Motivating Questions

Q: How many surjective functions are there from a set of size \( K \) to one of size \( N \)?

Q: (The Hatcheck Problem)

\( N \) people check their hats @ a fancy restaurant. However, the employee checking the hats is quitting soon & therefore gives everyone's hat back at random. What is the probability that nobody is given their own hat back?

I can't stand hats anymore!

Hey, this isn't my hat...

Not my problem anymore
Inclusion - Exclusion

Finding the cardinality of a union of sets

\[ |A_1 \cup A_2 | = \square + \square + \square + \square - \square \]

\[ |A_1| + |A_2| - |A_1 \cap A_2| \]

Includes Everything (but includes the intersection twice) \hspace{1cm} Excludes the Second copy of the intersection

\[ |A_1 \cup A_2 \cup A_3| = \square + \square + \square \]

\[ |A_1| + |A_2| + |A_3| = \square + 2\square + 3\square \]

double intersections are each counted twice \hspace{1cm} triple \( \cap \)s get counted three times

\[ |A_1 \cap A_2| + |A_2 \cap A_3| + |A_1 \cap A_3| = \square + 3\square \]

\[ |A_1| + |A_2| + |A_3| = \square + 2\square + 3\square \]

\[ |A_1 \cap A_2| + |A_2 \cap A_3| + |A_1 \cap A_3| = \square + 3\square \]

\[ \sum_{x \in \mathbb{B}} |\cap_j A_k| - \sum_{x \in \mathbb{B}} |\cap_j A_k| = \square + \square \]

\[ \text{But now we aren't counting the triple intersection} \]
\[ |A_1 \cup A_2 \cup A_3| = \sum_{x \in \mathbb{Z}} |\bigcap_{j \in x} A_j| - \sum_{x \in \mathbb{Z}} |\bigcap_{j \in x} A_j| + \sum_{x \in \mathbb{Z}} |\bigcap_{j \in x} A_j| \]

includes everything double counts double intersections & triple fixes the issue w/ double counts but takes away intersections from the count 

\[ (\begin{array}{c} 4 \\ 1 \end{array}) = 4, (\begin{array}{c} 4 \\ 2 \end{array}) = 6, (\begin{array}{c} 4 \\ 3 \end{array}) = 4, (\begin{array}{c} 4 \\ 1 \end{array}) = 1 \]

\[ \sum_{x \in \mathbb{Z}} |\bigcap_{j \in x} A_j| = \text{[Diagram]} + 2 \text{[Diagram]} + 3 \text{[Diagram]} + 4 \text{[Diagram]} \]

\[ \sum_{x \in \mathbb{Z}} |\bigcap_{j \in x} A_j| = \text{[Diagram]} + 3 \text{[Diagram]} + 6 \text{[Diagram]} \]

Double Intersections

\[ |A_1 \cap A_2| \]

\[ |A_1 \cap A_3| \]

\[ |A_1 \cap A_4| \]

\[ |A_2 \cap A_3| \]

\[ |A_2 \cap A_4| \]

\[ |A_3 \cap A_4| \]
**Triple Intersections**

\[
|A_1 \cap A_2 \cap A_3| = 2
\]

\[
|A_1 \cap A_2 \cap A_4| = 1
\]

\[
|A_1 \cap A_3 \cap A_4| = 1
\]

\[
|A_2 \cap A_3 \cap A_4| = 1
\]

\[
\sum_{x \in Y, j \in X} |\cap A_j| = +4
\]

\[
\sum_{x \in Y} |\cap A_j| = |A_1 \cap A_2 \cap A_3 \cap A_4| = 1
\]
\[
\sum_{x \in \mathbb{N}} |\bigcap_{j \in x} A_j| = \text{[Blue] } + 2 \text{[Blue] } + 3 \text{[Blue] } + 4 \text{[Blue] } \\
\text{[Blue]} + 1 = 1
\]

\[
\sum_{x \in \mathbb{N}} |\bigcap_{j \in x} A_j| = \text{[Purple] } + 3 \text{[Blue] } + 6 \text{[Blue] } \\
\text{[Purple]} + 1 = 2
\]

\[
\sum_{x \in \mathbb{N}} |\bigcap_{j \in x} A_j| = \text{[Blue] } + 4 \text{[Blue] } \\
\text{[Blue]} + 1 = 3
\]

\[
\sum_{x \in \mathbb{N}} |\bigcap_{j \in x} A_j| = |A_1 \cap A_2 \cap A_3 \cap A_4| = \text{[Red] } \\
\text{[Red]} + (2-1)\text{[Blue] } + (3-3+1)\text{[Blue] } + (4-6+4-1)\text{[Blue] } \\
\overbrace{8-7}^{8-7} \\
= \text{[Blue] } + \text{[Purple] } + \text{[Blue] } + \text{[Blue] } = |A_1 \cup A_2 \cup A_3 \cup A_4|
\]

**Theorem:** Inclusion-Exclusion principle

\[
|\bigcup_{k=1}^{n} A_k| = \sum_{i=1}^{n} (-1)^{i-1} \sum_{\substack{x \subseteq \mathbb{N} \ j \in x \ \text{[Red]} \ 1x1=i}} |\bigcap_{j=1}^{i} A_{i_j}| \\
\]

**Notation:** \( I(n, k) := \{ \bigcap_{j=1}^{k} A_{i_j} | \{i_1, ..., i_k \} \subseteq \mathbb{N} \} \)
\[ I_k := \bigcup_{X \in I(n,k)} X - \bigcup_{X \in I(n,k+1)} X \]  
\( \{ \text{Elements in a } K\text{-fold intersection which are not in a } K+1\text{-fold} \} \)

**Note:**  
\[ \sum_{k=1}^{n} \binom{n}{k} = \binom{n}{k} \]

**Proof:**  
\[ \sum_{X \in I(n,1)} |X| = \sum_{k=1}^{n} k \cdot |I_k| \]  
(elements in \( K\)-fold \( \cap \) get counted \( K \)-times if we count every element of every set)

\[ \sum_{X \in I(n,2)} |X| = |I_2| + \sum_{k=1}^{n-2} \binom{2+k}{k} \cdot |I_{k+2}| \]

(all elements in double intersections get counted)

\[ \vdots \]

\[ \sum_{X \in I(n,m)} |X| = |I_m| + \sum_{k=1}^{n-m} \binom{m+k}{k} \cdot |I_{m+k}| \]

\[ \vdots \]

\[ \sum_{X \in I(n,n)} |X| = |I_n| = \bigcap_{k=1}^{n} A_k \]

**Goal:**  
\[ \sum_{m=1}^{n} (-1)^{m-1} \sum_{X \in I(n,m)} |X| = |\bigcup_{k=1}^{n} A_k| \]
\[ \sum_{m=1}^{n} (-1)^{m-1} \sum_{x \in \mathcal{I}(n, k)} |X_x| = \sum_{m=1}^{n} (-1)^{m-1} \left( |X_m| + \sum_{k=1}^{m-\kappa} \binom{m}{k} |X_{m-k}| \right) \]

\[ = |X_1| + \binom{2}{1} |X_2| - |X_2| + \binom{3}{1} |X_3| - \binom{3}{1} |X_3| + |X_3| + \binom{4}{1} |X_4| - \binom{4}{1} |X_4| + \binom{4}{1} |X_4| - |X_4| \]

\[ + \sum_{\kappa=1}^{\binom{n}{\kappa}} (-1)^{\kappa+1} \binom{m}{k} |X_m| = |X_m| \]

\[ \sum_{m=1}^{n} |X_m| = \bigcup_{k=1}^{n} \mathcal{A}_k \]

Recall: \[ \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} = 0 \]
Counting Surjective Functions

\[ S(3,2) := \{ f : 3 \to 2 \mid f \text{ surjective} \} \]

(in general, the set of surjective functions from a set of size \( m \) to one of size \( n \) will be denoted \( S(m,n) \))

\[ \Rightarrow \text{Note: } |S(m,n)| = 0 \text{ if } m < n \]

Q: What is the cardinality of \( S(3,2) \)?

**Option 1:** list all possibilities

\[ \Rightarrow \text{won't help us find } |S(m,n)| \]

**Option 2:** use inclusion-exclusion

\[ P_1 := \{ f : 3 \to 2 \mid 1 \in \text{Range}(f) \} \]

\[ P_2 := \{ f : 3 \to 2 \mid 2 \in \text{Range}(f) \} \]

\[ F(3,2) := \{ f : 3 \to 2 \} \text{ all functions surjective or not} \]

\[ S(3,2) = F(3,2) - (P_1 \cup P_2) \]

\[ |S(3,2)| = |F(3,2)| - |P_1 \cup P_2| = 2^3 - |P_1| - |P_2| + |P_1 \cap P_2| \]
\[ = 8 - 1 - 1 = 6 \]

In General

\[ \forall n,m,i \text{ s.t. } m > n \geq i \text{ define the following sets} \]

\[ F(m,n) := \{ f : \mathbb{M} \rightarrow \mathbb{M} \} \quad \text{All functions} \]

\[ S(m,n) := \{ f : \mathbb{M} \rightarrow \mathbb{M} \mid f \text{ surjective} \} \quad \text{surjective functions} \]

\[ P_i := \{ f : \mathbb{M} \rightarrow \mathbb{M} \mid i \in \text{Range}(f) \} \quad \text{functions missing the } i^{th} \text{ element of the range} \]

Then, \[ S(m,n) = F(m,n) - \bigcup_{i=1}^{n} P_i \]

\[ \Rightarrow |S(m,n)| = |F(m,n)| - \bigcup_{i=1}^{n} |P_i| \]

use inclusion-exclusion
\[ \left| \bigcup_{i=1}^{n} P_i \right| = \sum_{\emptyset \neq X \subseteq \{1, \ldots, n\}} (-1)^{|X|-1} \left| \bigcap_{i \in X} P_i \right| \]

Because this is essentially just \( F(m, n-1\times1) \)

Since \( \bigcap_{i=1}^{n} P_i \)
\[ \Rightarrow \text{Range}(F) = \emptyset \]

\[ \therefore \text{there are } \binom{n}{1\times1} \text{ Subsets of size } 1\times1 \text{ in } \{1, \ldots, n\} \]

So the term \((-1)^{1\times1+1} (n-1\times1)^m\) appears \(\binom{n}{1\times1}\) times in the sum above.

\[ \therefore S(m,n) = \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} (n-k)^m \]

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**The Hat-check Problem**

Example: \( n = 3 \)

\( \exists \) bijection between

<table>
<thead>
<tr>
<th>Permutations</th>
<th>Hat Returns</th>
</tr>
</thead>
<tbody>
<tr>
<td>123</td>
<td>( \circ )</td>
</tr>
<tr>
<td>132</td>
<td>( \circ )</td>
</tr>
<tr>
<td>213</td>
<td>( \circ )</td>
</tr>
<tr>
<td>231</td>
<td>( \circ )</td>
</tr>
<tr>
<td>312</td>
<td>( \circ )</td>
</tr>
</tbody>
</table>
**Def:** A permutation leaving no objects in their original position is called a derangement.

Using Inclusion-Exclusion

\[ S(n) := \{ \text{permutations of } \{1, 2, \ldots, n\} \} \]

\[ D(n) := \{ \text{derangements of } \{1, 2, \ldots, n\} \} \]

\[ F_i := \{ \text{permutations of } \{1, 2, \ldots, n\} \mid i \text{ is in the } i^{th} \text{ position} \} \]

\[ D(n) = S(n) - \bigcup_{i=1}^{n} F_i \]
\[ |D(n)| = |S(n)| - \left| \bigcup_{i=1}^{n} F_i \right| \]

\[ = |S(n)| - \sum_{k=1}^{n} (-1)^{k+1} \left| F_{i_1} \cap \ldots \cap F_{i_k} \right| \]

**Lemma:** \(k\)-fold intersections of \(F_i\)

\[ \left| \bigcap_{j=1}^{k} F_{i_j} \right| = (n-k)! \]

for any \(i \in F_{i_1} \ldots F_{i_k}\)

\(F_{i_j}\) from \(F_1, \ldots, F_n\)

**proof:**

A permutation in \(\bigcap_{j=1}^{k} F_{i_j}\) has \(i_1\) in position \(i_1\), \(i_2\) in position \(i_2\), \ldots, \(i_k\) in position \(i_k\).

\[ 1 \ 2 \ \bar{i_1} \ i_1 \ i_{i+1} \ i_{i+2} \ldots \ i_{i_2} \ i_{i_2-1} \ i_{i_2} \ i_k \ i_{k+1} \ldots \ i_{n-1} \ i_n \]

\(K\) positions are determined, the rest can be anything.
\[ |S(n)| - \sum_{k=1}^{n} (-1)^{k+1} \left| F_{\tilde{z}_1} \cap \ldots \cap F_{\tilde{z}_k} \right| = |S(n)| - \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} (n-k)! \]

**Lemma**

\[ = n! - \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} (n-k)! \]

\[ = \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} (n-k)! = |D(n)| \]

**Note:** \( \binom{n}{k} (n-k)! = \frac{n!}{(n-k)!k!(n-k)!} = \frac{n!}{k!} \)

So \( |D(n)| = n! \sum_{k=0}^{n} \frac{(-1)^{k}}{k!} \)
probability
Nobody gets their own hat back

\[ \frac{|D(n)|}{|S(n)|} = \frac{n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \]

Taylor Series For \( e^x \) about \( x = 0 \)

\[ e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \]

\[ \frac{|D(n)|}{|S(n)|} = \text{finite approximation of } e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \]

\[ \lim_{n \to \infty} \frac{|D(n)|}{|S(n)|} = \frac{1}{e} \approx 0.368 \]

as \( n \) grows
\( \uparrow \)
probability
Nobody gets their hat back
\( \uparrow \)