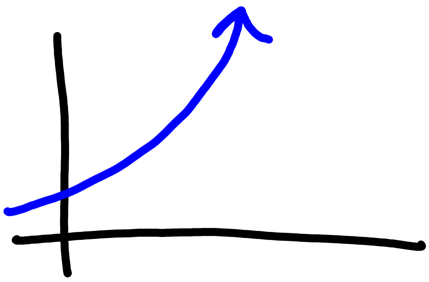
Calculus & Recurrence Relations

Continuous functions

$$f(x) = e^x$$



Derivatives

$$f'(x) = e^x$$

Slope @ a point

↳ captures the change of the function in a small interval around the point

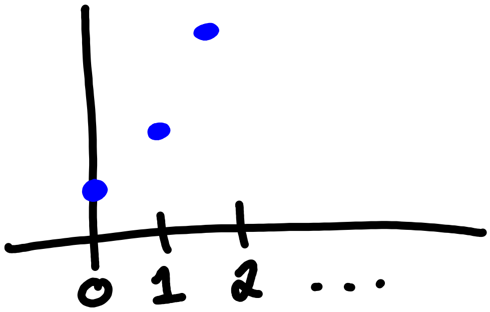
Antiderivatives (Integration)

$$\int f'(x)\,dx = f(x) + \underline{C}$$

Answers are not unique because of this arbitrary C

Sequences (discrete functions)

$$a_n = e^n$$



Recurrence Relations

$$a_{n+1} = e \cdot a_n$$

how to get to the next term

↳ tells how the sequence is changing from term to term

Solving Recurrence Relations

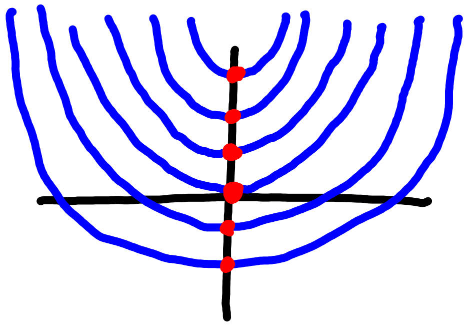
↳ Find a sequence that satisfies the recurrence relation

e.g. $7e^n = a_n$

Answers depend on an arbitrarily chosen initial value

$$f'(x) = 2x$$

$$\int 2x\,dx = x^2 + C$$



Each choice of C gives a different curve

$$a_n := 2a_{n-1}$$

$$\Rightarrow a_n = 2^n \cdot a_0$$

Starting with an arbitrary a_0 we can complete it to a sequence satisfying the recurrence relation. However, the choice of a_0 together with the recurrence relation uniquely determines the sequence

Initial Value Problems

differential equation

$$f''(x) = f'(x) + f(x)$$

initial conditions

$$f(0) = 1,\ f'(0) = 1$$

recurrence relation

$$F_n = F_{n-1} + F_{n-2}$$

base cases

$$F_0 = 1,\ F_1 = 1$$

Once enough information is given the solution becomes unique

Examples of Recurrence Relations

$$a_n = a_{n-1} + D \qquad a_n = a_{n-1} + n$$

$$a_n = r a_{n-1} \qquad a_n = n a_{n-1}$$

$$F_n = F_{n-1} + F_{n-2} \qquad H_n = 2H_{n-1} + 1$$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

Claim: $a_n := n^2$ does not satisfy $a_n = a_{n-1} + 12$

proof: $a_1 = 1^2 = 1 \neq 12 = 0^2 + 12 = a_0 + 12$ □

Claim: $a_n := n^2$ does not satisfy $a_n = a_{n-1} + D$ for any D.

proof: If it did (proof by contradiction) we would have

$$1 = a_1 = a_0 + D = 0^2 + D = D$$ but also

$$4 = a_2 = a_1 + D = 1 + D \Rightarrow 1 = D = 4 - 1 = 3$$

contradiction □

Review: We know how to find some solutions

$a_n := K + nD$ always solves $a_n = a_{n-1} + D$

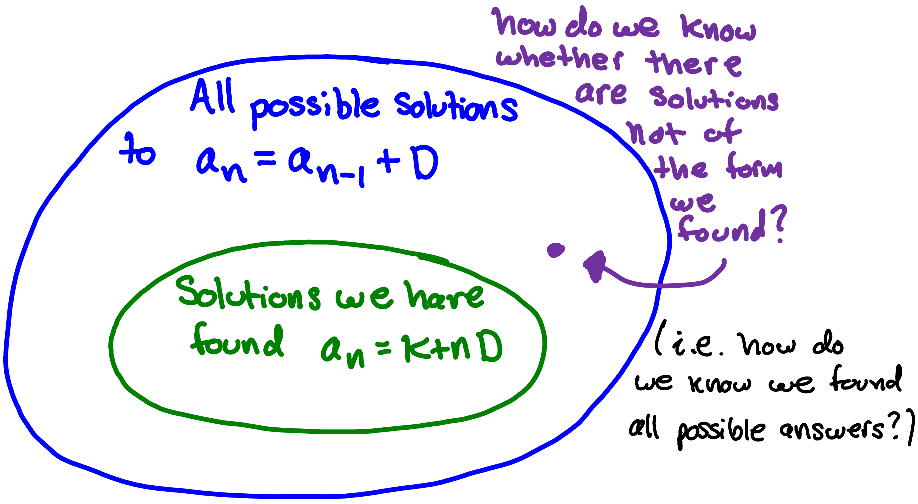
proof:

$$a_{n-1} + D \overset{\text{def}}{=} K + (n-1)D + D \overset{\text{Distributive property}}{=} K + (n-1+1)D = K + nD \overset{\text{def}}{=} a_n$$

this proves the sequence satisfies the recurrence relation □

This proves that recurrence relations alone don't have unique solutions

With induction (discussed at a later time) we can prove that recurrence relations have unique solutions once "enough" base cases are provided.



Idea: Accept the statement in red above

We now prove $\{a_n\}_{n=0}^{\infty}$ solves $a_n = a_{n-1} + D$

$\Longrightarrow a_n = K + nD$ for some K

indeed, if $\{a_n\}_{n=0}^{\infty}$ solves $a_n = a_{n-1} + D$ then

$(a_0 =: K,\ a_n = a_{n-1} + D)$ is an initial value problem

which gives rise to a unique solution. However,

our list of solutions contains $a_n = K + nD$

which gives rise to the same initial value

problem $(K,\ a_n = a_{n-1} + D)$. $\square$

Particular/General Solutions

Goal: Find All Solutions to a given recurrence relation

In general, this problem is too hard & we have no idea how to achieve this goal except in cases where the recurrence relation takes a particular form

First, we name a particularly nice class of recurrence relations to focus on.

↳ Look for a particular solution (any solution at all)

↳ Try to use particular solution(s) to find a generalized formula that describes all solutions

e.g. Finding a particular solution of $a_n = a_{n-1} + n$

As before $\{a_n\}_{n=0}^{\infty}$ solution $\Rightarrow (a_0, a_n = a_{n-1} + n)$

initial value problem

- pick the simplest possible base case.

  In this case, $a_0 = 0$ is a good choice

- Solve this initial value problem $(0, a_n = a_{n-1} + n)$

  ↳ We know (in this case) $p_n = \sum_{k=0}^{n} k = \frac{n(n+1)}{2} = \binom{n+1}{2}$

this is a particular solution to $a_n = a_{n-1} + n$

- Think about what happens if we change the initial conditions. $(K, a_n = a_{n-1} + n)$

  ↳ Since all we do is addition at every stage we guess the answer should be of the form

Guess: $a_n = K + p_n$

Check: $a_n := K + p_n = K + p_{n-1} + n = \underbrace{a_{n-1}}_{=:K+p_{n-1}} + n$

p_n solves $a_n = a_{n-1} + n$
meaning $p_n = p_{n-1} + n$

General Solution of $a_n = a_{n-1} + n$ is $a_0 + p_n$ $\square$

Similarly, (by trading 0 for 1 & + for x)
we can prove the following are the general solutions for each recurrence relation.

Name	Defining Relation	General Solution
Arithmetic Progression	"common difference" $a_n = a_{n-1} + D$	$a_n = K + nD$
Geometric Progression	"common ratio" $a_n = r \cdot a_{n-1}$	$a_n = Kr^n$
"Subsets of size 2"	$a_n = a_{n-1} + n$	$a_n = K + \binom{n+1}{2}$
Factorials	$a_n = n a_{n-1}$	$a_n = Kn!$

Linear Homogeneous Recurrence Relations of Degree K w/ Constant Coeff

(a broad category of recurrence relations we know how to solve)

$$\boxed{a_n := c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_K a_{n-K}}$$

Linear ~ "linear combination" sum of multiples of a_js

Homogeneous ~ Every term involves some a_j for $j<n$

Constant Coefficient ~ $c_i \in \mathbb{R}$ for all i (the coefficients of the a_j could, in general, be functions of n. This condition rules out all but the simplest such functions - the constant functions)

Degree K ~ $c_K \neq 0$ (the smallest index in some a_j is realized by a_{n-K}. Other a_j may have zero coefficients $\Rightarrow$ don't actually appear on RHS, but a_{n-K} must)

Examples	Non-Examples	Missing Property
• $a_n = r \cdot a_{n-1}$ ↳ compound Interest (solved above)	$a_n = a_{n-1} + D$	Homogeneous
• $a_n = a_{n-5}$ ↳ Degree 5 example	$a_n = a_{n-1} + n$	Homogeneous
• $F_n = F_{n-1} + F_{n-2}$ ↳ Gives Fibonacci #s w/ right base case (we have yet to introduce tools to solve this)	$a_n = n a_{n-1}$	Constant Coeff
• $a_n = r_1 a_{n-1} + r_2 a_{n-2}$ ↳ degree 2 generalization of the Fibonacci RR	$H_n = 2H_{n-1} + 1$	Homogeneous
	$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$	Linear
	$a_n = a_{n-1} + a_{n-2}^2$	Linear

$\therefore$ Once we learn how to solve Deg K-LHRRw/CC We can find all solutions to the recurrence relation defining the fibonacci numbers (& quite a bit more than that)

To explain how to solve Deg K-LHRRw/CC we work out the degree 2 example for the relation $F_n = F_{n-1} + F_{n-2}$ then an outline for the general procedure will be provided

Goal: Find all sequences $\{F_n\}_{n=0}^{\infty}$ which satisfy

$$F_n = F_{n-1} + F_{n-2}$$

Step 1: Find 2 "sufficiently different" particular solutions. (we look for 2 of them because the degree is 2)

Guess: $\exists$ solutions of the form $F_n = r^n$

(this may seem to come from nowhere, but this is nothing to worry about because we always make this guess for Deg K-LHRRw/CC)

Investigate the consequences of this guess

If $F_n = r^n$ for some $r \in \mathbb{R}$ Then

$$r^n = r^{n-1} + r^{n-2}$$ (by the recursive formula)

Thus, r is a zero of the polynomial $x^n - x^{n-1} - x^{n-2}$

So, finding a particular solution boils down to solving for x in the following equation

$$x^n - x^{n-1} - x^{n-2} = 0$$

(factor out common powers of x)

$$\Longleftrightarrow \quad x^{n-2}(x^2 - x - 1) = 0$$

(if a product is zero, one of the factors is zero)

$$\Longleftrightarrow \quad x^{n-2} = 0 \quad \vee \quad \underbrace{x^2 - x - 1}_{\text{"Characteristic Equation" for } F_n = F_{n-1} + F_{n-2}} = 0$$

(quadratic formula)

$$\Longleftrightarrow \quad x = 0 \quad \vee \quad x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(-1)}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

Summary of the work so far (the following implication is true)

$$\underbrace{F_n = r^n \wedge F_n = F_{n-1} + F_{n-2}} \Rightarrow r \in \left\{0, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right\}$$

it may be the case that these conditions never happen at the same time, in which case, the implication is vacuously true (something we would like to avoid)

the possibility of $r=0$ is rather uninteresting it means the sequence is constantly zeros (i.e. $F_n = 0 \ \forall n$) so we throw this boring solution away.

We now prove the converse ($\Longleftarrow$) by checking that

$$F_n = \left(\frac{1+\sqrt{5}}{2}\right)^n \quad \& \quad \tilde{F}_n = \left(\frac{1-\sqrt{5}}{2}\right)^n \quad \text{satisfy}$$

the recurrence relation $F_n = F_{n-1} + F_{n-2}$

<u>proof</u>:

$$F_{n-1} + F_{n-2} = \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} + \left(\frac{1+\sqrt{5}}{2}\right)^{n-2} = \left(\frac{1+\sqrt{5}}{2}\right)^{n-2}\left(\frac{1+\sqrt{5}}{2} + 1\right)$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^{n-2}\left(\frac{1+\sqrt{5}}{2} + \frac{2}{2}\right)$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^{n-2}\left(\frac{3+\sqrt{5}}{2}\right)$$

Lemma:

$$\boxed{\begin{aligned}\left(\frac{1+\sqrt{5}}{2}\right)^2 &= \frac{1}{2^2}(1+2\sqrt{5}+5)\\ &= \frac{6+2\sqrt{5}}{2^2} = \frac{2}{2}\cdot\frac{3+\sqrt{5}}{2} = \frac{3+\sqrt{5}}{2}\end{aligned}}$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^{n-2}\left(\frac{1+\sqrt{5}}{2}\right)^2$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^n = F_n \qquad \square$$

<u>Exercise</u>: Prove $\left(\frac{1-\sqrt{5}}{2}\right)^n = \left(\frac{1-\sqrt{5}}{2}\right)^{n-1} + \left(\frac{1-\sqrt{5}}{2}\right)^{n-2}$

<u>Step 2</u>: Combine the particular solutions to form the general solution.

<u>Relevant Fact 1</u>: $A_n = A_{n-1} + A_{n-2}$ & $\tilde{A}_n := \alpha A_n$ (in $\mathbb{R}$)

Then $\tilde{A}_n = \tilde{A}_{n-1} + \tilde{A}_{n-2}$ (i.e Real ※ multiples of solutions to DKLHRRCC are also solutions to the same relation)

<u>proof (of above case/example)</u>:

$$\tilde{A}_n := \alpha A_n = \alpha(A_{n-1} + A_{n-2}) = \alpha A_{n-1} + \alpha A_{n-2} = \tilde{A}_{n-1} + \tilde{A}_{n-2}$$

$\square$

Relevant Fact 2: $A_n = A_{n-1} + A_{n-2}$ & $B_n = B_{n-1} + B_{n-2}$

Then $C_n := A_n + B_n \implies C_n = C_{n-1} + C_{n-2}$

(i.e. the sum of two sequences that satisfy a DKLHRRCC also satisfies the same recurrence relation)

proof (of above case/example):

$$C_n := A_n + B_n = A_{n-1} + A_{n-2} + B_n = A_{n-1} + A_{n-2} + B_{n-1} + B_{n-2}$$
$$= (A_{n-1} + B_{n-1}) + (A_{n-2} + B_{n-2}) =: C_{n-1} + C_{n-2} \quad \square$$

$\underbrace{A_n = \left(\frac{1+\sqrt{5}}{2}\right)^n \text{ is a solution}}$ | $\underbrace{B_n = \left(\frac{1-\sqrt{5}}{2}\right)^n \text{ is a solution}}$

$\Downarrow$ Fact 1 | $\Downarrow$ Fact 1

$\alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n$ is a solution (for any $\alpha_1 \in \mathbb{R}$) | $\alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$ is a solution (for any $\alpha_2 \in \mathbb{R}$)

$\Downarrow$ Fact 2

$$\alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n \text{ is a solution}$$

Fact: (the proof of this claim in general is outlined below c.f. "Theorem 1")

Since $\left(\frac{1+\sqrt{5}}{2}\right) \neq \left(\frac{1-\sqrt{5}}{2}\right)$, every solution to $F_n = F_{n-1} + F_{n-2}$ is of the form $\alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$ for some $\alpha_1, \alpha_2 \in \mathbb{R}$.

Fibonacci Numbers

Solve the following initial value problem

$$\left(F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2}\right)$$

We know $F_n = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)^n$

So it suffices to find the values of α_1 & α_2 that give the correct initial conditions

$$\begin{cases} 0 =: F_0 = \alpha_1 + \alpha_2 & (\Leftrightarrow \alpha_1 = -\alpha_2) \\ 1 =: F_1 = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right) \end{cases}$$

Combining gives

$$\begin{aligned} 1 &= -\alpha_2\left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right) \\ &= \alpha_2\left(\frac{1-\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2}\right) \\ &= -\sqrt{5}\,\alpha_2 \end{aligned}$$

$$\Leftrightarrow \alpha_2 = \frac{-1}{\sqrt{5}} = -\alpha_1 \quad \left(\text{so } \alpha_1 = \frac{1}{\sqrt{5}}\right)$$

Note: These initial conditions yield the sequence

$0, 1, 1, 2, 3, 5, 8, \ldots$ (always whole numbers!!)

Tying up a loose end

a while back we claimed that the n^{th} Fibonacci # is given by $F_n = \dfrac{\eta^n + (1-\eta)^n}{\sqrt{5}}$ where $\eta = \dfrac{1+\sqrt{5}}{2}$. We finally prove this claim

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

$$\overset{*}{=} \frac{1}{\sqrt{5}}\left(\eta^n - (1-\eta)^n\right)$$

Lemma:

$$\boxed{\begin{aligned} 1-\eta &= 1 - \frac{1+\sqrt{5}}{2} \\ &= \frac{2}{2} - \frac{1+\sqrt{5}}{2} \\ &= \frac{1-\sqrt{5}}{2} \end{aligned}}$$

$\square$

Theorem 1: Degree 2 LHRRw/CC (distinct root case)

Suppose $r^2 - c_1 r - c_2$ has two distinct roots $r_1 \neq r_2$

Then $\{x_n\}_{n=0}^{\infty}$ is a solution to $a_n = c_1 a_{n-1} + c_2 a_{n-2}$

$\iff$

$$x_n = \alpha_1 r_1^n + \alpha_2 r_2^n \quad \text{for some } \alpha_1, \alpha_2 \in \mathbb{R}$$

Real number coefficients whose values determine the initial conditions

Unequal zeros of the "characteristic equation" $r^2 - c_1 r - c_2$

Idea behind the proof

$$\boxed{r_1 \neq r_2 \text{ roots of } r^2 - c_1 r - c_2} \xrightarrow[\text{1 \& 2 or direct proof}]{\text{can use ``relevant facts''}} \boxed{x_n = \alpha_1 r_1^n + \alpha_2 r_2^n \text{ is a solution to } a_n = c_1 a_{n-1} + c_2 a_{n-2}}$$

$\{x_n\}_{n=0}^{\infty}$ is a solution to
$a_n = c_1 a_{n-1} + c_2 a_{n-2}$

$\Downarrow$

The initial value problem $(x_0, x_1, a_n = c_1 a_{n-1} + c_2 a_{n-2})$ has $\{x_n\}_{n=0}^{\infty}$ as a solution

One can then check that

The initial value problem also happens to have

$$a_n = \left(\frac{x_1 - x_0 r_2}{r_1 - r_2}\right) r_1^n + \left(\frac{x_0 r_1 - x_1}{r_1 - r_2}\right) r_2^n$$

$$\bigcirc \wedge \square \Longrightarrow x_n = \underbrace{\left(\frac{x_1 - x_0 r_2}{r_1 - r_2}\right)}_{\alpha_1} r_1^n + \underbrace{\left(\frac{x_0 r_1 - x_1}{r_1 - r_2}\right)}_{\alpha_2} r_2^n$$

relies on a fact we won't prove. You will have the tools to prove suitable initial value problems have unique solutions when we learn induction

$\square$

Solving Deg 2 LHRRw/CC in practice

Step 1: Find the characteristic equation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} \longmapsto r^2 - c_1 r - c_2$$

Step 2: Find the zeros of the characteristic equation

$$r^2 - c_1 r - c_2 \longmapsto r_1, r_2 = \frac{c_1 \pm \sqrt{c_1^2 + 4c_2}}{2}$$

(if $c_1^2 + 4c_2 = 0$ then the two roots (zeros) are equal so Thm 1 does not help. We will learn how to deal with this soon enough)

Step 3: Thm 1 $\Rightarrow a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$

If no initial conditions were given we are done

If initial conditions are given we move on to the final step

Step 4: Use the initial conditions to solve for α_1, α_2

$$\begin{cases} a_0 = \alpha_1 + \alpha_2 \\ a_1 = \alpha_1 r_1 + \alpha_2 r_2 \end{cases}$$

two equations in two unknowns

$$\Longleftrightarrow \begin{cases} \alpha_1 = \dfrac{a_1 - a_0 r_2}{r_1 - r_2} \\[2ex] \alpha_2 = \dfrac{a_0 r_1 - a_1}{r_1 - r_2} \end{cases}$$

Solution