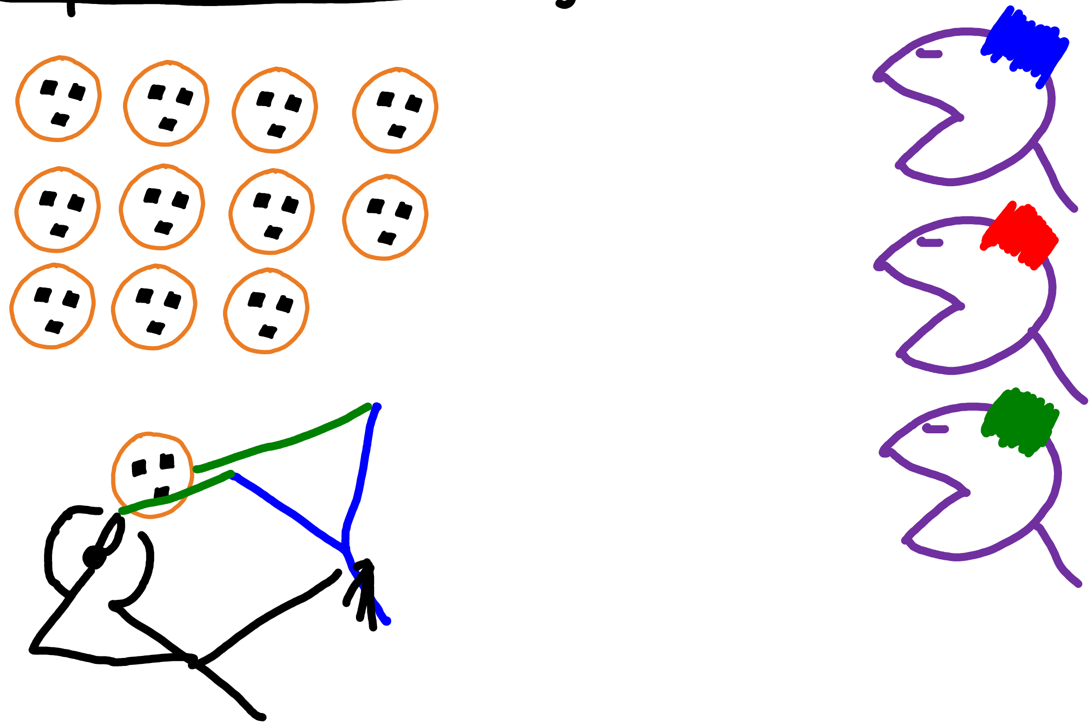
A problem we already know how to answer



How many ways can we distribute 12 identical cookies among 3 children?

Cookies = indistinguishable objects
Children = distinguishable boxes

$\Big\}$ Thus, we are counting combinations w/ repetition

$$\Rightarrow \binom{12+3-1}{3-1} = \binom{12+3-1}{12}$$

the answer is a binomial coefficient

Q: What does this question have to do with binomials?

A: Here is another solution to the problem

Child $\longleftrightarrow$ polynomial

\# cookies $\longleftrightarrow$ exponents

$$1^0+x^1+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10}+x^{11}+x^{12}$$

↳ we have a degree 12 polynomial in x for each child

the term x^k in the polynomial for child n represents the possibility that they are given exactly k cookies

Child 1 Child 2 Child 3

$$(1^0+x^1+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10}+x^{11}+x^{12})(1^0+x^1+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10}+x^{11}+x^{12})(1^0+x^1+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10}+x^{11}+x^{12})$$

then we multiply the polynomials & collect like terms. This is like combining all possible ways of giving 12 or fewer cookies to each child. We have exactly 12 cookies to hand out so we are only interested in finding the coefficient of x^{12} (after combining)

Note: this is the number of natural number solutions to the equation $A+B+C=12$

exponent of x in factor chosen from first child's polynomial

Second child's x exponent

third child's exponent

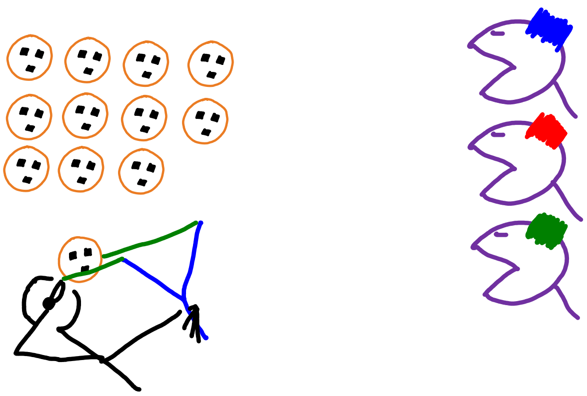
Exponent of the product $x^A x^B x^C$

which we know to be

$$\binom{12+3-1}{3-1}=\binom{12+3-1}{12}$$

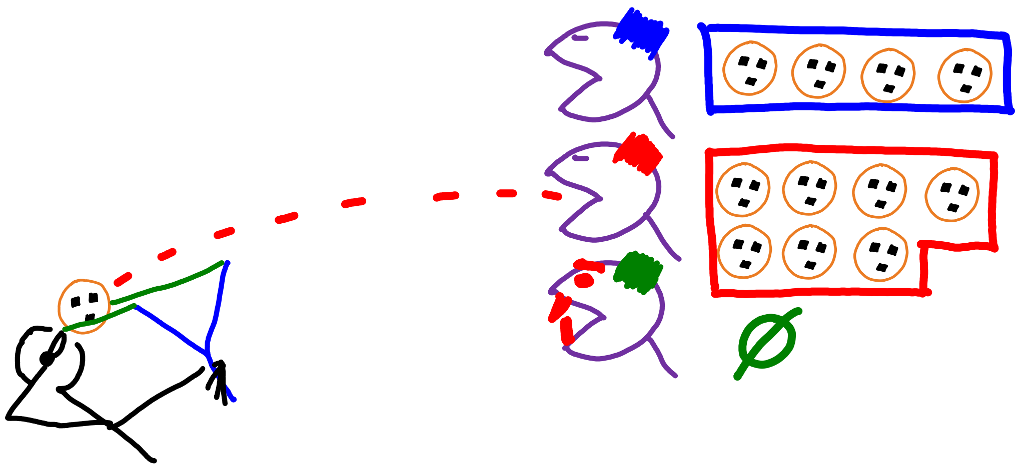
SO WHAT?

A New Type of Problem



Same basic question as before

Except this time we don't want to upset any children



($\star$) We impose the extra condition that all children receive at least 2, but no more than 5 cookies

$$(x^2+x^3+x^4+x^5)(x^2+x^3+x^4+x^5)(x^2+x^3+x^4+x^5)$$

$$=(x^2+x^3+x^4+x^5)^3$$ multiply out

$$=(x^2+x^3+x^4+x^5)(x^4+x^5+x^6+x^7+x^5+x^6+x^7+x^8+x^6+x^7+x^8+x^9+x^7+x^8+x^9+x^{10})$$

$$=(x^2+x^3+x^4+x^5)(x^4+2x^5+3x^6+4x^7+3x^8+2x^9+x^{10})$$

Can forget about terms that wont contribute an x^{12} when multiplied w/ any term in the other factor.

$$(x^2+x^3+x^4+x^5)(4x^7+3x^8+2x^9+x^{10})$$

$$= 4x^9+3x^{10}+2x^{11}+x^{12}+4x^{10}+3x^{11}+2x^{12}+x^{13}$$
$$+4x^{11}+3x^{12}+2x^{13}+x^{14}+4x^{12}+3x^{13}+2x^{14}+x^{15}$$

Hence, the answer is $10 = 1+2+3+4$

Another Old Question

If $|X|=n$ Then $|P(X)|=?$

element 1   element 2   ... element n

$$(1^0+x^1)\cdot(1^0+x^1)\cdot\ \ldots\ \cdot(1^0+x^1)$$

the first element is not in the subset

the first element is in the subset

the subset does not include element 2

the subset does include element 2

...

this factor is here to record membership of the last element

Multiply out & combine like terms

$\Rightarrow$ the coefficient of $x^k = |\{S \mid S \in P(X), |S|=k\}|$
$=: P_k(X)$

Subsets of cardinality k

But we know $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

By Binomial Theorem $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ (w/ $y=1$)

The above does two things

↳ provides yet another proof that $|P_k([n])| = \binom{n}{k}$

↳ Shows that the ?generating function? for the finite sequence $\{\binom{n}{k}\}_{k=0}^{n}$ is $(1+x)^n$

Solving Recurrence Relations (Another application we already have the tools for)

Geometric Progressions: $\{a_k\}_{k=0}^{\infty}$ defined via recurrence

$$a_0 := 4, \quad a_n := 2a_{n-1}$$

$$G(x) := \sum_{k=0}^{\infty} a_k x^k$$

"Generating function for the sequence"

↳ Formal power series w/ k^{th} coeff equal to the k^{th} term in sequence

("Formal" here means don't worry about convergence)

Recall: With Calculus we can prove

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \quad |x| < 1$$

(a proof is provided in 8 pages)

Shift all coeff up one

Consider $$G(x) - 2xG(x) = \sum_{k=0}^{\infty} a_k x^k - 2\sum_{k=1}^{\infty} a_{k-1} x^k$$

$$= a_0 + \sum_{k=1}^{\infty} (\underbrace{a_k - 2a_{k-1}}_{=0}) x^k$$

recurrence relation

$$= a_0 = 4$$

Solve for $G(x)$

$$4 = G(x) - 2xG(x) = (1-2x)G(x)$$

$$\therefore \quad \sum_{k=0}^{\infty} a_k x^k =: G(x) = \frac{4}{1-2x} \underset{\text{Calculus}}{=} 4\sum_{k=0}^{\infty} 2^k x^k$$

$$\Rightarrow \quad a_k = 4 \cdot 2^k$$

Proving Identities (polynomial equalities become combinatorial identities)

(I) (old fact) $$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}, \quad k < n$$

(new fact) (II) $$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{k-i}\binom{n}{i}, \quad k < \min\{m,n\}$$

proof of (I): Binomial Theorem

Begin w/ a basic fact about multiplication

$$(1+x)^{n-1} + x(1+x)^{n-1} = (1+x)(1+x)^{n-1} = (1+x)^n$$

$\uparrow$ factor out $(1+x)^{n-1}$

$$(1+x)^{n-1} = \left[\sum_{k=0}^{n-1}\binom{n-1}{k}x^k\right], \quad x(1+x)^{n-1} = \left[x\sum_{k=0}^{n-1}\binom{n-1}{k}x^k\right], \quad (1+x)^n = \left[\sum_{k=0}^{n}\binom{n}{k}x^k\right]$$

$$\left(1+\sum_{k=1}^{n-1}\binom{n-1}{k}x^k\right) + \left(\sum_{k=1}^{n-1}\binom{n-1}{k-1}x^k + x^n\right) = \underline{\sum_{k=0}^{n}\binom{n}{k}x^k}$$

$$= \underline{1+\sum_{k=1}^{n-1}\left[\binom{n-1}{k}+\binom{n-1}{k-1}\right]x^k + x^n}$$

Matching up coefficients of x^k we get the equality
$$\binom{n-1}{k}+\binom{n-1}{k-1} = \binom{n}{k} \qquad \square$$

proof of (II)

Begin w/ a basic fact about multiplication

$$(1+x)^{m+n} = (1+x)^m(1+x)^n$$

$$\underbrace{\sum_{k=0}^{m+n}\binom{m+n}{k}x^k} \qquad \left(\sum_{k=0}^{m}\underline{\binom{m}{k}}x^k\right)\left(\sum_{k=0}^{n}\underline{\binom{n}{k}}x^k\right)$$

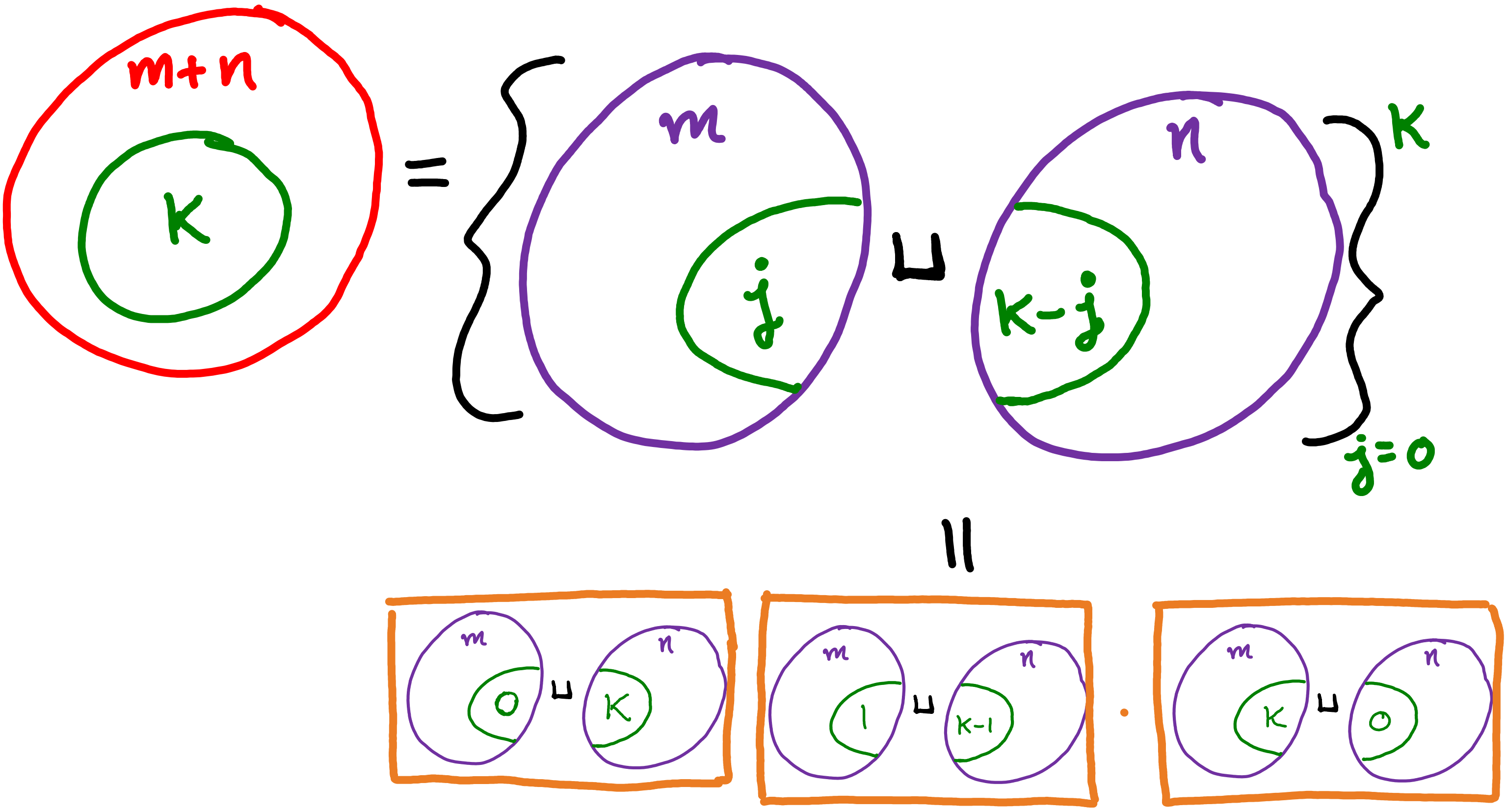
only care about coeff of x^K

$$= \sum_{K=0}^{m+n}\left(\sum_{j=0}^{K} a_j b_{K-j}\right) \underbrace{x^j \cdot x^{K-j}}_{= x^K}$$

When we equate the coeff of x^K on both Sides of the equation we see

$$\binom{m+n}{K} = \sum_{j=0}^{K} \binom{m}{K-j}\binom{n}{j} \qquad \square$$

A combinatorial proof of (II)



$\square$

Corollary of (II): $\displaystyle\binom{2n}{n} = \sum_{j=0}^{n} \binom{n}{j}^2$

proof: Set $m = n = k$ in (II)

$$\binom{2n}{n} = \sum_{j=0}^{n} \binom{n}{n-j}\binom{n}{j} = \sum_{j=0}^{n} \binom{n}{j}^2 \qquad \square$$

these are equal

More Combinatorial Proofs

(i) $k\binom{n}{k} = n\binom{n-1}{k-1}$, $1 \leq k \leq n$

(ii) $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$, $k \leq r \leq n$

(iii) $\binom{2n}{2} = 2\binom{n}{2} + n^2$

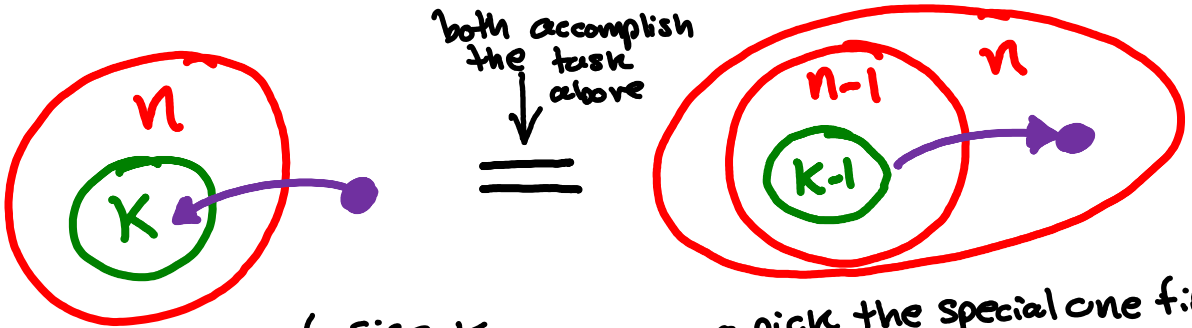
(iv) $\binom{n+1}{k+1} = \sum_{j=k}^{n} \binom{j}{k}$, $k \leq n$ "Hockeystick Identity"

(v) $\binom{n+1}{k} = \frac{n+1}{k}\binom{n}{k-1}$

Note: (i) $\Leftrightarrow$ (v)

proof of (i): $k\binom{n}{k} = n\binom{n-1}{k-1}$

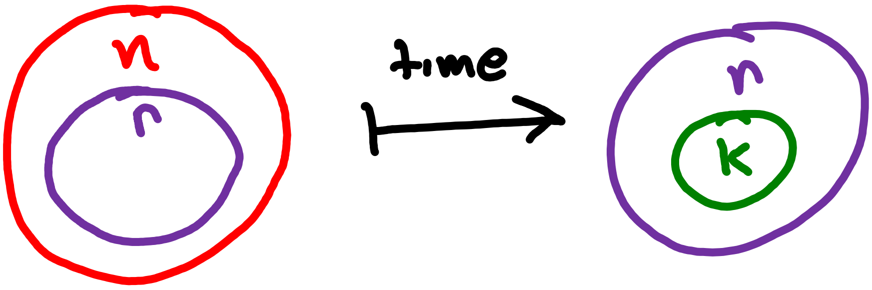
task: Choose a subset of size k & a special element from this subset



- choose a subset of size k
- then pick one to be special

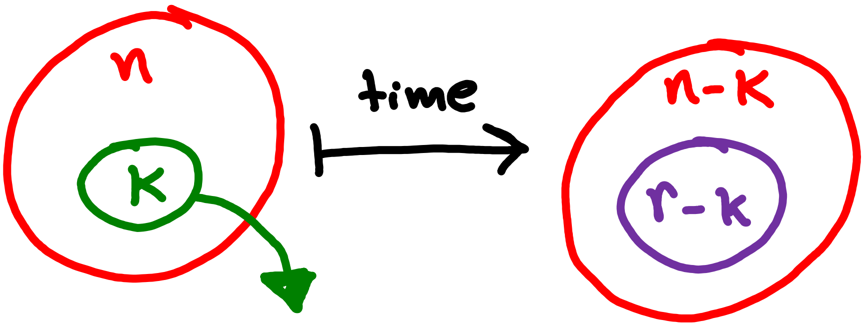
- pick the special one first
- then pick a subset of size $k-1$ from the remaining $n-1$ $\square$

proof of (ii): $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$



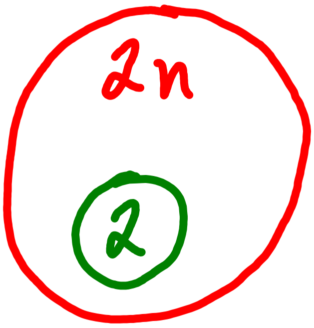
- Choose a subset
- choose a subset of this subset

V.S.



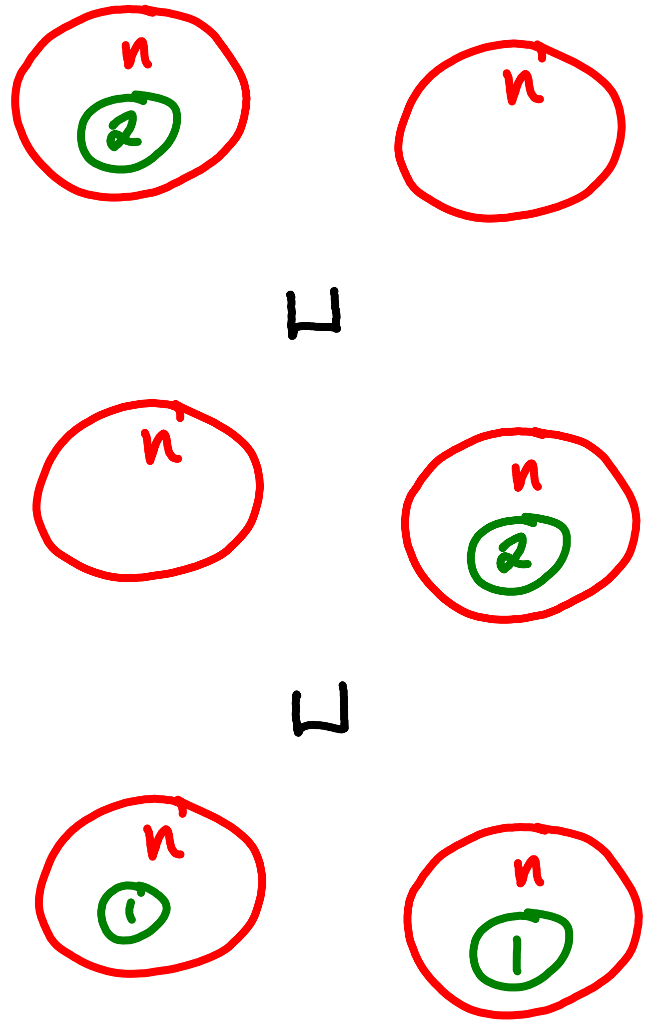
- remove a subset
- choose a subset of what remains $\square$

proof of (iii): $\binom{2n}{2} = 2\binom{n}{2} + n^2$



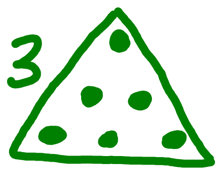
Choose 2 elements from a set of size 2n

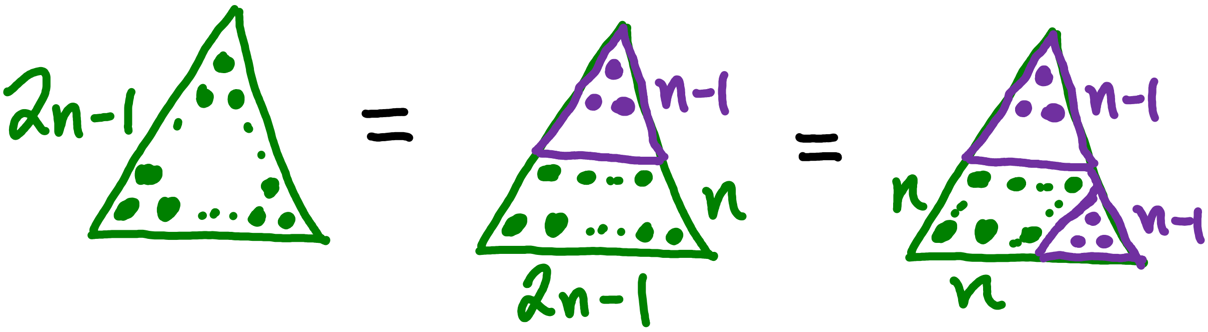
OR



Alternate proof:

Recall: $\binom{n+1}{2} = \sum_{k=1}^{n} k$





Full triangle is divided up into two small triangles & a parallelogram (can be straightened out to a square)

$\square$

<u>proof of</u> (iv): $\binom{n+1}{k+1} = \sum_{j=k}^{n} \binom{j}{k}$

$$\text{LHS} = \underbrace{|B(n+1,k+1)|}_{\substack{\text{bit strings of length}\\ n+1 \text{ w/ exactly } k+1\\ \text{ones}}} = \sum_{j=k}^{n} \underbrace{|B(n+1,k+1,j+1)|}_{\substack{\text{bit strings of length } n+1\\ \text{w/ exactly } k+1 \text{ ``ones'' where}\\ \text{the last ``one'' occurs in position}\\ \text{number } j+1}}$$

$$|B(n+1,k+1,j+1)| = \binom{j}{k}$$

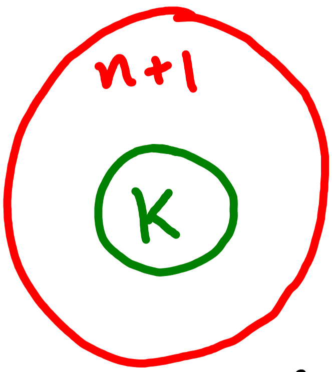
$$\underset{1}{\_\_}\;\underset{2}{\_\_}\;\underset{3}{\_\_}\;\underset{4}{\_\_}\;\cdots\;\underset{k}{\_\_}\;\underbrace{\underset{k+1}{\_\_}\;\underset{k+2}{\_\_}\;\cdots\;\underset{n+1}{\_\_}}_{\text{final 1 is placed somewhere here}}$$

if the final bit is placed in position $j+1$ then k 1s must be placed in the preceeding j positions

in conclusion $\binom{n+1}{k+1} = \sum_{j=k}^{n} \binom{j}{k}$ as claimed

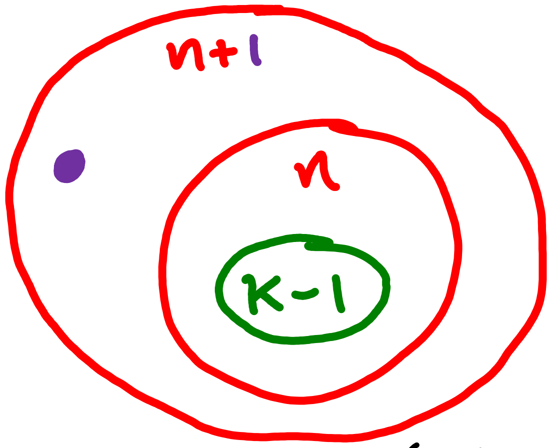
$\square$

proof of (✓):



Choose K elements from a collection of n+1

OR



pick out the first element & choose K-1 from the remaining n elements

Note: on the left, our collections are unordered but on the right, there is a designated "first" element w/ an unordered collection of K-1 elements

there are K possible choices for this "first" element meaning we count the same unordered collection of K elements K times on the right

$$\binom{n+1}{K} = \frac{n+1}{K}\binom{n}{K-1}$$

# of ways to pick first element

correct for over counting

# of ways to pick K-1 elements from what remains

Calculus Fact Used Above

proof:

$$\sum_{k=0}^{\infty} x^k := \lim_{n\to\infty} \sum_{k=0}^{n} x^k = \lim_{n\to\infty} \frac{1-x^{n+1}}{1-x}$$

Sum of a (finite) geometric series

$|x|<1$

$$\Rightarrow \lim_{n\to\infty} x^{n+1} = 0$$

$$= \frac{1}{1-x} \quad \square$$