You can have cake **or** you can have ice cream

↳ We understand that "or" here means you can have exactly one of the options, not both.

↳ Later we will call this "exclusive or" because it excludes the possibility of having both options.

If it is your birthday **or** you are over the age of 65, then you can eat for free.

↳ We understand that a person celebrating their 80^{th} birthday can eat for free. Thus "or" is being used in an inclusive sense.

If you study, **then** your grade will improve.

↳ We understand that it is possible for your grade to improve even without studying

↳ Later we will call this a "conditional statement"

If you graduate with a 4.0, **then** I will buy you a car.

↳ We do not expect to get a car graduating with any GPA less than a 4.0

↳ Later we will call this a "biconditional statement"

The English language can be ambiguous.

As shown above, context can change the meanings of words. We want to use English when writing proofs, but we cannot tolerate ambiguity. Therefore we must choose a meaning for words/phrases like "or"/"If... ...Then..." & agree to stick with our choice

Propositional Logic: (The language of proofs)

Definition: A "proposition" is defined to be a statement which is either True $\underbrace{\text{xor}}_{\text{exclusive or}}$ False.

$1+2=3$ is a True proposition

$1+2=4$ is a False proposition

Is it windy? is a question not a proposition

This statement is False. Cannot be assigned a truth value, so it is not a proposition.

	Variables Represent	operators	order of operations
Algebra	Real Numbers	$+, -, \times, /, \ldots$	PEMDAS $( ), (-)^x, \times, /, +, -$
Propositional Logic	<u>propositions</u> • Statement • True xor False	<u>Logical operators</u> $\neg$ <u>Connectives</u> $\wedge, \vee, \oplus, \rightarrow/\Rightarrow$ $\leftrightarrow/\Leftrightarrow$	PNCDIB $( ), \neg, \wedge, \vee, \rightarrow, \leftrightarrow$ Negation, Conjunction, Disjunction, Implication, Biconditionals

In CS we would write

!	$\neg$
&&	$\wedge$
||	$\vee$
if	$\rightarrow$
==	$\leftrightarrow$

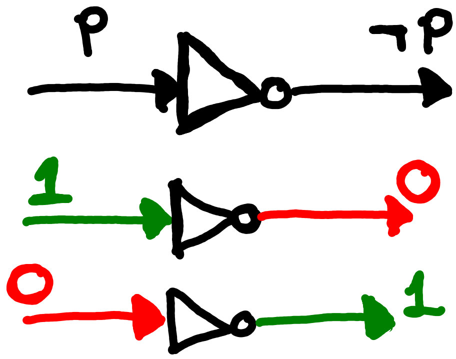
$$\left.\begin{array}{l} \neg(\neg P) = P \\ -(-5) = 5 \end{array}\right\}\text{double negatives}$$

$$\left.\begin{array}{l} P\vee(Q\wedge R) = (P\vee Q)\wedge(P\vee R) \\ P\wedge(Q\vee R) = (P\wedge Q)\vee(P\wedge R) \\ 5\times(3+2) = (5\times3)+(5\times2) \end{array}\right\}\text{Distributive Law}$$

Negation ¬

$\neg P$:= "It is not the case that" P

P	$\neg P$
T	F
F	T

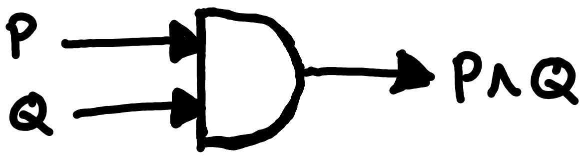


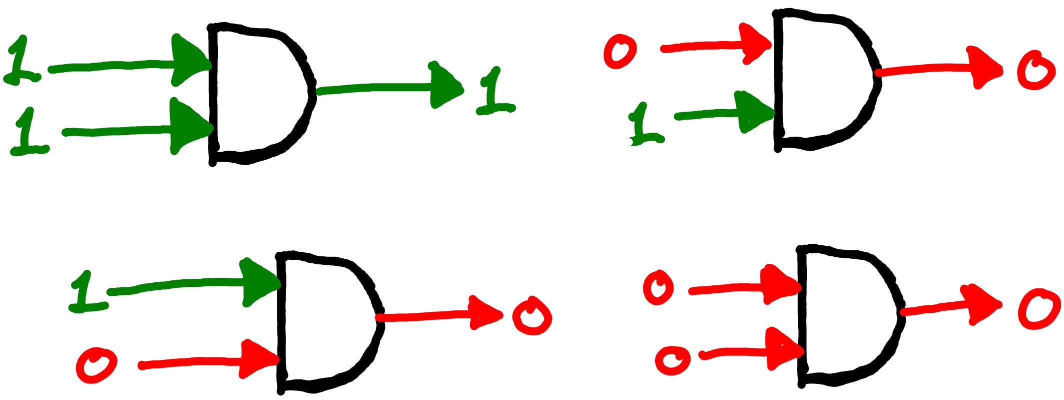
Conjunction ∧

$P \wedge Q$:= P "and" Q

P "but also" Q

P	Q	$P \wedge Q$
T	T	T
F	T	F
T	F	F
F	F	F

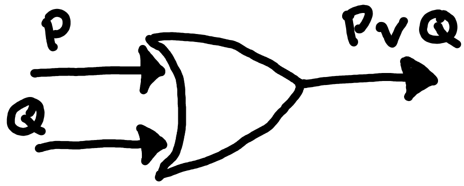


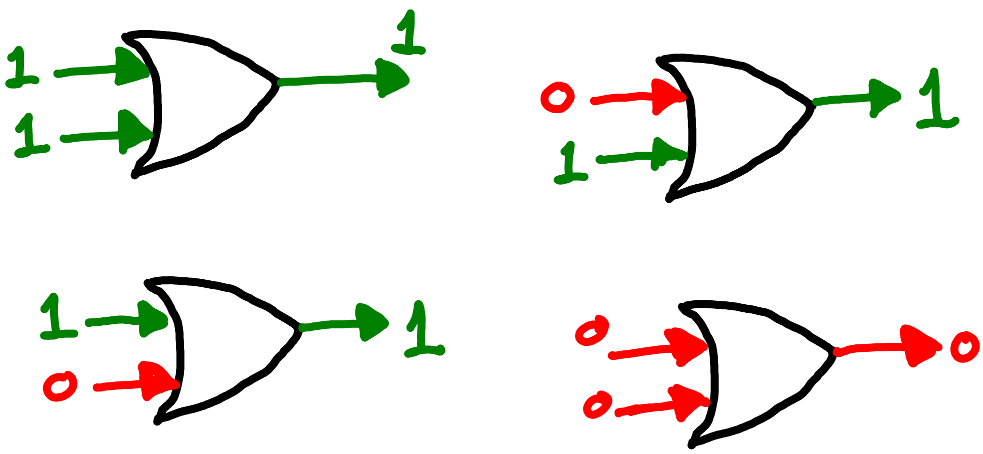


Disjunction $\vee$

$P \vee Q :=$ P "(inclusive) or" Q

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F





Conditional $\rightarrow$ ($\Rightarrow$)

$P \rightarrow Q$ can be read in many ways

(i) "If" P "Then" Q

(ii) P "implies" Q

(iii) P "only if" Q

(iv) P is "sufficient" for Q

(v) A "sufficient condition" for Q is P

(vi) Q "whenever" P / Q "when" P

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

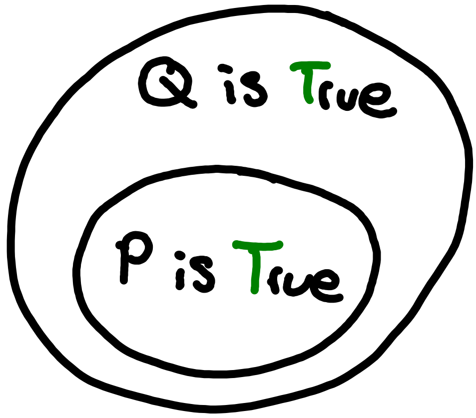
(vii) Q is "necessary" for P

(viii) A "necessary condition" for P is Q

(ix) Q "follows from" P

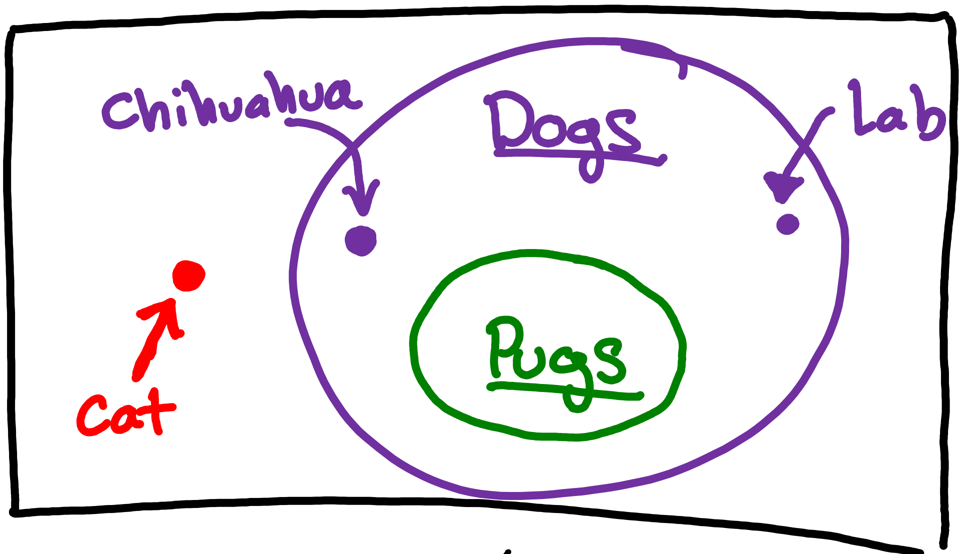
(x) Q "provided" P

(xi) Q "unless" $\neg$P



e.g. Pug $\longrightarrow$ Dog

"If you are a pug, then you are a dog"



- You can be a dog while not being a pug (compare w/ line 3 of the above truth table & discussion of converse below)

- Since the "pug" circle is entirely inside the "Dog" circle, you cannot be a pug if you are not a dog.

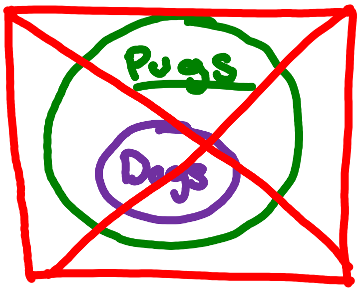
The Contrapositive

$\neg$Dog $\longrightarrow$ $\neg$Pug

- the truth value of a conditional & its contrapositive always agree

The Converse

Dog $\rightarrow$ Pug



- Although the original implication was true, its converse is clearly absurd.

The Inverse

$\neg$ Pug $\rightarrow$ $\neg$ Dog

- Inverse = Converse + Contrapositive

Summary of $\Rightarrow$

logically equivalent $\neq$ logically equivalent

		conditional	contrapositive	converse	inverse
P	Q	$P \Rightarrow Q$	$\neg Q \Rightarrow \neg P$	$Q \Rightarrow P$	$\neg P \Rightarrow \neg Q$
T	T	T	T	T	T
T	F	F	F	T	T
F	T	T	T	F	F
F	F	T	T	T	T

Affirming the Consequent: (The Converse fallacy)

- Sometimes it can be difficult to differentiate between an implication & its converse, but they are different. Confusing an implication & its converse is a common logical fallacy.

Q: Is the following argument Logical?

Sentence 1:

If the potato crops fail, then there will be famine.

Hence

Sentence 2:

If we can successfully grow potatoes, then there will be no famine.

A: We translate both sentences into propositional logic to find out (The answer is "No")

Propositions: $p =$ we can successfully grow potatoes

$f =$ there will be famine

First translate:

Sentence 1 $\quad \neg p \Rightarrow f$

Sentence 2 $\quad p \Rightarrow \neg f$

Determine relationship:

$$\begin{tikzcd}
\overset{\text{Sentence 1}}{\boxed{\neg p \Rightarrow f}} \arrow[rr, leftrightarrow, "\text{contrapositive}", "\text{(logical equivalence)}"'] \arrow[d, phantom, "\not\Updownarrow"] & & \boxed{\neg f \Rightarrow \neg(\neg p)} \arrow[d, leftrightarrow, "\text{Simplify}"] \\
\underset{\text{Sentence 2}}{\boxed{p \Rightarrow \neg f}} & & \boxed{\neg f \Rightarrow p} \arrow[ll, leftrightarrow, dashed, "\text{Converse}"', "\text{(changes meaning)}"]
\end{tikzcd}$$

The Biconditional $\longleftrightarrow$ ($\Longleftrightarrow$)

$P \longleftrightarrow Q$

(i) P is "necessary and sufficient" for Q

(ii) "If" P "Then" Q "and conversely"

(iii) P "if and only if" Q / P "iff" Q

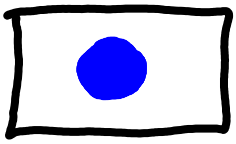
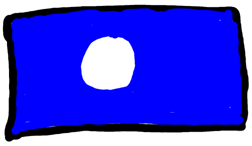
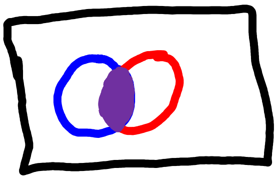
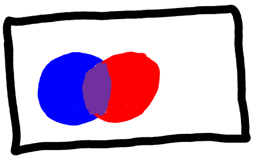
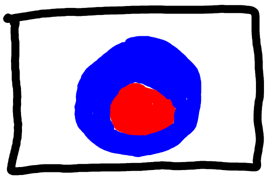
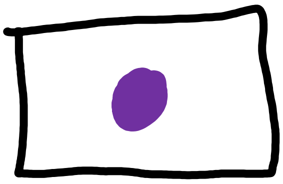
(iv) P "exactly when" Q

P	Q	$P \longleftrightarrow Q$	$(P \to Q) \wedge (Q \to P)$
T	T	T	$T \wedge T = T$
T	F	F	$F \wedge T = F$
F	T	F	$T \wedge F = F$
F	F	T	$T \wedge T = T$

logically equivalent
(i.e their truth tables agree)

- the above table says biconditionals are built of two conditionals, an implication & the converse.

Notation: If P is logically equivalent to Q we write $P \equiv Q$

Logic	Picture	Sets
P		A
$\neg P$		$\{x \mid \neg(x \in A)\} =: \overline{A}$ $\{x \mid x \notin A\} =: A^c$
$P \wedge Q$		$\{x \mid x \in A \wedge x \in B\} = A \cap B$
$P \vee Q$		$\{x \mid x \in A \vee x \in B\} = A \cup B$
$P \rightarrow Q$		$A \subseteq B \Leftrightarrow x \in A \rightarrow x \in B$
$P \leftrightarrow Q$		$A = B \Leftrightarrow x \in A \leftrightarrow x \in B$

A true proposition is like an element of a set

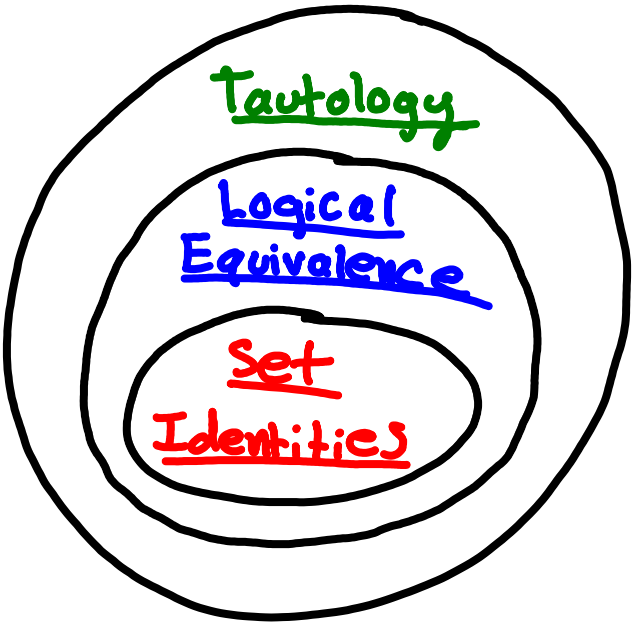
A false proposition is like a non-element of a set

Truth Table

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Membership Table

A	B	$A \cap B$
1	1	1
1	0	0
0	1	0
0	0	0



logical equivalence $\equiv$
↳ Identical Truth Tables

Tautology
↳ Truth table is all Ts
(ie. A proposition that is always true)

Fact: $P \equiv Q \iff P \leftrightarrow Q$ is a tautology

e.g.

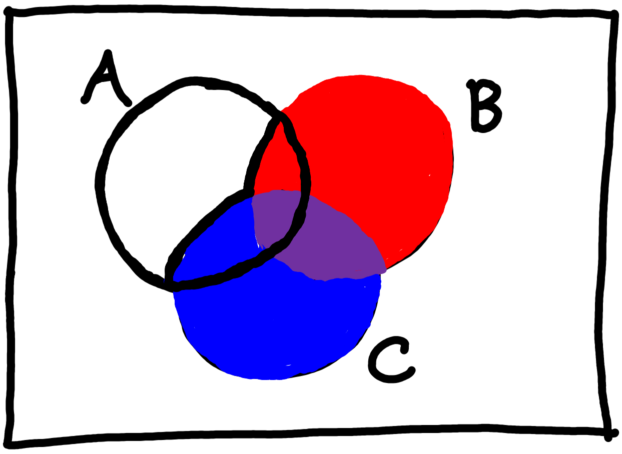
P	Q	$\overbrace{P \to Q}^{\text{"P implies Q"}}$	$\overbrace{Q \vee \neg P}^{\text{"Q unless } \neg P\text{"}}$	$(P \to Q) \leftrightarrow (Q \vee \neg P)$
T	T	T	T	$T \leftrightarrow T = T$
T	F	F	F	$F \leftrightarrow F = T$
F	T	T	T	$T \leftrightarrow T = T$
F	F	T	T	$T \leftrightarrow T = T$

logically equivalent — tautology

Exercise: Show $(P \to Q) \wedge (Q \to P) \leftrightarrow (P \leftrightarrow Q)$ is a tautology.

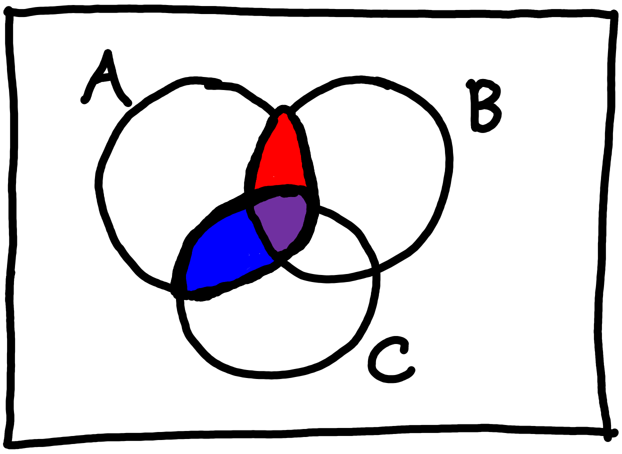
Set Identities

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$



$B \cup C$

intersect with A $\longrightarrow$



$(A \cap B)$ $(A \cap C)$

$(A \cap B \cap C)$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \qquad P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$$

Set Identity $\Longrightarrow$ Logical Equivalence

proof:

Membership Table

Truth Table

$1 \leftrightarrow T$

$A \cap (B \cup C) \longleftrightarrow P \wedge (Q \vee R)$

$0 \leftrightarrow F$

$\updownarrow$ agree

$1 \leftrightarrow T$

$(A \cap B) \cup (A \cap C) \longleftrightarrow (P \wedge Q) \vee (P \wedge R)$

$0 \leftrightarrow F$

thus these must agree

List of Logical Equivalences & Set Identities/Facts

$p \wedge T \equiv p$	$A \cap U = A$
$p \vee F \equiv p$	$A \cup \emptyset = A$
$p \vee T \equiv T$	$A \cup U = U$
$p \wedge F \equiv F$	$A \cap \emptyset = \emptyset$
$p \vee p \equiv p$	$A \cup A = A$
$p \wedge p \equiv p$	$A \cap A = A$
$\neg(\neg p) \equiv p$	$(A^c)^c = A$
$p \vee Q \equiv Q \vee p$	$A \cup B = B \cup A$
$p \wedge Q \equiv Q \wedge p$	$A \cap B = B \cap A$
$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$	$(A \cup B) \cup C = A \cup (B \cup C)$
$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$	$(A \cap B) \cap C = A \cap (B \cap C)$

De Morgan's Laws

$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$	$(A \cap B)^c = A^c \cup B^c$
$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$	$(A \cup B)^c = A^c \cap B^c$

$$P \vee (P \wedge Q) \equiv P \qquad A \cup (A \cap B) = A$$

$$P \wedge (P \vee Q) \equiv P \qquad A \cap (A \cup B) = A$$

$$P \vee \neg P \equiv T \qquad A \cup A^c = \mathcal{U}$$

$$P \wedge \neg P \equiv F \qquad A \cap A^c = \varnothing$$

$$P \to Q \equiv \neg Q \to \neg P \qquad A \subseteq B \iff B^c \subseteq A^c$$

$$\boxed{(P \to Q) \wedge (Q \to R) \equiv P \to (Q \wedge R) \qquad A \subseteq B,\ B \subseteq C \Rightarrow A \subseteq (B \cap C)}$$

$$(P \to R) \wedge (Q \to R) \equiv (P \vee Q) \to R \qquad A, B \subseteq C \iff (A \cup B) \subseteq C$$

$$\boxed{(P \to Q) \vee (P \to R) \equiv P \to (Q \vee R) \qquad A \subseteq B \text{ or } A \subseteq C \Rightarrow A \subseteq (B \cup C)}$$

$$\boxed{(P \to R) \vee (Q \to R) \equiv (P \wedge Q) \to R \qquad A \subseteq C \text{ or } B \subseteq C \Rightarrow (A \cap B) \subseteq C}$$

$$\boxed{P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P) \qquad A = B \iff A \subseteq B \ \& \ B \subseteq A} \ \star$$

$$P \leftrightarrow Q \equiv \neg P \leftrightarrow \neg Q \qquad A = B \iff A^c = B^c$$

Remark: Note that the implications in the first three boxes don't go both ways on the level of sets. Therefore, one has to be slightly careful with this analogy, especially when conditionals/subsets are involved (detailed explanation below). The purple ★ box above provides us with a useful strategy for proving equalities of sets.

Methods of Proving Set Identities

Subset Method (prove 2 implications)	— Show that each side of the identity is a subset of the other side.	★ most common approach
Apply Existing Identities (equivalences)	— Start with one side, transform it into the other side using a sequence of steps by applying an established identity in each step	
Membership Table (Truth table)	— For each possible combination of the atomic sets, show that an element in exactly these atomic sets must either belong to both sides or belong to neither side	

Truth Sets

Surprisingly, the analogy between propositions & sets failed us despite loads of evidence that a logical equivalence produces a set identity / fact & conversely that a set identity/fact can be translated into a logical equivalence, so what's the deal? Well, it happens that there is a precise connection between sets & propositions, but it is slightly more nuanced than the naive correspondence wherein we simply replace

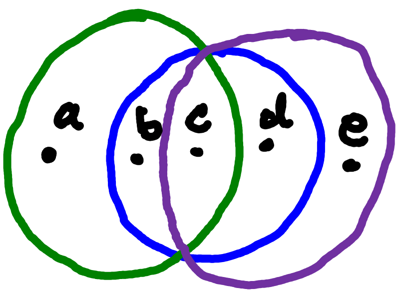
	T	with	$\mathcal{U}$	
	F	with	$\emptyset$	
	$\neg$	with	$(-)^c$	(complements)
	$\wedge$	with	$\cap$	
	$\vee$	with	$\cup$	
	$\rightarrow$	with	$\subseteq$	
&	$\longleftrightarrow$	with	$=$	(equalities)

The honest connection comes from the notion of a "truth set" Before getting to the definition of a truth set, lets prove, by way of (counter)-example, that the two implications

$$A \subseteq B \text{ or } A \subseteq C \Rightarrow A \subseteq (B \cup C)$$

$$\& \quad A \subseteq C \text{ or } B \subseteq C \Rightarrow (A \cap B) \subseteq C$$

cannot be reversed.

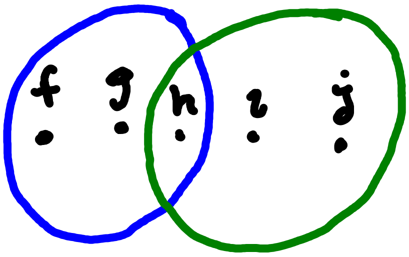


$A = \{b, c, d\}$
$B = \{a, b, c\}$
$C = \{c, d, e\}$

$A \subseteq (B \cup C)$

but it is not the case that

$A \subseteq B$ or $A \subseteq C$



$A = \{f, g, h\}$
$B = \{h, i, j\}$
$C = \{g, h, i\}$

$A \cap B = \{h\} \subseteq C$ but it is not the case that $A \subseteq C$ or $B \subseteq C$

to correct for this apparent failure we now introduce truth sets. First, fix a "universe" i.e. a set over which the possible values of variables are allowed to range, and call it $\mathcal{U}$. Next, suppose P is a statement (or string of characters) involving a single variable x & whenever an element $t \in \mathcal{U}$ is plugged into P (by replacing all occurrences of x) the result is a proposition. The subset $T_P \subseteq \mathcal{U}$, of all elements such that P becomes a TRUE proposition upon substituting these elements in for x, is called the truth set of P. As we will soon see such a P (also denoted $P(x)$ to indicate that x is the variable) is called a <u>predicate</u>. Now, given a logical equivalence such as

$$\big(P(x) \to Q(x)\big) \vee \big(P(x) \to R(x)\big)$$
$$\equiv$$
$$P(x) \to \big(Q(x) \vee R(x)\big)$$

Valid for all x in the chosen universe $\mathcal{U}$

we may view this as a logical equivalence of two predicates $\mathbb{P}(x) := (P(x) \to Q(x)) \vee (P(x) \to R(x))$ & $\mathbb{Q}(x) = P(x) \to (Q(x) \vee R(x))$

$$\mathbb{P}(x) \equiv \mathbb{Q}(x)$$

whose truth sets are equal as sets, i.e.

$$T_{\mathbb{P}} = T_{\mathbb{Q}} \subseteq \mathcal{U}$$

Thus, every logical equivalence becomes an equality of sets once a universe is fixed & all propositional variables are replaced with (single variable) predicates (all of which take the same variable) ranging over the universe. It may now seem as if the naive correspondence outline above has been destroyed, and in a certain sense it has been, but recall that this was necessary - we proved above that no link between sets & propositions can be this simple as it is possible to find counterexamples However, not all is lost in the true version of the analogy. Loosely speaking, the naieve analogy is true not on the level of sets, but on the level of <u>elements</u> of sets. This will be demonstrated by way of example now

Let $\mathcal{U} := \mathbb{Z}$

$P(x) := x \in \mathbb{Z}$, $Q(x) := x \in 2\mathbb{Z} := \{\text{even integers}\}$

$R(x) := x \notin 2\mathbb{Z}$ & consider the logical equivalence

$$\underbrace{(P(x) \to Q(x)) \vee (P(x) \to R(x))}_{\mathbb{P}(x)} \equiv \underbrace{P(x) \to (Q(x) \vee R(x))}_{\mathbb{Q}(x)}$$

$\left(\underline{\text{note}}:\ T_{\mathbb{P}} = \mathbb{Z}\right)$ $\quad (T_{\mathbb{Q}} = \mathbb{Z})$

of course $T_P \nsubseteq T_Q$, $T_P \nsubseteq T_R$ although $T_P = T_Q \cup T_R$

but it is the case that $z \in T_P \Rightarrow z \in T_Q$ or $z \in T_R$

or equivalently $\underbrace{(z \in T_P \Rightarrow z \in T_Q)}$ or $(z \in T_P \Rightarrow z \in T_R)$.

$\underline{\text{note}}$: if this is true for every $z \in T_P$ then $T_P \subseteq T_Q$

As a second & final example we take $\mathcal{U} := \mathbb{R}$

$P(x) := x > 100$, $Q(x) := x \in \mathbb{Q}$, $R(x) := x \in \mathbb{Z}$

& consider $\underbrace{(P(x) \to R(x)) \vee (Q(x) \to R(x))}_{\mathbb{P}(x)} \equiv \underbrace{(P(x) \wedge Q(x)) \to R(x)}_{\mathbb{Q}(x)}$

$\mathbb{P}(\sqrt{2}) = T$, $\mathbb{Q}(\sqrt{2}) = T$ so $\sqrt{2} \in T_{\mathbb{P}} \cap T_{\mathbb{Q}}$

$\mathbb{P}(100 + 1/3) = F$ & $\mathbb{Q}(100 + 1/3) = F$ so $100 + 1/3 \notin T_{\mathbb{P}} \cup T_{\mathbb{Q}}$

but $(r \in T_P \Rightarrow r \in T_R)$ or $(r \in T_Q \Rightarrow r \in T_R)$ is equivalent to

$(r \in T_P \cap T_Q) \Rightarrow r \in T_R$ for all $r \in \mathbb{R} =: \mathcal{U}$ (but the equivalence now holds for somewhat silly reasons - check yourself!)