Main Actors: Sets (i.e. collections of "elements")

Sets play 2 main roles

(i) Counting
(ii) Home Base for Math

Finite Sets

\[ \emptyset := \{ \} \quad \text{empty set} \]
\[ \{1\} := \{1\} \]
\[ \{1,2\} := \{1,2\} \]
\[ \{1,2,3\} := \{1,2,3\} \]

\[ \vdots \]

Infinite Sets

\[ \mathbb{N} := \{0,1,2,3,\ldots\} \quad \text{Natural Numbers (Counting Numbers)} \]
\[ \mathbb{Z} := \{\ldots,-2,-1,0,1,2,\ldots\} \quad \text{Integers} \]
\[ \mathbb{Q} := \{a/b \mid a \in \mathbb{Z}, b \in \mathbb{N}, b \neq 0\} \quad \text{Rational Numbers (fractions)} \]
\[ \mathbb{R} := \text{All Real Numbers} \]
2 Problem Solving: Morals

(I) Drawing Pictures Helps You Reason

- We can represent 3 arbitrary sets $A, B, C$ as a Venn Diagram.

- Elements of $A$ are located inside the blue circle & similarly for elements of $B$ and $C$.

(II) Naming Things Gives You Power Over Them

- $A \cap B$ represents the set of elements common to the sets $A$ & $B$.

- $A \cup B$ represents the set of elements in $A$ or $B$ or both.
A minus B

\[ A \setminus B \]

The set of elements in A but not in B

Complement of A

\[ \bar{A} \text{ or } A^c \]

The set of elements from the universe except the elements of A

B is a Subset of A

\[ B \subseteq A \]

All elements of B are also elements of A

Note: \( \emptyset \subseteq X \) for any set \( X \)

Definition: Two sets A & B are disjoint

if \( A \cap B = \emptyset \) (i.e. they have no elements in common)
Picture of disjoint sets:

A ∩ B = A ∪ B

the symbol for “disjoint union”

Basics of Counting

Given: A finite set X

Q: How many elements does X have?
A: There are |X| elements in X.

Def: The natural number |X| is called the cardinality of X.

Addition Rule: (Combining groups)

Given: A, B two disjoint sets w/ |A| = n, |B| = m

Q: |A ∪ B| = ?
A: Draw a picture

\[ A \cap B = A \cup B \]

How many elements are in the large circle on RHS? \( n+m \)

\[ |A \cup B| = |A| + |B| - |A \cap B| = 3 + 7 = 10 \]

Addition Rule:
If \( A \cap B = \emptyset \), then \( |A \cup B| = |A| + |B| \)

Subtraction Rule: (Inclusion-Exclusion principle)
Given: Two sets \( A, B \) (which are not necessarily disjoint)

Q: \( |A \cup B| = ? \)
**A:** Draw a picture

\[ |A| = \text{blue} + \text{purple} \]
\[ |B| = \text{purple} + \text{red} \]
\[ |A \cap B| = \text{purple} \]

\[ |A \cup B| = \text{blue} + \text{purple} + \text{red} \]

Therefore

\[ |A| + |B| = |A \cup B| + |A \cap B| \]

Equivalently, subtract

\[ |A| + |B| - |A \cap B| = |A \cup B| \]

**Multiplication Rule:** (Cartesian Product of sets)

**Tall Rectangle**

\[ \mathcal{A} \times \mathcal{B} = \begin{pmatrix} (1,3) & (2,3) \\ (1,2) & (2,2) \\ (1,1) & (2,1) \end{pmatrix} \]

**Wide Rectangle**

\[ \mathcal{B} \times \mathcal{A} = \begin{pmatrix} (1,2) & (2,2) & (3,2) \\ (1,1) & (2,1) & (3,1) \end{pmatrix} \]
\[ A \times B \neq B \times A \text{ unless } A = B \]

**Cartesian Product** = ordered pairs of elements
(first element comes from first set
second element comes from second set)

\[ \mathbb{R} \times \mathbb{R} \quad \text{or} \quad \mathbb{R}^2 \]
\[ \xrightarrow{y} \quad \xrightarrow{x} \quad \text{2D plane} \quad \text{(continuous plane)} \]

\[ A \times B \]
\[ \text{2D arrangement of points} \quad \text{(discrete plane)} \]

**Multiplication Rule:**
\[ |A \times B| = |A| \cdot |B| \]

\[ \text{e.g.} \quad |2 \times 3| = 6 = |3 \times 2| \]

C.f. the statement @ the top of this page
Exponential Rule: \((\text{Power Sets})\)

the power set of \(A\), denoted \(\mathcal{P}(A)\),
is the set of all subsets of \(A\)

\[\begin{align*}
\mathcal{P}(\emptyset) &= \{\emptyset\} \\
\mathcal{P}(\{1\}) &= \mathcal{P}(\{1,1\}) = \{\emptyset, \{1\}, \{1,1\}\} \\
\mathcal{P}(\{2\}) &= \mathcal{P}(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\} \\
\mathcal{P}(\{3\}) &= \mathcal{P}(\{1,2,3\}) \\
&= \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}
\end{align*}\]

Notice: \[|\mathcal{P}(\emptyset)| = 1 = 2^0 = 2^{|\emptyset|}\]
\[|\mathcal{P}(\{1\})| = 2 = 2^1 = 2^{|1|}\]
\[|\mathcal{P}(\{2\})| = 4 = 2^2 = 2^{|2|}\]
\[|\mathcal{P}(\{3\})| = 8 = 2^3 = 2^{|3|}\]

Fact: \[|\mathcal{P}(X)| = 2^{|X|}\]

We will see multiple proofs of the above fact later on.
Sets = Home Base for Relations

Relations

Functions

Graphs

Partitions

\( f: \mathbb{B} \rightarrow 2 \)

\begin{align*}
  1 & \mapsto 1 \\
  2 & \mapsto 1 \\
  3 & \mapsto 2
\end{align*}

\( f(1) = 1 \)
\( f(2) = 1 \)
\( f(3) = 2 \)

\( g: \mathbb{B} \rightarrow 6 \)

\begin{align*}
  1 & \mapsto 4 \\
  2 & \mapsto 1 \\
  3 & \mapsto 6 \\
  4 & \mapsto 3 \\
  5 & \mapsto 6
\end{align*}

\( g(1) = 4 \)
\( g(2) = 1 \)
\( g(3) = 6 \)
\( g(4) = 3 \)
\( g(5) = 6 \)
Functions & (In)equalities

Q: Given two sets, People & Hats
   how can we prove
   \[ |P| \leq |H|, \quad \text{"There are at least as many hats as there are people"} \]
   \[ |H| \leq |P|, \quad \text{or} \]
   \[ |H| = |P|? \]

Function: \( \phi: H \rightarrow P \)
   every hat is placed on someone’s head

Injective: \( \psi: H \leftarrow P \)
   \[ |H| \leq |P| \]
   every person is wearing at most 1 hat

Surjective: \( \theta: H \rightarrow P \)
   \[ |H| \geq |P| \]
   every person is wearing at least 1 hat

\[ \phi(h_1) = \phi(h_2) \iff h_1 = h_2 \]

Alice has a cowboy hat on
Bijective: \[ \alpha : H \rightarrow P \]

\[ |H| = |P| \]

every person is wearing exactly 1 hat.

**Graphs**

**Vertices & Edges**

Complete graphs

\[ \rightarrow \text{all pairs of vertices are connected by an edge} \]

Trees

\[ \rightarrow \text{no loops} \]
**Food For Thought**

we can define multiplication of natural numbers as follows

\[ N \times M = \times \text{ of objects in } n \text{ groups of } m \text{ objects} \]

& we can use graphs to help us compute products

4 \times 6

we see that there are 4 groups of 6 green dots

Therefore 4 \times 6 = \times green dots in the above picture

6 \times 4

we see that there are 6 groups of 4 purple dots

Therefore 6 \times 4 = \times purple dots in the above picture

These two pictures seem very different, so it is not obvious, starting w/ this definition of multiplication, that

\[ 6 \times 4 = 4 \times 6 \]

(or more generally that \( n \times m = m \times n \))

Using this definition of multiplication, how can we prove

\[ n \times m = m \times n \]?

(Hint: Find a bijection between green dots in picture 1 & purple dots in picture 2.)