3	5/4	~~5/5~~	
4	4/1	~~4/2~~	4/3	~~4/4~~	4/5	
3	3/1	3/2	~~3/3~~	3/4	3/5	
2	2/1	~~2/2~~	2/3	~~2/4~~	2/5	
1	1/1	1/2	1/3	1/4	1/5	
	1	2	3	4	5	...

Denominator →

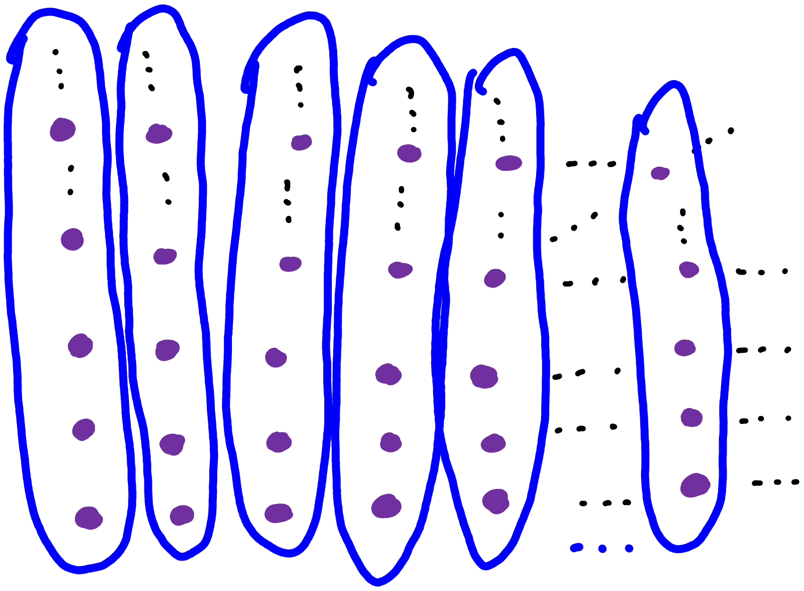
Omit fractions that reduce to ones we have already listed

$$|\mathbb{Q}^+| = |\mathbb{N}| \implies |\mathbb{Q}| = 2|\mathbb{Q}^+| + |\{0\}|$$
$$= |\mathbb{N}| \qquad \square$$

Counting Technique

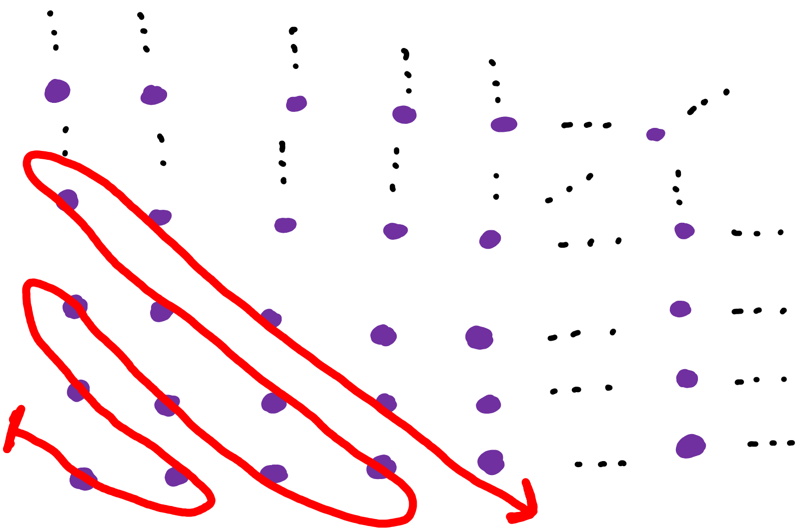
Count in two ways

$$\#\bullet = \infty \cdot \infty$$

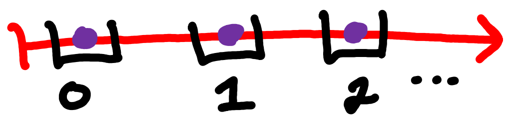


each column has ∞ dots & there are ∞ many columns.

This provides our first count. For the second count we invoke "diagonalization"



$$\#\bullet = \infty$$



Thus $\infty \cdot \infty = \#\bullet = \infty$

"$\infty = \infty \cdot \infty = \infty \cdot \infty \cdot \infty = \dots = \infty^n$"

$$\left| \bigsqcup_{i=1}^{\infty} \mathbb{N}_i \right| = \aleph_0$$

Uncountable Sets

$$B_\infty := \{\text{Sequences of } 1\text{s} \ \& \ 0\text{s}\}$$

$$= \{\infty \text{ bit strings}\} \qquad (\text{a "bit" is a } 1 \text{ or } 0)$$

$$= \{f : \mathbb{N} \to \{0,1\}\}$$

Suppose we could list all elements of B_∞

"complete list"

$$\begin{bmatrix} \boxed{1}\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 0 \ \ldots \\ 1\ \boxed{0}\ 0\ 0\ 1\ 0\ 0\ 1\ 0 \ \ldots \\ 1\ 1\ \boxed{1}\ 0\ 0\ 1\ 1\ 1\ 1 \ \ldots \\ 0\ 1\ 0\ \boxed{0}\ 0\ 0\ 0\ 1\ 0 \ \ldots \\ \vdots \qquad \ddots \end{bmatrix}$$

new string not on list above

$$\begin{bmatrix} \boxed{0}\ \boxed{1}\ \boxed{0}\ \boxed{1} \ \ldots \end{bmatrix}$$

This new string differs from the i^{th} string of the list in exactly the i^{th} bit.

– Similar proof shows $\mathbb{R}$ is uncountable (use decimal expansion of #s)

$$|\{f : \mathbb{N} \to \{0,1\}\}| = 2^{\aleph_0}$$

$$n \to 0, \quad n \to 1$$

two choices for each number