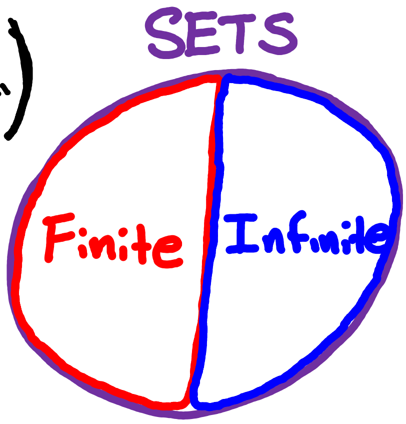
Discrete Math

Main Actors: Sets (i.e. collections of "elements")



Sets play 2 main roles

(i) Counting

(ii) Home Base for Math

Finite Sets

$\varnothing := \{\}$ empty set

$\mathbb{1} := \{1\}$

$\mathbb{2} := \{1,2\}$

$\mathbb{3} := \{1,2,3\}$

$\vdots$

the "elementhood" symbol

$3 \in \mathbb{3}$

e.g. $a \in \{a,b,c,d\}$

$x \notin \{a,b,c,d\}$

x is NOT an element of this set

Infinite Sets

$\mathbb{N} := \{0,1,2,3,\ldots\}$ Natural Numbers (Counting Numbers)

$\mathbb{Z} := \{\ldots,-2,-1,0,1,2,\ldots\}$ Integers

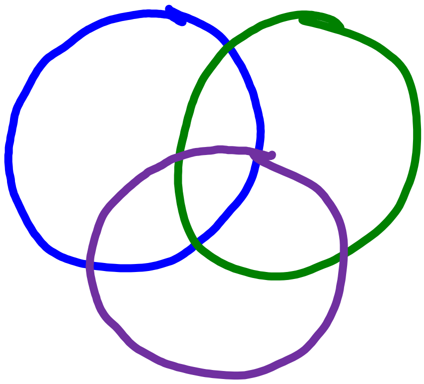
$\mathbb{Q} := \{a/b \mid a \in \mathbb{Z}, b \in \mathbb{N}, b \neq 0\}$ Rational Numbers (fractions)

$\mathbb{R} :=$ All Real Numbers

2 Problem Solving Morals

(I) Drawing Pictures Helps You Reason

↳ We can represent 3 arbitrary sets A, B, C as a Venn Diagram

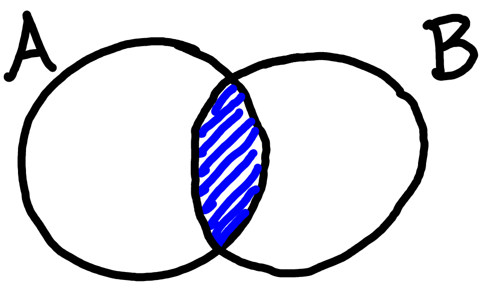


– Elements of A are located inside the blue circle & similarly for elements of B and C.

(II) <u>Naming</u> Things Gives You Power Over Them

Notation

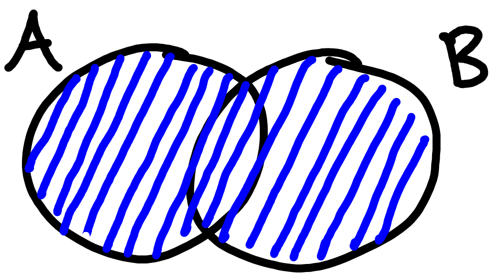
A intersect B



$A \cap B$

↳ The set of elements common to the sets A & B

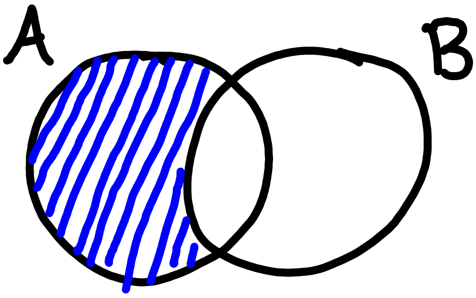
A union B



$A \cup B$

↳ The set of elements in A or B or both

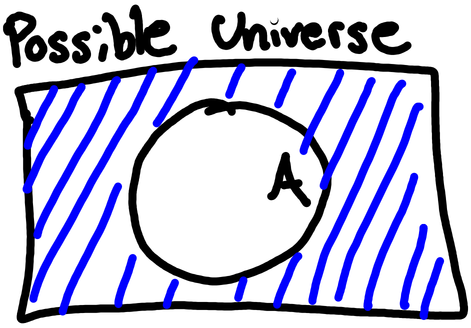
A **minus** B



$A - B$

↳ The set of elements in A but not in B

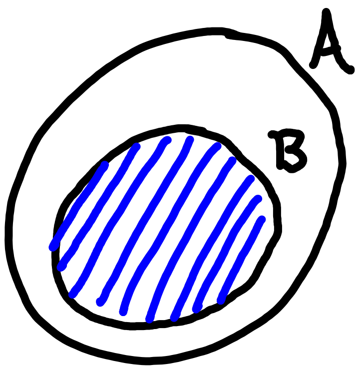
Complement of A



$\bar{A}$ or A^c

↳ The set of elements from the universe except the elements of A

B is a **Subset** of A



$B \subset A$

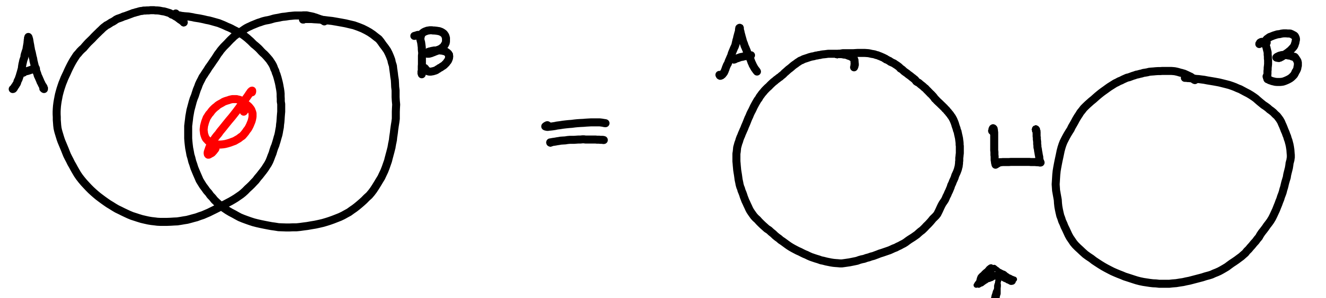
$B \subseteq A$

↳ All elements of B are also elements of A

<u>Note:</u> $\emptyset \subseteq X$ for any set X

<u>Definition:</u> Two sets A & B are **disjoint** if $A \cap B = \emptyset$ (i.e they have no elements in common)

Picture of disjoint sets:



the symbol for "disjoint union"

Basics of Counting

Given: A finite set X

Q: How many elements does X have?

A: There are $|X|$ elements in X.

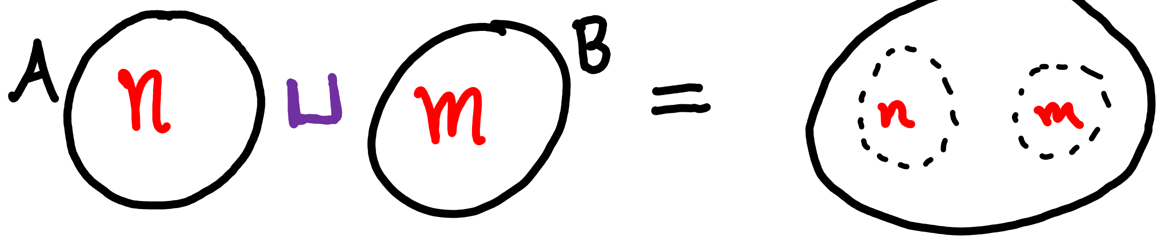
Def: The natural number $|X|$ is called the *cardinality* of X.

Addition Rule: (Combining groups)

Given: A, B two *disjoint* sets w/
$|A| = n$, $|B| = m$

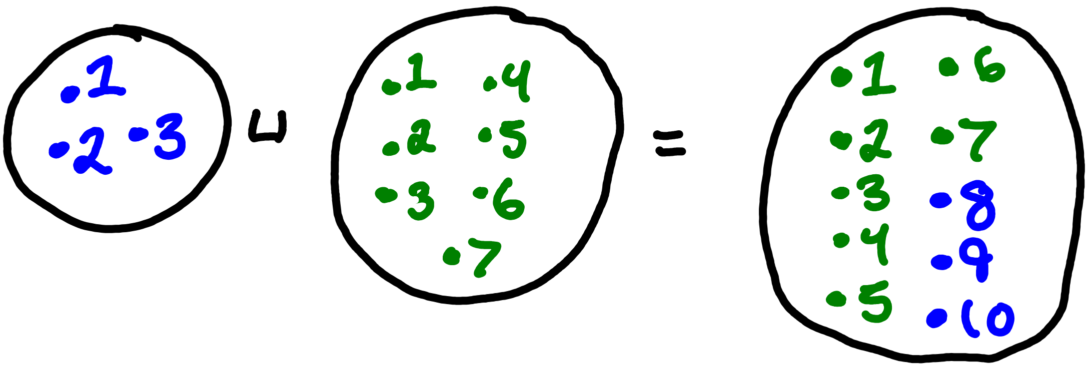
Q: $|A \cup B| = ?$

A: Draw a picture



How many elements are in the large circle on RHS? $n+m$

e.g. $|3 \sqcup 7| = |3| + |7| = 3 + 7 = 10$



Addition Rule:

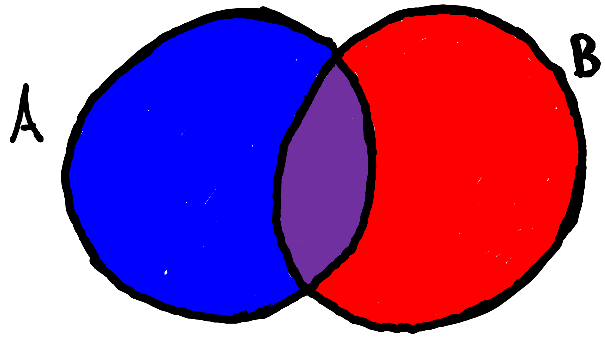
If $A \cap B = \emptyset$, Then $|A \cup B| = |A| + |B|$

Subtraction Rule: (Inclusion - Exclusion principle)

Given: Two sets A, B (which are not necessarily disjoint)

Q: $|A \cup B| = ?$

A: Draw a picture



$|A| = \blacksquare + \blacksquare$

$|B| = \blacksquare + \blacksquare$

$|A \cap B| = \blacksquare$

$|A \cup B| = \blacksquare + \blacksquare + \blacksquare$

Therefore

$$|A| + |B| = |A \cup B| + \underbrace{|A \cap B|}_{\text{Subtract}}$$

Equivalently,

$$|A| + |B| - |A \cap B| = |A \cup B|$$

<u>Multiplication Rule</u>: (Cartesian Products of sets)

Tall Rectangle

$2 \times 3 =$

(1,3) (2,3)

(1,2) (2,2)

(1,1) (2,1)

$3 \times 2 =$

(1,2) (2,2) (3,2)

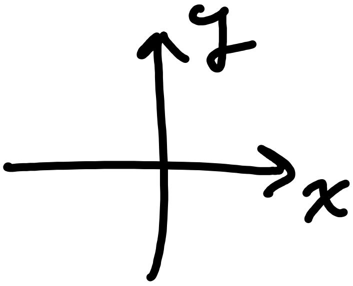
(1,1) (2,1) (3,1)

Wide Rectangle

⚠ $A \times B \neq B \times A$ unless $A = B$

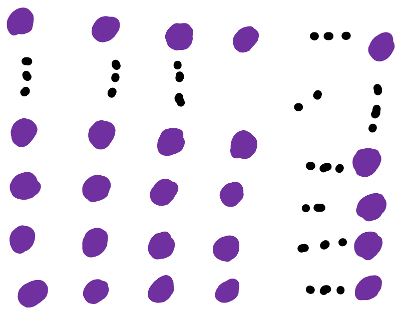
Cartesian Product = ordered pairs of elements
(first element comes from first set
Second element comes from second set)

$\mathbb{R} \times \mathbb{R}$ or $\mathbb{R}^2$



2D plane
(continuous plane)

$A \times B$



2D arrangement of points
(Discrete plane)

<u>Multiplication Rule</u>:

$$|A \times B| = |A| \cdot |B|$$

<u>e.g.</u> $|2 \times 3| = 6 = |3 \times 2|$

c.f. the statement @ the top of this page

Exponential Rule: (Power Sets)

the power set of A, denoted $\mathcal{P}(A)$, is the set of all subsets of A

eg. $\mathcal{P}(\emptyset) = \{\emptyset\}$

$\mathcal{P}(\mathbb{1}) = \mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}$

$\mathcal{P}(\mathbb{2}) = \mathcal{P}(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \mathbb{2}\}$

$\mathcal{P}(\mathbb{3}) = \mathcal{P}(\{1,2,3\})$

$= \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \mathbb{3}\}$

Notice: $|\mathcal{P}(\emptyset)| = 1 = 2^0 = 2^{|\emptyset|}$

$|\mathcal{P}(\mathbb{1})| = 2 = 2^1 = 2^{|\mathbb{1}|}$

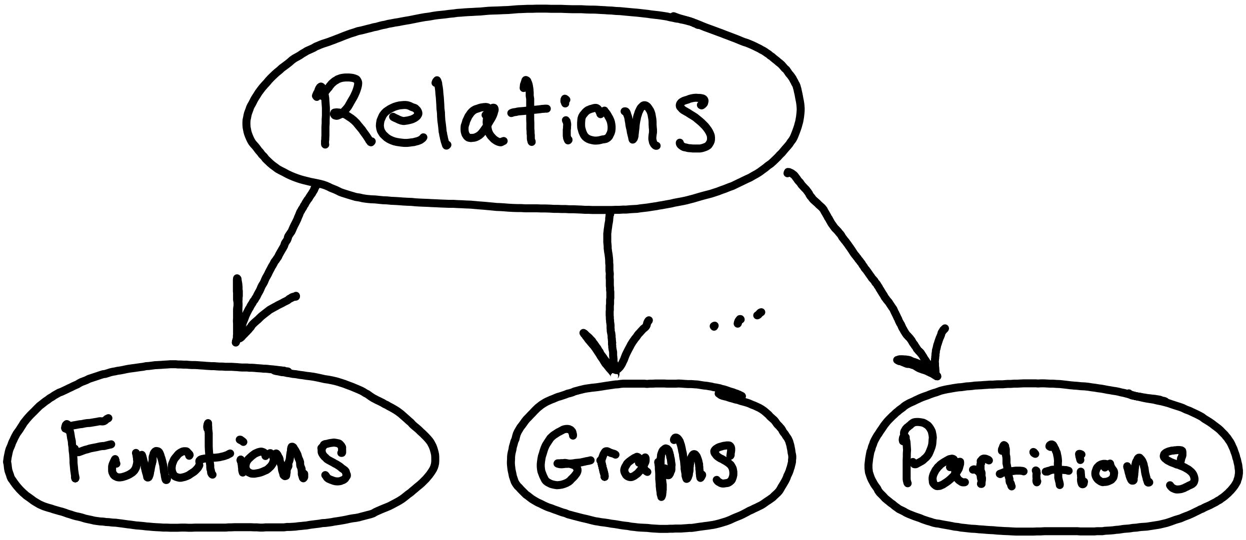
$|\mathcal{P}(\mathbb{2})| = 4 = 2^2 = 2^{|\mathbb{2}|}$

$|\mathcal{P}(\mathbb{3})| = 8 = 2^3 = 2^{|\mathbb{3}|}$

Fact: $|\mathcal{P}(X)| = 2^{|X|}$

We will see multiple proofs of the above fact later on.

Sets = Home Base for Relations

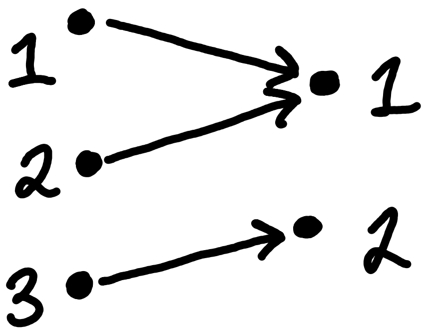


Functions

$f : \mathbf{3} \longrightarrow \mathbf{2}$

$1 \longmapsto 1$
$2 \longmapsto 1$
$3 \longmapsto 2$

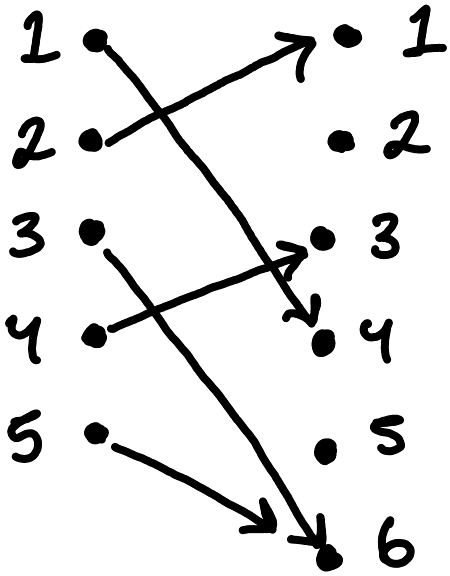
$f(1) = 1$
$f(2) = 1$
$f(3) = 2$



$g : \mathbf{5} \longrightarrow \mathbf{6}$

$1 \longmapsto 4$
$2 \longmapsto 1$
$3 \longmapsto 6$
$4 \longmapsto 3$
$5 \longmapsto 6$

$g(1) = 4$
$g(2) = 1$
$g(3) = 6$
$g(4) = 3$
$g(5) = 6$



Functions & (In)equalities

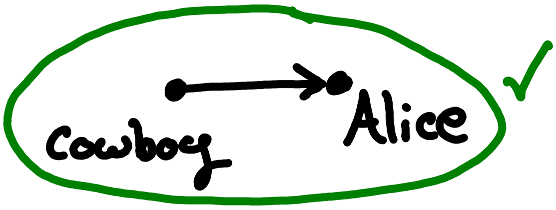
Q: Given two sets, People & Hats, how can we prove

$|P| \leq |H|$, ← "There are at least as many hats as there are people"

$|H| \leq |P|$, ⋮

OR $|H| = |P|$?

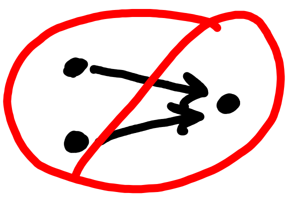
Function: $\varphi : H \longrightarrow P$ every hat is placed on someone's head



Alice has a cowboy hat on

Injective: $\psi : H \hookrightarrow P$ $\psi(h_1) = \psi(h_2) \iff h_1 = h_2$

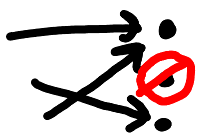
$|H| \leq |P|$



every person is wearing at most 1 hat

Surjective: $\theta : H \twoheadrightarrow P$

$|H| \geq |P|$



every person is wearing at least 1 hat

bijective: $\alpha : H \xrightarrow{\sim} P$

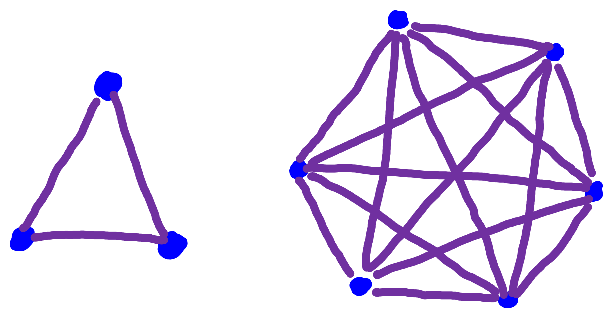
$$\text{bijective} = \boxed{\begin{array}{c}\text{injective} \\ + \\ \text{surjective}\end{array}}$$

$$|H| = |P|$$

every person is wearing exactly 1 hat

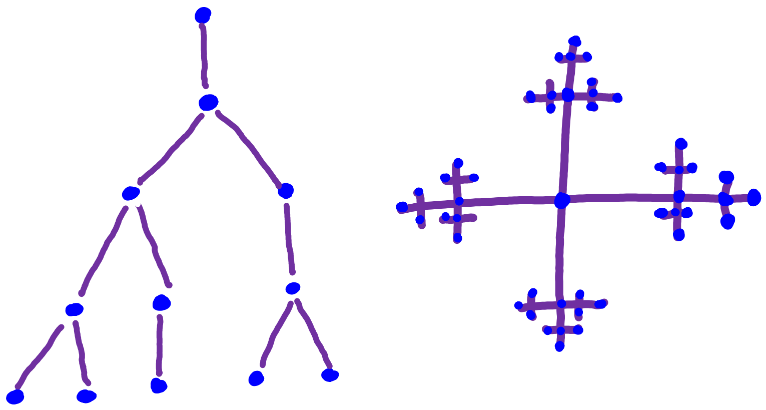
Graphs

Vertices & Edges



Complete graphs

↳ all pairs of vertices are connected by an edge



Trees

↳ No loops

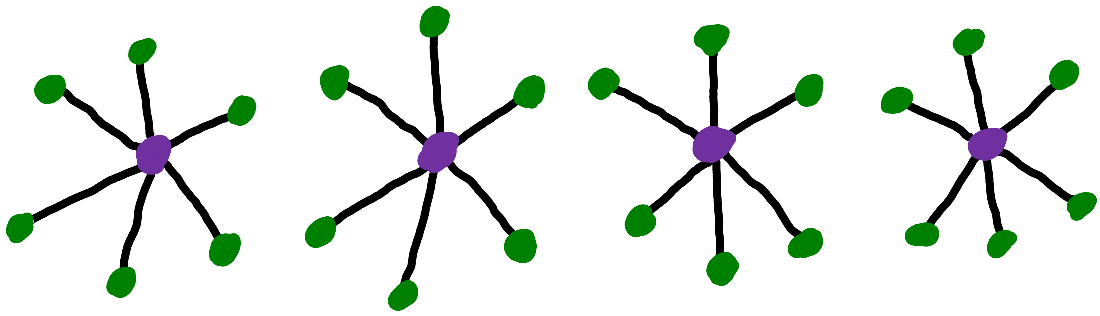
Food For Thought

we can define multiplication of natural numbers as follows

$$n \times m = \# \text{ of objects in } n \text{ groups of } m \text{ objects}$$

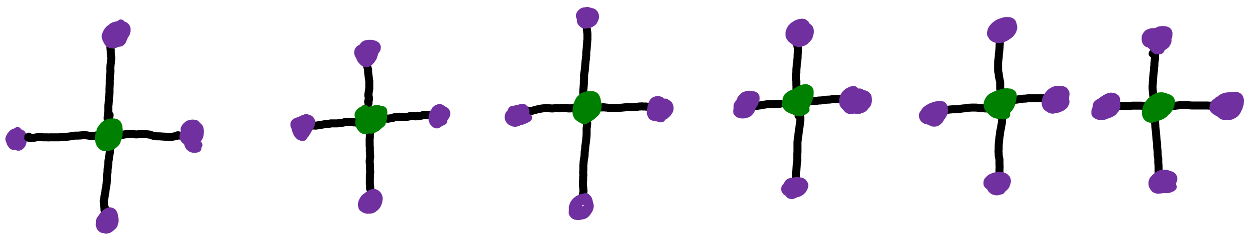
& we can use graphs to help us compute products

4×6



we see that there are 4 groups of 6 green dots
Therefore $4 \times 6 = \#$ green dots in the above picture

6×4



we see that there are 6 groups of 4 purple dots
Therefore $6 \times 4 = \#$ purple dots in the above picture

These two pictures seem very different, so it is not obvious, starting w/ this definition of multiplication, that $6 \times 4 = 4 \times 6$ (or more generally that $n \times m = m \times n$)

Using this definition of multiplication, how can we prove $n \times m = m \times n$? (Hint: Find a bijection between green dots in picture 1 & purple dots in picture 2)