Continuous functions

\[ f(x) = e^x \]

Sequences (discrete functions)

\[ a_n = e^n \]

Derivatives

\[ f'(x) = e^x \]

Slope @ a point

\[ \text{captures the change of the function in a small interval around the point} \]

Recurrent Relations

\[ a_{n+1} = e \cdot a_n \]

how to get to the next term

\[ \text{tells how the sequence is changing from term to term} \]

Antiderivatives (Integration)

\[ \int f'(x) \, dx = f(x) + C \]

Solving Recurrence Relations

\[ \text{Find a sequence that satisfies the recurrence relation} \]

\[ e.g. \ 7e^n = a_n \]

Answers are not unique because of this arbitrary \( C \)

Answers depend on an arbitrarily chosen initial value
\[ f'(x) = 2x \]
\[ \int 2x \, dx = x^2 + C \]

Each choice of C gives a different curve.

\[ a_n = 2a_{n-1} \]
\[ \Rightarrow a_n = 2^n \cdot a_0 \]

Starting with an arbitrary \( a_0 \) we can complete it to a sequence satisfying the recurrence relation. However, the choice of \( a_0 \) together with the recurrence relation uniquely determines the sequence.

**Initial Value Problems**

**differential equation**
\[ f''(x) = f'(x) + f(x) \]

**recurrence relation**
\[ F_n = F_{n-1} + F_{n-2} \]

**base cases**
\[ F_0 = 1, \ F_1 = 1 \]

Once enough information is given the solution becomes unique.

**Examples of Recurrence Relations**
\[ a_n = a_{n-1} + D \]
\[ a_n = a_{n-1} + n \]
\[ a_n = r a_{n-1} \]
\[ a_n = n a_{n-1} \]
\[ F_n = F_{n-1} + F_{n-2} \]
\[ H_n = 2H_{n-1} + 1 \]
\[ C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1} \]
Claim: \( a_n := n^2 \) does not satisfy \( a_n = a_{n-1} + 12 \)

Proof: \( a_1 = 1^2 = 1 \neq 12 = 0^2 + 12 = a_0 + 12 \) \( \Box \)

Claim: \( a_n := n^2 \) does not satisfy \( a_n = a_{n-1} + D \) for any \( D \).

Proof: If it did (proof by contradiction) we would have \( 1 = a_1 = a_0 + D = 0^2 + D = D \) but also \( 4 = a_2 = a_1 + D = 1 + D \Rightarrow 1 = D = 4 - 1 = 3 \) \( \underline{\text{contradiction}} \) \( \Box \)

Review: We know how to find Some Solutions

\[ a_n := k + nD \] always solves \( a_n = a_{n-1} + D \)

Proof: \( a_{n-1} + D \xrightarrow{\text{def}} k + (n-1)D + D = k + (n-1+1)D = k + nD = a_n \)

This proves the sequence satisfies the recurrence relation \( \Box \)

This proves that recurrence relations alone don't have unique solutions

With induction (discussed at a later time) we can prove that recurrence relations have unique solutions once "enough" base cases are provided.
Idea: Accept the statement in red above

We now prove $\sum_{n=0}^{\infty} a_n$ solves $a_n = a_{n-1} + D$

$\Rightarrow a_n = K + nD$ for some $K$

indeed, if $\sum_{n=0}^{\infty} a_n$ solves $a_n = a_{n-1} + D$ then

$(a_0 = K, a_n = a_{n-1} + D)$ is an initial value problem which gives rise to a unique solution. However, our list of solutions contains $a_n = K + nD$ which gives rise to the same initial value problem $(K, a_n = a_{n-1} + D)$. $\square$

**Particular/General Solutions**

**Goal:** Find All Solutions to a given recurrence relation

In general, this problem is too hard & we have no idea how to achieve this goal except in cases where the recurrence relation takes a particular form
First, we name a particularly nice class of recurrence relations to focus on.

1. Look for a particular solution (any solution at all)
2. Try to use particular solution(s) to find a generalized formula that describes all solutions

**Example:** Finding a particular solution of \( a_n = a_{n-1} + n \)

As before, \( \{a_n\}_{n=0}^\infty \) solution \( \Rightarrow (a_0, a_n = a_{n-1} + n) \)

**Initial value problem**

- **pick the simplest possible base case.**
  
  In this case, \( a_0 = 0 \) is a good choice

- **Solve this initial value problem** (0, \( a_n = a_{n-1} + n \))

  \[ \Rightarrow \text{We know } \phi_n = \sum_{k=0}^{n} k = \frac{n(n+1)}{2} = \binom{n+1}{2} \]

- **this is a particular solution to** \( a_n = a_{n-1} + n \)

- **Think about what happens if we change the initial conditions.** (K, \( a_n = a_{n-1} + n \))

  \[ \Rightarrow \text{Since all we do is addition at every stage we guess the answer should be of the form} \]
Guess: $a_n = K + p_n$

Check: $a_n = K + p_n = K + p_{n-1} + n = a_{n-1} + n$

$p_n$ solves $a_n = a_{n-1} + n$

meaning $p_n = p_{n-1} + n$

General Solution of $a_n = a_{n-1} + n$ is $a_0 + p_n$

Similarly, (by trading $\bigcirc$ for $1$ & $+$ for $\times$)

we can prove the following are the general solutions for each recurrence relation.

<table>
<thead>
<tr>
<th>Name</th>
<th>Defining Relation</th>
<th>General Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Arithmetic Progression</td>
<td>$a_n = a_{n-1} + D$</td>
<td>$a_n = K + nD$</td>
</tr>
<tr>
<td>Geometric Progression</td>
<td>$a_n = r \cdot a_{n-1}$</td>
<td>$a_n = K \cdot r^n$</td>
</tr>
<tr>
<td>&quot;Subsets of size 2&quot;</td>
<td>$a_n = a_{n-1} + n$</td>
<td>$a_n = K + (\binom{n+1}{2})$</td>
</tr>
<tr>
<td>factorials</td>
<td>$a_n = n \cdot a_{n-1}$</td>
<td>$a_n = K \cdot n!$</td>
</tr>
</tbody>
</table>
Linear Homogeneous Recurrence Relations of Degree K w/ Constant Coeff

(a broad category of recurrence relations we know how to solve)

\[ a_n := c_1a_{n-1} + c_2a_{n-2} + \ldots + c_k a_{n-k} \]

Linear ~ "linear combination" sum of multiples of \( a_j \)s
Homogeneous ~ Every term involves some \( a_j \) for \( j < n \)

Constant Coefficient ~ \( c_j \in \mathbb{R} \) for all \( i \) (the coefficients of the \( a_j \) could, in general, be functions of \( n \). This condition rules out all but the simplest such functions - the constant functions).

Degree K ~ \( c_K \neq 0 \) (the smallest index in some \( a_j \) is realized by \( a_{n-k} \). Other \( a_j \) may have zero coefficients \( \Rightarrow \) don't actually appear on RHS, but \( a_{n-k} \) must)

<table>
<thead>
<tr>
<th>Examples</th>
<th>Non-Examples</th>
<th>Missing Property</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_n = r \cdot a_{n-1} )</td>
<td>( a_n=a_{n-1}+D )</td>
<td>Homogeneous</td>
</tr>
<tr>
<td>( a_n=a_{n-1}+N )</td>
<td>( a_n = N a_{n-1} )</td>
<td>Constant Coeff</td>
</tr>
<tr>
<td>( F_n = F_{n-1} + F_{n-2} )</td>
<td>( H_n = 2H_{n-1} + 1 )</td>
<td>Homogeneous</td>
</tr>
<tr>
<td>( a_n = r_1a_{n-1} + r_2a_{n-2} )</td>
<td>( C_n = \sum_{k=0}^{n-1} c_k C_{n-k-1} )</td>
<td>Linear</td>
</tr>
<tr>
<td>( a_n = a_{n-1} + a_{n-2} )</td>
<td>( a_n = a_{n-1} + a_{n-2} )</td>
<td>Linear</td>
</tr>
</tbody>
</table>
Once we learn how to solve $\deg K - \text{LHRR w/CC}$ we can find all solutions to the recurrence relation defining the Fibonacci numbers (and quite a bit more than that).

To explain how to solve $\deg K - \text{LHRR w/CC}$ we work out the degree 2 example for the relation $F_n = F_{n-1} + F_{n-2}$ and then an outline for the general procedure will be provided.

**Goal:** Find all sequences $\{F_n\}_{n=0}^{\infty}$ which satisfy

$$F_n = F_{n-1} + F_{n-2}$$

**Step 1:** Find 2 "sufficiently different" particular solutions. (We look for 2 of them because the degree is 2.)

**Guess:** 3 solutions of the form $F_n = r^n$

(This may seem to come from nowhere, but this is nothing to worry about because we always make this guess for $\deg K - \text{LHRR w/CC}$.)

Investigate the consequences of this guess.

If $F_n = r^n$ for some $r \in \mathbb{R}$ then

$$r^n = r^{n-1} + r^{n-2}$$

(by the recursive formula.)
Thus, $r$ is a zero of the polynomial $x^n - x^{n-1} - x^{n-2}$.

So, finding a particular solution boils down to solving for $x$ in the following equation:

$$x^n - x^{n-1} - x^{n-2} = 0$$

(factoring out common powers of $x$)

$$x^{n-2}(x^2 - x - 1) = 0$$

(if a product is zero, one of the factors is zero)

$$x^{n-2} = 0 \lor (x^2 - x - 1) = 0$$

"Characteristic Equation" for $F_n = F_{n-1} + F_{n-2}$

$$x = 0 \lor x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Summary of the work so far (the following implication is true)

$$F_n = r^n \land F_n = F_{n-1} + F_{n-2} \Rightarrow r \in \{0, \frac{1 + \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2}\}$$

- the possibility of $r = 0$ is rather uninteresting
- it means the sequence is constantly zero
- (i.e. $F_n = 0 \forall n$)
- so we throw this boring solution away.

it may be the case that these conditions never happen at the same time, in which case, the implication is vacuously true (something we would like to avoid)
We now prove the converse \( \iff \) by checking that
\[
F_n = \left(\frac{1 + \sqrt{5}}{2}\right)^n \quad \text{and} \quad \tilde{F}_n = \left(\frac{1 - \sqrt{5}}{2}\right)^n
\]
satisfy the recurrence relation \( F_n = F_{n-1} + F_{n-2} \).

**Proof:**
\[
F_{n-1} + F_{n-2} = \left(\frac{1 + \sqrt{5}}{2}\right)^{n-1} + \left(\frac{1 + \sqrt{5}}{2}\right)^{n-2} = \left(\frac{1 + \sqrt{5}}{2}\right)^{n-2} \left(\frac{1 + \sqrt{5}}{2} + 1\right)
\]
\[
= \left(\frac{1 + \sqrt{5}}{2}\right)^{n-2} \left(\frac{1 + \sqrt{5}}{2} + \frac{2}{2}\right)
\]
\[
= \left(\frac{1 + \sqrt{5}}{2}\right)^{n-2} \left(\frac{3 + \sqrt{5}}{2}\right)
\]
\[
= \left(\frac{1 + \sqrt{5}}{2}\right)^{n-2} \left(\frac{1 + \sqrt{5}}{2}\right)^2
\]
\[
= \left(\frac{1 + \sqrt{5}}{2}\right)^n = F_n \quad \square
\]

**Lemma:**
\[
\left(\frac{1 + \sqrt{5}}{2}\right)^2 = \frac{1}{2^2} (1 + 2 \sqrt{5} + 5)
\]
\[
= \frac{6 + 2 \sqrt{5}}{2^2} = \frac{3 + \sqrt{5}}{2}
\]

**Exercise:** Prove \( \left(\frac{1 - \sqrt{5}}{2}\right)^n = \left(\frac{1 - \sqrt{5}}{2}\right)^{n-1} + \left(\frac{1 - \sqrt{5}}{2}\right)^{n-2} \)

**Step 2:** Combine the particular solutions to form the general solution.

**Relevant Fact 1:** \( A_n = A_{n-1} + A_{n-2} \) \& \( \tilde{A}_n = \alpha A_n \)

Then \( \tilde{A}_n = \tilde{A}_{n-1} + \tilde{A}_{n-2} \) (i.e. Real \& multiples of

Solutions to DKLHRRCC are also solutions to the same relation)

**Proof:** (of above case/example):
\[
\tilde{A}_n = \alpha A_n = \alpha (A_{n-1} + A_{n-2}) = \alpha A_{n-1} \cup \alpha A_{n-2} = \tilde{A}_{n-1} + \tilde{A}_{n-2} \quad \square
\]
Relevant Fact 2: \( A_n = A_{n-1} + A_{n-2} \) & \( B_n = B_{n-1} + B_{n-2} \)

Then \( C_n := A_n + B_n \Rightarrow C_n = C_{n-1} + C_{n-2} \)

(i.e. the sum of two sequences that satisfy a DKLHRRCC also satisfies the same recurrence relation)

Proof (of above case/example):
\[ C_n := A_n + B_n = A_{n-1} + A_{n-2} + B_n = A_{n-1} + A_{n-2} + B_{n-1} + B_{n-2} \]
\[ = (A_{n-1} + B_{n-1}) + (A_{n-2} + B_{n-2}) =: C_{n-1} + C_{n-2} \]

\[ A_n = \left(\frac{1 + \sqrt{5}}{2}\right)^n \quad \text{is a solution} \]
\[ B_n = \left(\frac{1 - \sqrt{5}}{2}\right)^n \quad \text{is a solution} \]

\[ \alpha_1 \left(\frac{1 + \sqrt{5}}{2}\right)^n \quad \text{is a solution (for any} \ \alpha_1 \in \mathbb{R} \text{)} \]
\[ \alpha_2 \left(\frac{1 - \sqrt{5}}{2}\right)^n \quad \text{is a solution (for any} \ \alpha_2 \in \mathbb{R} \text{)} \]

\[ \alpha_1 \left(\frac{1 + \sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1 - \sqrt{5}}{2}\right)^n \quad \text{is a solution} \]

Fact: (the proof of this claim in general is outlined below c.f. “Theorem 1”)

Since \( \left(\frac{1 + \sqrt{5}}{2}\right) \neq \left(\frac{1 - \sqrt{5}}{2}\right) \), every solution to \( F_n = F_{n-1} + F_{n-2} \)

is of the form \( \alpha_1 \left(\frac{1 + \sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1 - \sqrt{5}}{2}\right)^n \)

for some \( \alpha_1, \alpha_2 \in \mathbb{R} \).
Fibonacci Numbers

Solve the following initial value problem

\[
(F_0 = 0, \ F_1 = 1, \ F_n = F_{n-1} + F_{n-2})
\]

we know \( F_n = \alpha_1 \left( \frac{1 + \sqrt{5}}{2} \right)^n + \alpha_2 \left( \frac{1 - \sqrt{5}}{2} \right)^n \)

So it suffices to find the values of \( \alpha_1 \) & \( \alpha_2 \) that give the correct initial conditions

\[
\begin{aligned}
0 & = F_0 = \alpha_1 + \alpha_2 \quad (\iff \alpha_1 = -\alpha_2) \\
1 & = F_1 = \alpha_1 \left( \frac{1 + \sqrt{5}}{2} \right) + \alpha_2 \left( \frac{1 - \sqrt{5}}{2} \right)
\end{aligned}
\]

Combining gives

\[
1 = -\alpha_2 \left( \frac{1 + \sqrt{5}}{2} \right) + \alpha_2 \left( \frac{1 - \sqrt{5}}{2} \right)
\]

\[
= \alpha_2 \left( \frac{1 - \sqrt{5}}{2} - \frac{1 + \sqrt{5}}{2} \right)
\]

\[
= -\sqrt{5} \alpha_2
\]

\(\iff\) \( \alpha_2 = \frac{-1}{\sqrt{5}} = -\alpha_1 \) (so \( \alpha_1 = \frac{1}{\sqrt{5}} \))

Note: These initial conditions yield the sequence

\(0, 1, 1, 2, 3, 5, 8, ... \) (always whole numbers !!)
Tying up a loose end

A while back we claimed that the $n^{th}$ Fibonacci number is given by $F_n = \frac{\eta^n + (1-\eta)^n}{\sqrt{5}}$ where $\eta = \frac{1+\sqrt{5}}{2}$.

We finally prove this claim.

Lemma:

$1 - \eta = 1 - \frac{1+\sqrt{5}}{2} = \frac{1-\sqrt{5}}{2} = \frac{\sqrt{5} - 1}{2}$

$F_n = \frac{1}{\sqrt{5}} \left( \left( \frac{1+\sqrt{5}}{2} \right)^n - \left( \frac{1-\sqrt{5}}{2} \right)^n \right) = \frac{1}{\sqrt{5}} \left( \eta^n - (1-\eta)^n \right)$

\[\square\]

**Theorem 1**: Degree 2 LHRRw/CC (distinct root case)

Suppose $r^2 - C_1 r - C_2$ has two distinct roots $r_1 \neq r_2$.

Then $\{x_n\}_{n=0}^{\infty}$ is a solution to $a_n = C_1 a_{n-1} + C_2 a_{n-2}$ if and only if

$x_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for some $\alpha_1, \alpha_2 \in \mathbb{R}$

Real number coefficients whose values determine the initial conditions.

\[\text{Unequal zeroes of the characteristic equation } r^2 - C_1 r - C_2\]

\[\text{Idea behind the proof} \quad r_1 \neq r_2 \text{ roots of } r^2 - C_1 r - C_2 \Rightarrow x_n = \alpha_1 r_1^n + \alpha_2 r_2^n \text{ is a solution to } a_n = C_1 a_{n-1} + C_2 a_{n-2}\]

\[\text{can use } \text{"relevant facts" } 1 \& 2 \text{ or direct proof}\]
$\{x_n\}_{n=0}^{\infty}$ is a solution to $a_n = c_1 a_{n-1} + c_2 a_{n-2}$

The initial value problem $$(x_0, x_1, a_n = c_1 a_{n-1} + c_2 a_{n-2})$$ has $\{x_n\}_{n=0}^{\infty}$ as a solution.

One can then check that

$$a_n = \left(\frac{x_n - x_0 f_2}{f_1 - f_2}\right) f_1^n + \left(\frac{x_0 f_1 - x_1}{f_1 - f_2}\right) f_2^n$$

relies on a fact we won't prove. You will have the tools to prove suitable initial value problems have unique solutions when we learn induction.

Solving Deg 2 LHRRew/CC in practice

**Step 1:** Find the characteristic equation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} \quad \rightarrow \quad r^2 - c_1 r - c_2$$

**Step 2:** Find the zeros of the characteristic equation

$$r^2 - c_1 r - c_2 \quad \rightarrow \quad r_1, r_2 = \frac{c_1 \pm \sqrt{c_1^2 + 4c_2}}{2}$$

(if $c_1^2 + 4c_2 = 0$ then the two roots (zeros) are equal so Thm 1 does not help. We will learn how to deal with this soon enough.)
Step 3: Thm 1 \[ a_n = \alpha_1 r_1^n + \alpha_2 r_2^n \]

If no initial conditions were given we are done.

If initial conditions are given we move on to the final step.

Step 4: Use the initial conditions to solve for \( \alpha_1, \alpha_2 \)

\[
\begin{aligned}
\alpha_0 &= \alpha_1 + \alpha_2 \\
\alpha_1 &= \alpha_1 r_1 + \alpha_2 r_2
\end{aligned}
\]

\[
\begin{aligned}
\alpha_1 &= \frac{a_1 - a_0 s_2}{r_1 - r_2} \\
\alpha_2 &= \frac{a_0 s_1 - a_1}{r_1 - r_2}
\end{aligned}
\]