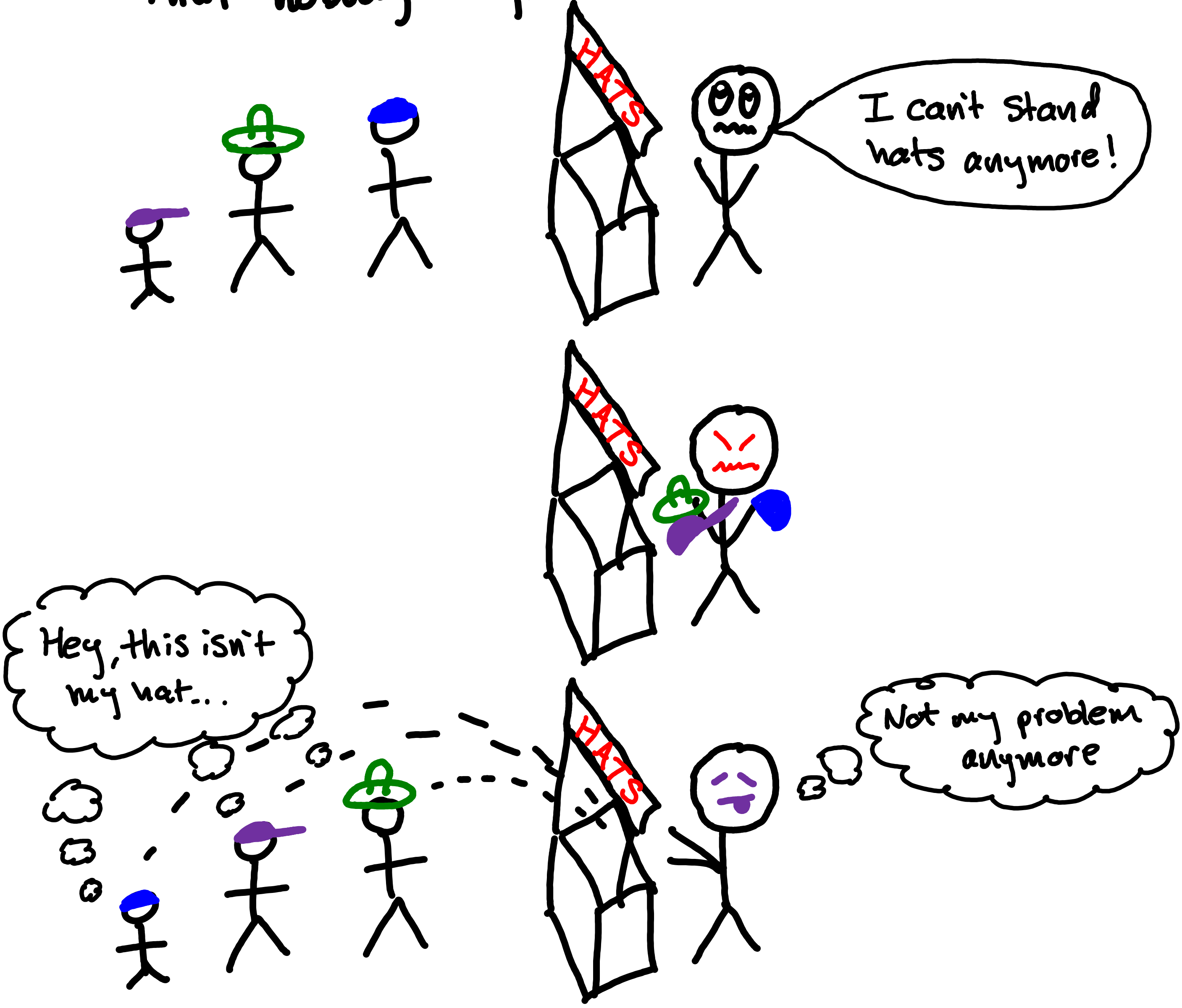
Motivating Questions

Q: How many surjective functions are there from a set of size k to one of size n?

Q: (The Hatcheck Problem)

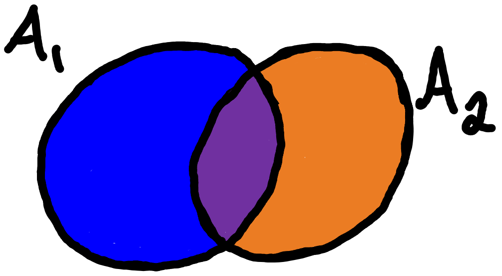
n people check their hats @ a fancy restaurant

However, the employee checking the hats is quitting soon & therefore gives everyone's hat back at random. What is the probability that nobody is given their own hat back?



Inclusion − Exclusion

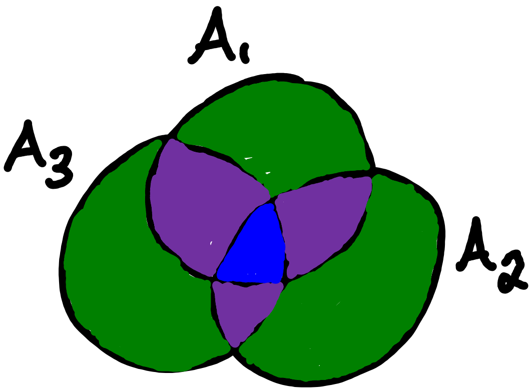
Finding the cardinality of a union of sets



$$|A_1 \cup A_2| = \underbrace{{\color{blue}\blacksquare} + {\color{purple}\blacksquare}}_{|A_1|} + \underbrace{{\color{orange}\blacksquare} + {\color{purple}\blacksquare}}_{|A_2|} - \underbrace{{\color{purple}\blacksquare}}_{|A_1 \cap A_2|}$$

$$\underbrace{{\color{green}|A_1| + |A_2|}}_{\text{Includes Everything (but includes the intersection twice)}} - \underbrace{{\color{red}|A_1 \cap A_2|}}_{\text{Excludes the second copy of the intersection}}$$

$$|A_1 \cup A_2 \cup A_3| = {\color{green}\blacksquare} + {\color{purple}\blacksquare} + {\color{blue}\blacksquare}$$



$$|A_1| + |A_2| + |A_3| = {\color{green}\blacksquare} + \underbrace{2{\color{purple}\blacksquare}}_{\text{double intersections are each counted twice}} + \underbrace{3{\color{blue}\blacksquare}}_{\text{triple } \cap\text{s get counted three times}}$$

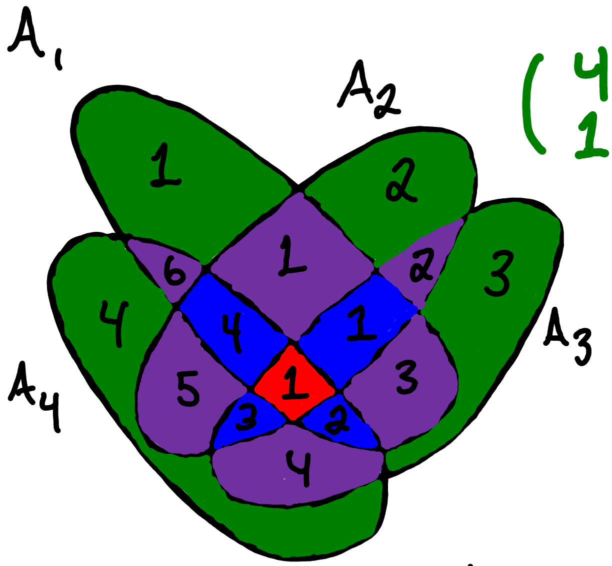
$$\underbrace{|A_1 \cap A_2| + |A_2 \cap A_3| + |A_1 \cap A_3|}_{\text{All Double Intersections}} = {\color{purple}\blacksquare} + 3{\color{blue}\blacksquare}$$

$$
\begin{array}{rl}
 & |A_1| + |A_2| + |A_3| = {\color{green}\blacksquare} + 2{\color{purple}\blacksquare} + 3{\color{blue}\blacksquare} \\
{\color{red}-} & |A_1 \cap A_2| + |A_2 \cap A_3| + |A_1 \cap A_3| = {\color{purple}\blacksquare} + 3{\color{blue}\blacksquare} \\
\hline
\end{array}
$$

$${\color{green}\sum_{\substack{X \subseteq 3 \\ |X|=1}} \left|\bigcap_{j \in X} A_j\right|} - {\color{red}\sum_{\substack{X \subseteq 3 \\ |X|=2}} \left|\bigcap_{j \in X} A_j\right|} = {\color{green}\blacksquare} + {\color{purple}\blacksquare}$$

But now we aren't counting the triple intersection

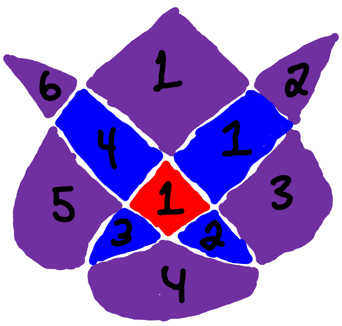
$$|A_1 \cup A_2 \cup A_3| = \underbrace{\sum_{\substack{x \subseteq 3 \\ |x|=1}} \left|\bigcap_{j\in x} A_j\right|}_{\text{includes everything double counts double intersections \& triple counts triple intersections}} - \underbrace{\sum_{\substack{x \subseteq 3 \\ |x|=2}} \left|\bigcap_{j\in x} A_j\right|}_{\text{fixes the issue w/ double counts but takes away triple intersections from the count}} + \underbrace{\sum_{\substack{x \subseteq 3 \\ |x|=3}} \left|\bigcap_{j\in x} A_j\right|}_{\text{add back the triple intersection}}$$



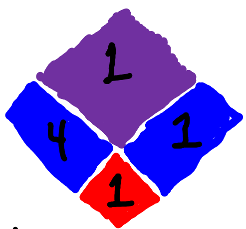
$$\binom{4}{1} = 4,\ \binom{4}{2} = 6,\ \binom{4}{3} = 4,\ \binom{4}{4} = 1$$

$$\sum_{\substack{x \subseteq 4 \\ |x|=1}} \left|\bigcap_{j=1}^{|x|} A_j\right| = \blacksquare + 2\blacksquare + 3\blacksquare + 4\blacksquare$$

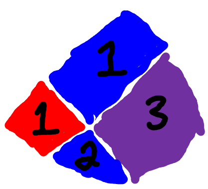
$$\sum_{\substack{x \subseteq 4 \\ |x|=2}} \left|\bigcap_{j=1}^{|x|} A_j\right| = \blacksquare + 3\blacksquare + 6\blacksquare$$



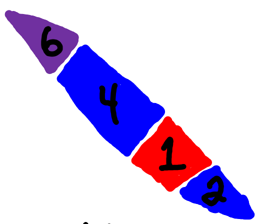
Double Intersections



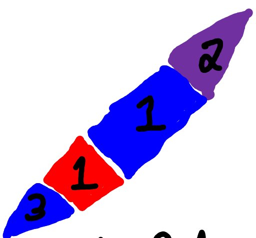
$|A_1 \cap A_2|$



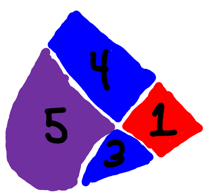
$|A_1 \cap A_3|$



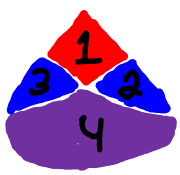
$|A_1 \cap A_4|$



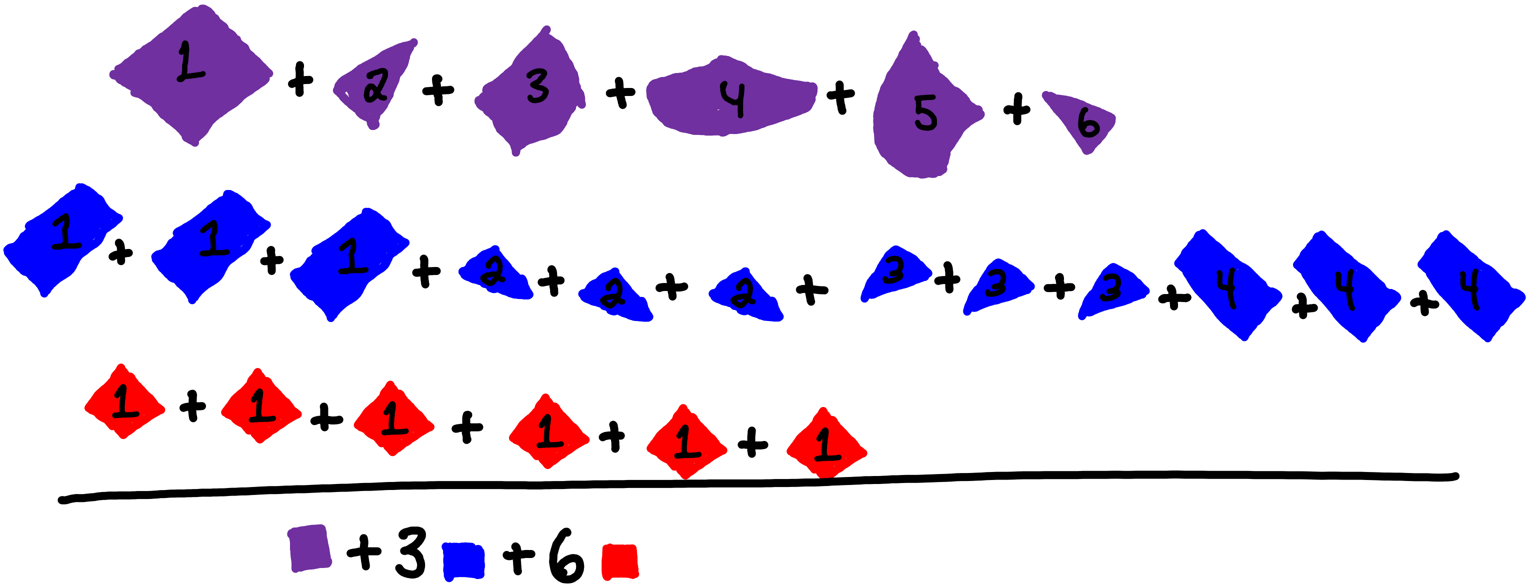
$|A_2 \cap A_3|$



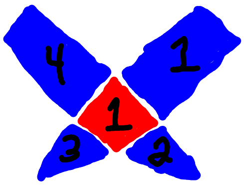
$|A_2 \cap A_4|$



$|A_3 \cap A_4|$



Triple Intersections

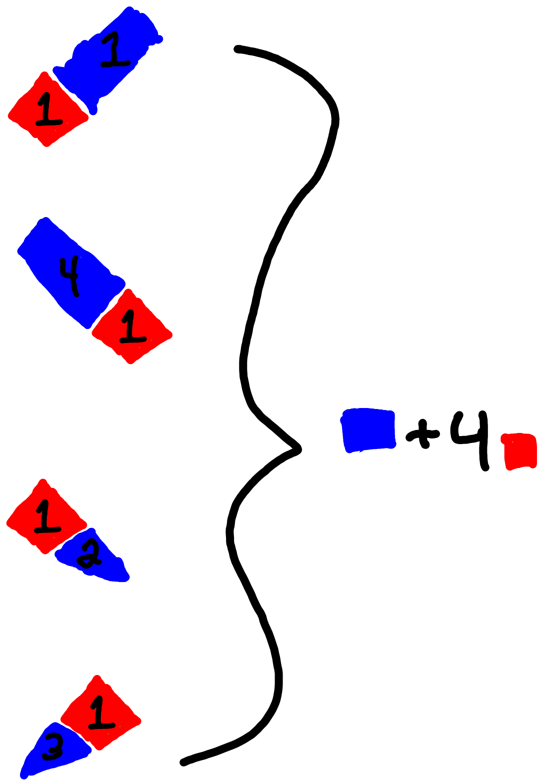


$|A_1 \cap A_2 \cap A_3| =$

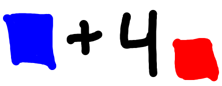
$|A_1 \cap A_2 \cap A_4| =$

$|A_1 \cap A_3 \cap A_4| =$

$|A_2 \cap A_3 \cap A_4| =$



$$\sum_{\substack{X \subseteq 4 \\ |X|=3}} \left| \bigcap_{j \in X} A_j \right| =$$



$$\sum_{\substack{X \subseteq 4 \\ |X|=4}} \left| \bigcap_{j \in X} A_j \right| = |A_1 \cap A_2 \cap A_3 \cap A_4| =$$

$$\sum_{\substack{X\subseteq \mathbb{4}\\ |X|=1}} \left|\bigcap_{j\in X} A_j\right| = {\color{green}\blacksquare} + 2{\color{purple}\blacksquare} + 3{\color{blue}\blacksquare} + 4{\color{red}\blacksquare}$$

$-$

$$\sum_{\substack{X\subseteq \mathbb{4}\\ |X|=2}} \left|\bigcap_{j\in X} A_j\right| = {\color{purple}\blacksquare} + 3{\color{blue}\blacksquare} + 6{\color{red}\blacksquare}$$

$+$

$$\sum_{\substack{X\subseteq \mathbb{4}\\ |X|=3}} \left|\bigcap_{j\in X} A_j\right| = {\color{blue}\blacksquare} + 4{\color{red}\blacksquare}$$

$-$

$$\sum_{\substack{X\subseteq \mathbb{4}\\ |X|=4}} \left|\bigcap_{j\in X} A_j\right| = |A_1\cap A_2\cap A_3\cap A_4| = {\color{red}\blacksquare}$$

$$ {\color{green}\blacksquare} + (2-1){\color{purple}\blacksquare} + (3-3+1){\color{blue}\blacksquare} + (\underbrace{4-6+4-1}_{8-7}){\color{red}\blacksquare}$$

$$= {\color{green}\blacksquare} + {\color{purple}\blacksquare} + {\color{blue}\blacksquare} + {\color{red}\blacksquare} = |A_1\cup A_2\cup A_3\cup A_4|$$

Theorem: Inclusion-Exclusion principle

$$\left|\bigcup_{k=1}^{n} A_k\right| = \sum_{i=1}^{n} (-1)^{i-1} \sum_{\substack{X\subseteq \mathbb{n}\\ |X|=i}} \left|\bigcap_{j\in X} A_j\right|$$

Notation: $I(n,k) := \left\{ \bigcap_{j=1}^{k} A_{i_j} \;\middle|\; \{i_1,\dots,i_k\}\subseteq \mathbb{n} \right\}$

(when n is understood)

$$\mathcal{I}_k = \bigcup_{X \in I(n,k)} X - \bigcup_{X \in I(n,k+1)} X$$

} Elements in a k-fold intersection which are not in a $k+1$-fold $\cap$

<u>Note:</u> $$|\mathcal{I}_k| = \binom{n}{k}$$

<u>proof:</u> $$\sum_{X \in I(n,1)} |X| = \sum_{k=1}^{n} k\,|\mathcal{I}_k|$$

(elements in k-fold $\cap$ get counted k times if we count every element of every set)

$$\underbrace{\sum_{X \in I(n,2)} |X|}_{\text{all elements in double intersections get counted}} = |\mathcal{I}_2| + \sum_{k=1}^{n-2} \binom{2+k}{k} |\mathcal{I}_{k+2}|$$

times a $2+k$ fold $\cap$ appears in a 2 fold $\cap$

$\vdots$

times an $m+k$-fold $\cap$ appears in a m-fold $\cap$

$$\sum_{X \in I(n,m)} |X| = |\mathcal{I}_m| + \sum_{k=1}^{n-m} \binom{m+k}{k} |\mathcal{I}_{m+k}|$$

$\vdots$

$$\sum_{X \in I(n,n)} |X| = |\mathcal{I}_n| = \left|\bigcap_{k=1}^{n} A_n\right|$$

<u>Goal:</u> $$\sum_{m=1}^{n} (-1)^{m-1} \sum_{X \in I(n,m)} |X| = \left|\bigcup_{k=1}^{n} A_k\right|$$

$$\sum_{m=1}^{n} (-1)^{m-1} \sum_{X \in I(n,k)} |X| = \sum_{m=1}^{n} (-1)^{m-1} \left(|\mathcal{I}_m| + \sum_{k=1}^{n-m} \binom{m+k}{k} |\mathcal{I}_{m+k}| \right)$$

$$= |\mathcal{I}_1|$$

$$+ \binom{2}{1}|\mathcal{I}_2| - |\mathcal{I}_2|$$

$$+ \binom{3}{2}|\mathcal{I}_3| - \binom{3}{1}|\mathcal{I}_3| + |\mathcal{I}_3|$$

$$+ \binom{4}{3}|\mathcal{I}_4| - \binom{4}{2}|\mathcal{I}_4| + \binom{4}{1}|\mathcal{I}_4| - |\mathcal{I}_4|$$

$$\vdots$$

$$\boxed{\sum_{k=1}^{m} (-1)^{k+1} \binom{m}{k} = \binom{m}{0} = 1}$$

$$+ \sum_{k=1}^{m} (-1)^{k+1} \binom{m}{k} |\mathcal{I}_m| = |\mathcal{I}_m|$$

$$\vdots$$

$$\sum_{m=1}^{n} |\mathcal{I}_m| = \left| \bigcup_{k=1}^{n} A_k \right| \qquad \square$$

Recall: $$\sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} = 0$$

Counting Surjective Functions

$$S(3,2) := \{ f : \mathbf{3} \longrightarrow \mathbf{2} \mid f \text{ surjective} \}$$

(in general, the set of surjective functions from a set of size m to one of size n will be denoted $S(m,n)$

↳ Note: $|S(m,n)| = 0$ if $m < n$)

Q: What is the cardinality of $S(3,2)$?

Option 1: list all possibilities

↳ won't help us find $|S(m,n)|$

Option 2: Use inclusion - exclusion

$$P_1 := \{ f : \mathbf{3} \longrightarrow \mathbf{2} \mid 1 \notin \text{Range}(f) \}$$

$$P_2 := \{ f : \mathbf{3} \longrightarrow \mathbf{2} \mid 2 \notin \text{Range}(f) \}$$

$$F(3,2) := \{ f : \mathbf{3} \longrightarrow \mathbf{2} \}$$ all functions surjective or not

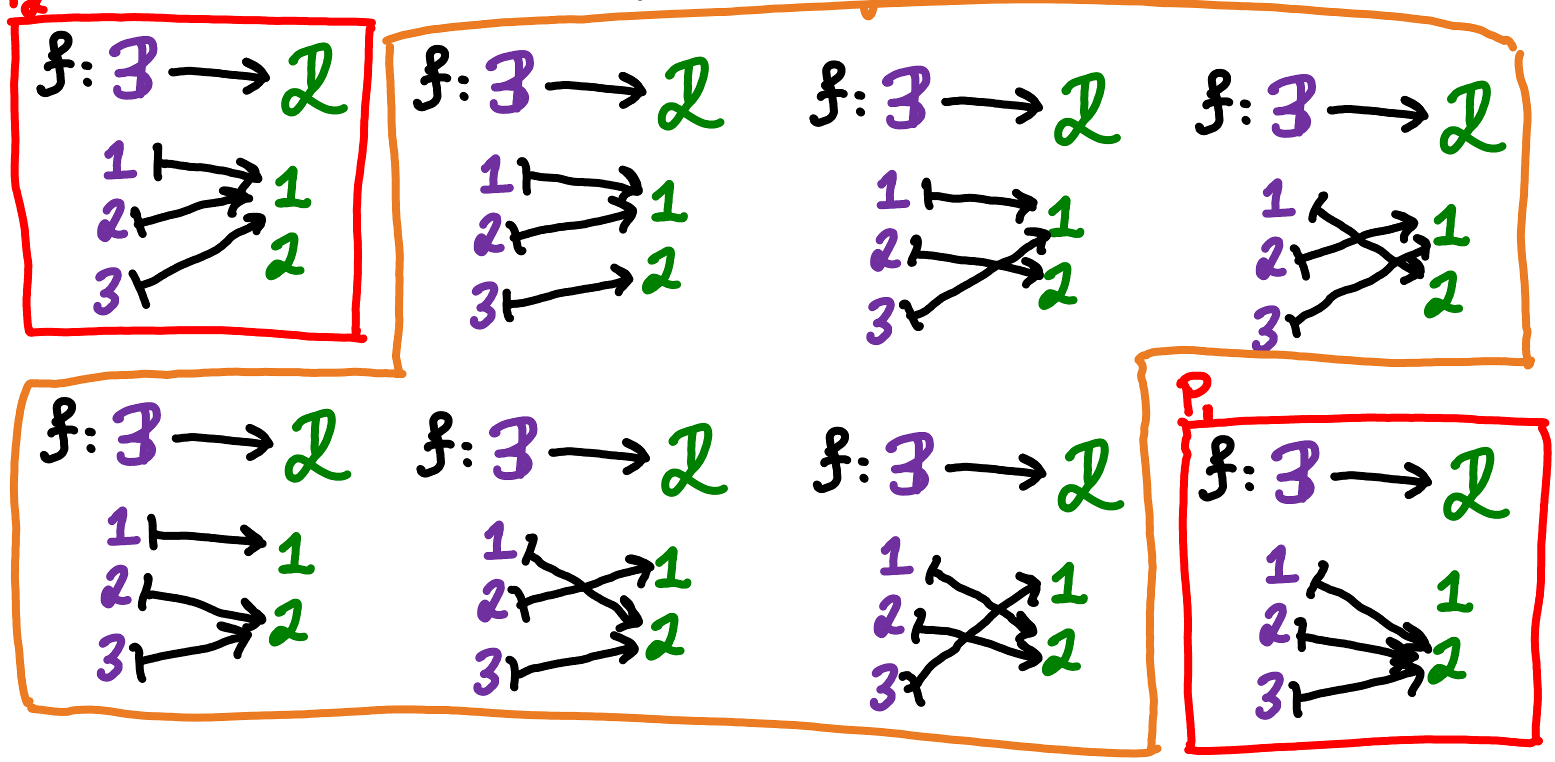
$$\underbrace{S(3,2)}_{\text{surjective functions}} = \underbrace{F(3,2)}_{\text{are functions}} - (P_1 \cup P_2)$$

that do not "miss" the value 1 or "miss" the value 2

$$|S(3,2)| = |F(3,2)| - |P_1 \cup P_2| = 2^3 - |P_1| - |P_2| + |\overset{= \emptyset}{P_1 \cap P_2}|$$

$= 8 - 1 - 1 = 6$

P_2 double check $F(3,2)$ Surjective



In General

$\forall n, m, i$ s.t. $m > n \geq i$ define the following sets

$F(m,n) := \{ f : m \to n \}$ All functions

$S(m,n) := \{ f : m \to n \mid f \text{ surjective} \}$ surjective functions

$P_i := \{ f : m \to n \mid i \notin \text{Range}(f) \}$ functions missing the i^{th} element of the range

Then, $S(m,n) = F(m,n) - \bigcup_{i=1}^{n} P_i$

$$\Rightarrow |S(m,n)| = |F(m,n)| - \underbrace{\left| \bigcup_{i=1}^{n} P_i \right|}_{\text{use inclusion-exclusion}}$$

$$\left|\bigcup_{i=1}^{n} P_i\right| = \sum_{\substack{\emptyset \neq X \subseteq \mathbb{n} \\ X \neq \mathbb{n}}} (-1)^{|X|+1} \left|\bigcap_{i \in X} P_i\right|$$

$$\left|\bigcap_{i \in X} P_i\right| = (n-|X|)^m$$

because this is essentially just $|F(m, n-|X|)|$

Since $\bigcap_{i=1}^{n} P_i$

$\Rightarrow \text{Range}(f) = \emptyset$

Note: there are $\binom{n}{|X|}$ subsets of size $|X|$ in $\mathbb{n}$

So the term $(-1)^{|X|+1}(n-|X|)^m$ appears $\binom{n}{|X|}$ times in the sum above.

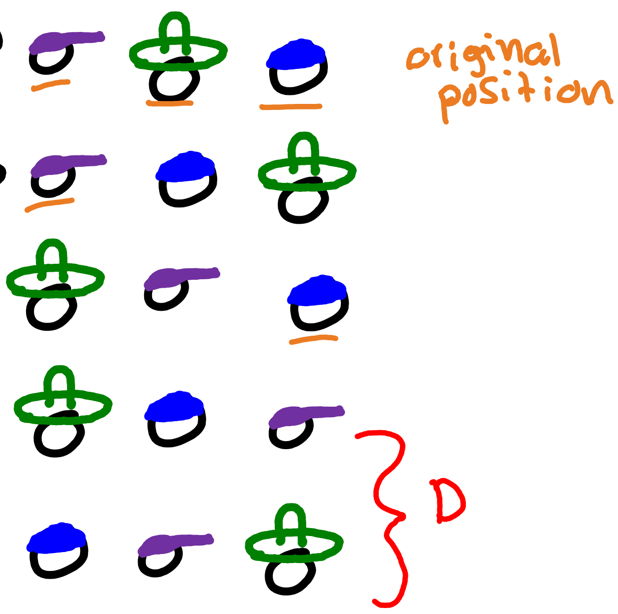
therefore,

$$\boxed{|S(m,n)| = \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} (n-k)^m}$$

The Hatcheck Problem

Example: $n=3$ $\quad\exists$ bijection between

Permutations		Hat Returns
123	$\longleftrightarrow$	
132	$\longleftrightarrow$	
213	$\longleftrightarrow$	
231	$\longleftrightarrow$	
312	$\longleftrightarrow$	



321 $\longleftrightarrow$ 

Def: A permutation leaving no objects in their original position is called a **derrangement**

Hatcheck problem $\longleftrightarrow$ Counting the # of derrangements

Using Inclusion-Exclusion

permutations $-$ (union of sets which are not derrangements)

$S(n) := \{\text{permutations of } [n]\}$

$D(n) := \{\text{Derrangements of } [n]\}$

$F_i := \{\text{permutations of } [n] \mid i \text{ is in the } i^{th} \text{ position}\}$

$$D(n) = S(n) - \bigcup_{i=1}^{n} F_i$$

$$|D(n)| = |S(n)| - \left|\bigcup_{i=1}^{n} F_i\right|$$

$$= |S(n)| - \sum_{k=1}^{n} (-1)^{k+1} |F_{i_1} \cap \dots \cap F_{i_k}|$$

Lemma: k-fold intersections of F_i

$$\left|\bigcap_{j=1}^{k} F_{i_j}\right| = (n-k)! \qquad \underbrace{\text{for any } \{F_{i_j}\}_{j=1}^{k}}_{\text{collection of } k \ F_i \text{ from } \{F_1,\dots,F_n\}}$$

proof:

A permutation in $\bigcap_{j=1}^{k} F_{i_j}$ has i_1 in position i_1, i_2 in position $i_2, \dots,$ i_k in position i_k

$$\underset{1}{\underline{\quad}}\ \underset{2}{\underline{\quad}}\ \cdots\ \underset{i_1}{\underline{i_1}}\ \underset{i_1+1}{\underline{\quad}}\ \cdots\ \underset{i_2-1}{\underline{\quad}}\ \underset{i_2}{\underline{i_2}}\ \cdots\ \underset{i_k}{\underline{i_k}}\ \cdots\ \underset{n-1}{\underline{\quad}}\ \underset{n}{\underline{\quad}}$$

k positions are determined, the rest can be anything

This is like a permutation on $n-k$ objects. $\square$

$$= |S(n)| - \sum_{k=1}^{n} (-1)^{k+1} |F_{i_1} \cap \ldots \cap F_{i_k}|$$

lemma

$$= |S(n)| - \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} (n-k)!$$

\# of k-fold intersections

$$= n! - \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} (n-k)!$$

$$= \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} (n-k)! = |D(n)|$$

<u>Note:</u> $\binom{n}{k}(n-k)! = \frac{n!}{\cancel{(n-k)!}\,k!}\cancel{(n-k)!} = \frac{n!}{k!}$

So $|D(n)| = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$

probability
Nobody gets their own hat back

$$\frac{\# \text{ Derrangements}}{\# \text{ permutations}}$$

nobody gets own hat

all possible ways to give hats back

$$\frac{|D(n)|}{|S(n)|} = \frac{\cancel{n}! \sum_{k=0}^{n} \frac{(-1)^k}{k!}}{\cancel{n}!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

Taylor Series For e^x about $x=0$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\frac{|D(n)|}{|S(n)|} = \text{finite approximation of } e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$$

$$\lim_{n \to \infty} \frac{|D(n)|}{|S(n)|} = \frac{1}{e} \approx 0.368$$

as # people grows

probability nobody gets their hat back