Counting "Anagrams"

BINARY is a 6 letter word. So is BRAINY. Notice that these two words use exactly the same set of letters.

Usually

An anagram is a word formed by rearranging the letters from another word

For Us

An "anagram" will be a string formed by rearranging the symbols from another string

A problem we already know how to answer

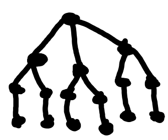
How many anagrams of BAT are there? $(3! = 6)$

(i) Each letter in this word is different from each of the other letters

$\Downarrow$

(ii) Anagrams in this case simply correspond to permutations of the letters

We can write them all in a list



BAT, BTA, ABT, ATB, TAB, TBA

A New Type of Problem

How many anagrams of RABBIT are there?

(i) A good First Guess would be $6!$

Since there are 6 letters in the word

(ii) However, some of the $6!$ permutations result in the same string since there are 2 Bs in our starting word

Solution #1: Use the division Rule

$\#$Anagrams of RABBIT $< 6!$

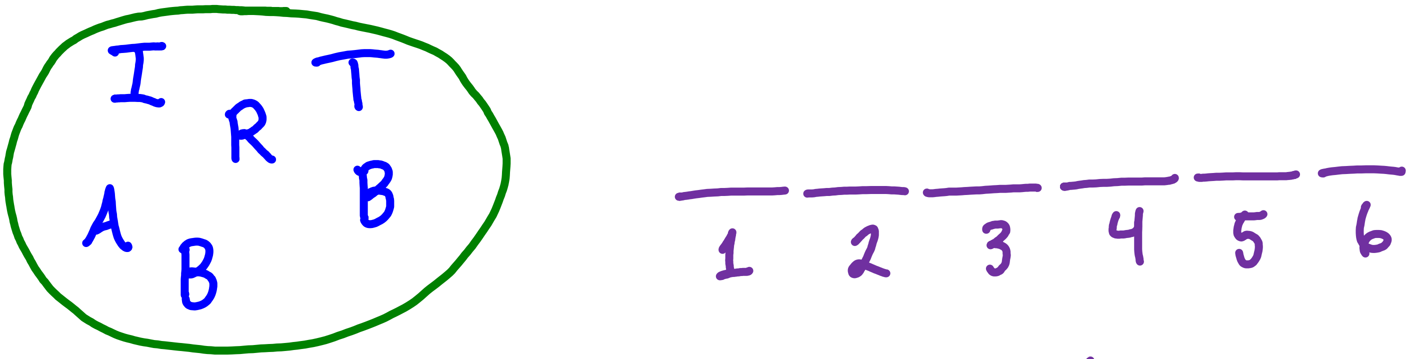
↳ As noted above, some strings are counted more than once. In fact, each string is counted exactly twice (Swap the position of the two Bs & you get the same string)

Therefore

$$\#\text{Anagrams of RABBIT} = \frac{6!}{2}$$

Solution #1: Use the multiplication rule

- We need to place these letters in the 6 slots below to form an anagram.



- place the Bs. This can be done in $\binom{6}{2}$ ways
- place the A next The Bs take up 2 slots so there are only $\binom{6-2}{1} = \binom{4}{1}$ ways
- the I can then be placed in $\binom{3}{1}$ ways
- the T can then be placed in $\binom{2}{1}$ ways
- the position of the R is then forced

Therefore

$$\text{\#Anagrams of RABBIT} = \binom{6}{2}\binom{4}{1}\binom{3}{1}\binom{2}{1}\cdot 1$$

$$= \frac{6!}{\cancel{4!}\,2!}\cdot\frac{\cancel{4!}}{\cancel{3!}\,1!}\cdot\frac{\cancel{3!}}{\cancel{2!}\,1!}\cdot\frac{\cancel{2!}}{1!\,1!}$$

$$= \frac{6!}{2}$$

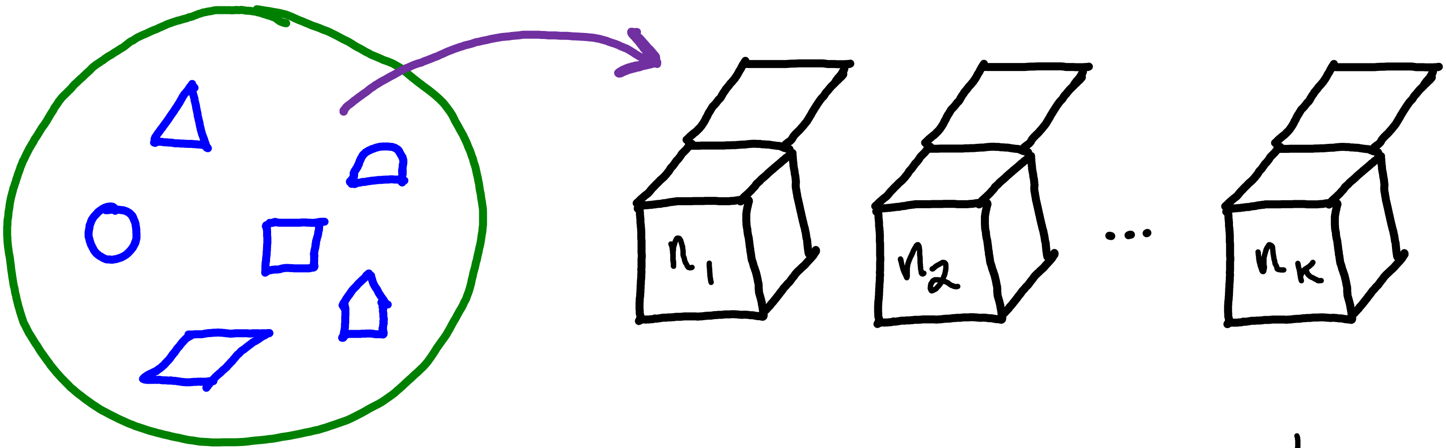
Permutations w/ Indistinguishable Objects

$$\underbrace{A, A, A, \ldots, A}_{n_1 \text{ copies}}, \underbrace{B, B, \ldots, B}_{n_2 \text{ copies}}, \ldots, \underbrace{X, X, \ldots, X}_{n_k \text{ copies}}$$

$$n := \sum_{i=1}^{k} n_i$$

$$\boxed{\#\text{different permutations of the above letters} = \frac{n!}{n_1!\, n_2! \cdots n_k!}}$$

Something Seemingly unrelated



Suppose we have n (distinguishable) objects

& we wish to place them into k different boxes so that there are n_i objects in the i^{th} box

Solution: Use the product rule

$$\binom{n}{n_1} \cdot \binom{n-n_1}{n_2} \cdot \binom{n-n_1-n_2}{n_3} \cdot \ldots \cdot \binom{n-\sum_{i=1}^{k-1} n_i}{n_k}$$

Box #	1	2	3	k

$$\binom{n}{n_1}\cdot\binom{n-n_1}{n_2}\cdot\binom{n-n_1-n_2}{n_3}\cdot\ldots\cdot\binom{n-\sum_{i=1}^{k-1}n_i}{n_k} = \frac{n!}{\underbrace{n_1!n_2!\cdot\ldots\cdot n_k!}_{\text{looks familiar}}}$$

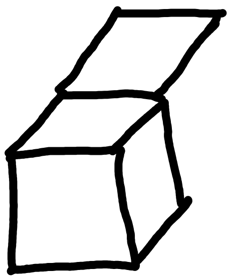
$$\frac{n!}{n_1!n_2!\cdot\ldots\cdot n_k!}$$

$\|$ # permutations of n symbols where there are k distinct symbols & n_i repetitions of the i^{th} symbol

$\overset{?}{\longleftrightarrow}$

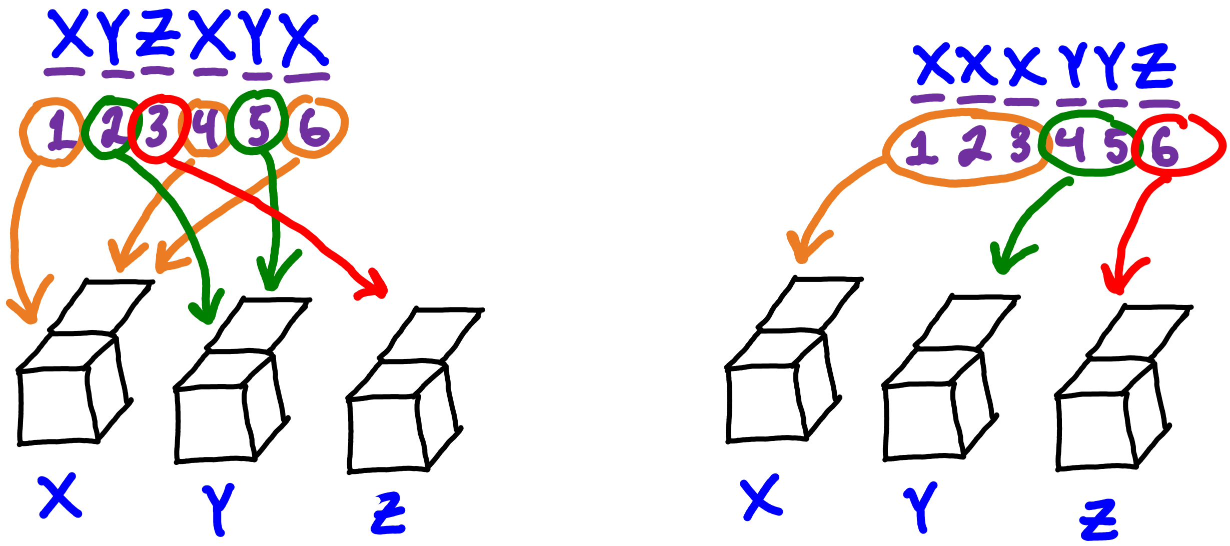
$\|$ # ways to place n distinguishable objects into k boxes so that the i^{th} box has n_i objects

Since $\frac{n!}{n_1!n_2!\ldots n_k!}$ Counts the # of possabilities in both situations we should be able to find a bijection with which to translate between these two points of view.

Symbol type $\longleftrightarrow$ 

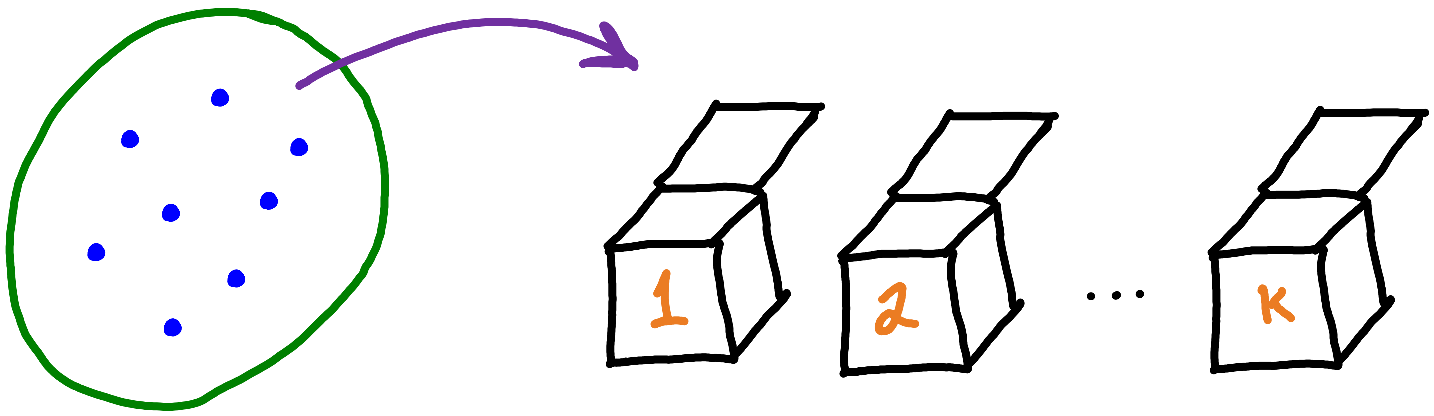
# of repetitions of the i^{th} symbol $\longleftrightarrow$ n_i

position in permutation $\longleftrightarrow$ an object



What we have just counted is the number of ways to place Distinguishable Objects into Distinguishable Boxes.

Indistinguishable Objects & Distinguishable Boxes



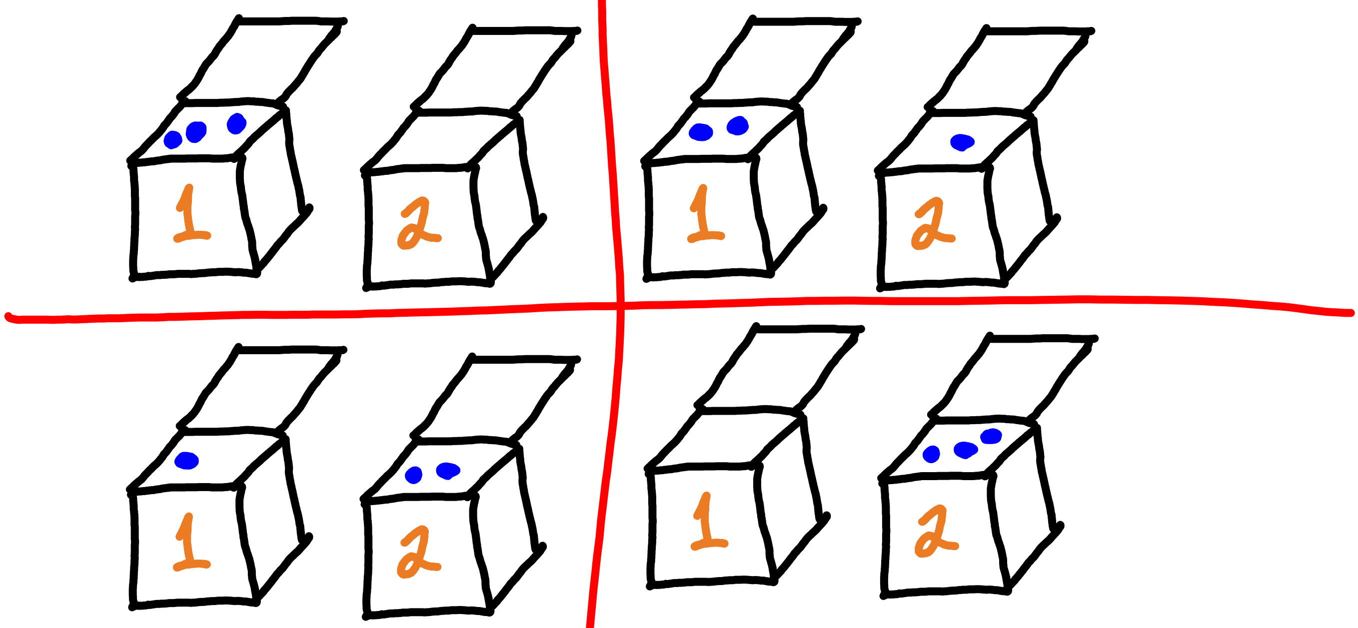
n indistinguishable objects

k labeled boxes (distinguishable)

The only thing that matters is the number of objects in each box.

e.g placing 3 identical balls in 2 numbered boxes

$***| \approx 0001$ $\qquad$ $**|* \approx 0010$



$*|** \approx 0100$ $\qquad$ $|*** \approx 1000$

# of different configurations corresponds to # of bit strings of length 4 with exactly one 1 (equivalently, bit strings of length 4 with exactly 3 zeros)

$$4 = \binom{4}{1} = \binom{4}{3}$$

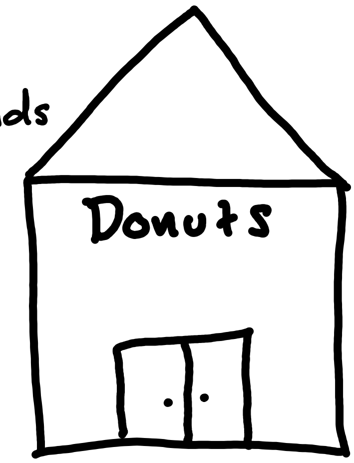
Distributing n indistinguishable objects into k distinguishable boxes

$\longleftrightarrow$

k-combinations from a set of n elements w/ repetition allowed

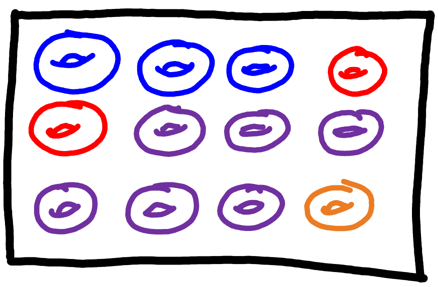
$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

K-Combinations w/ repetition (Stars & bars)



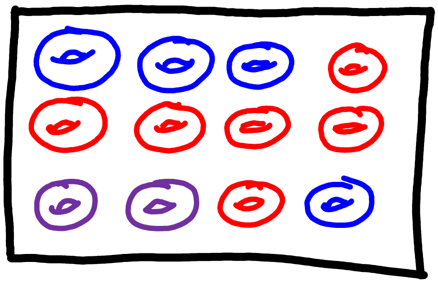
A donut shop offers 4 different kinds of donut

Q: How many different ways can you buy a dozen donuts? (order of donuts does not matter, we only care how many of each flavor are bought)



$$* * * \mid * * \mid * * * * * * \mid *$$

We have 4 varieties of donuts so we use 3 dividing lines (bars) to separate 12 donuts (stars)



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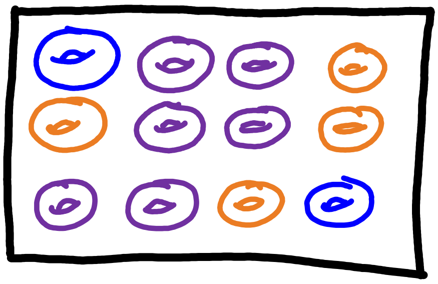
No plain donuts



||****|****

No glazed donuts

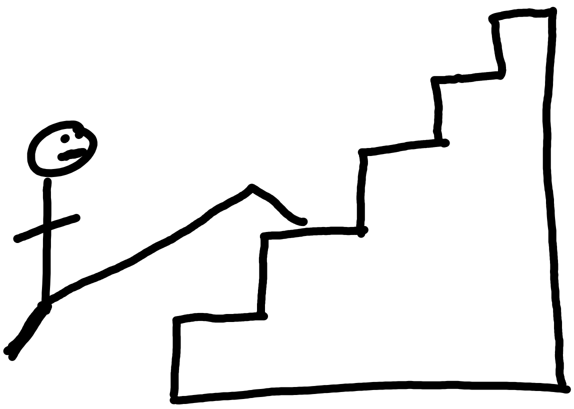
A: # of different boxes of a dozen donuts

= # of arrangements of 12 stars & 3 bars

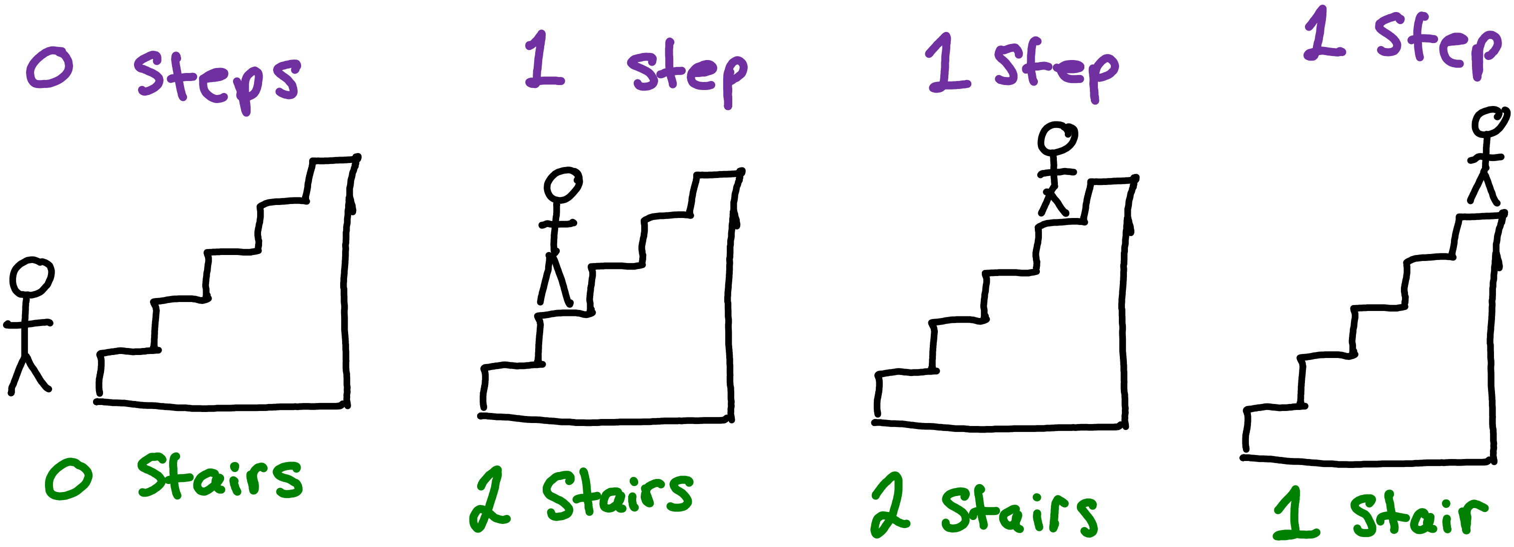
= # of bit strings of length 15 (= 12 + K − 1) with exactly 3 1s (or 12 0s).

Application 2: Ascending a staircase in a fixed number of steps.

Jeff likes going up the stairs using exactly 3 "steps." He is okay with the idea of not ascending any stairs on a given "step" & he has really long legs so he can climb any number of stairs in a single "step."



Q: How many different ways can Jeff climb a staircase w/ 5 stairs in the particular way he likes?



$$* * | * * | *$$

A: use stars & bars! $\binom{5+3-1}{2} = \binom{5+3-1}{5}$

Application 3: Solving equations over $\mathbb{N}$

Q: How many solutions to the equation

$x + y + z = 11$ are there if

x, y, and z are all in $\mathbb{N}$?

e.g. $x=6$, $y=2$, $z=3$

$$6+2+3=11$$

$$******|**|***$$

A: $\binom{13}{2}=\binom{13}{11}$ different solutions

__permutations w/ repetition__

\# of license plates

(i) Each license plate is 7 characters long

(ii) Each character can be a lowercase letter or a digit (0-9)

How many possible license plates are there?

$$\underset{\substack{\uparrow\\36\\\text{possibilities}}}{\_\_}\ \underset{\substack{\uparrow\\36\\\text{possibilities}}}{\_\_}\ \underset{\substack{\uparrow\\36\\\text{possibilities}}}{\_\_}\ \underset{\substack{\uparrow\\36\\\text{possibilities}}}{\_\_}\ \underset{\substack{\uparrow\\36\\\text{possibilities}}}{\_\_}\ \underset{\substack{\uparrow\\36\\\text{possibilities}}}{\_\_}\ \underset{\substack{\uparrow\\36\\\text{possibilities}}}{\_\_}$$

A: 36^7

Summary

Type	Repetition Allowed?	Formula
r-permutations	NO	$\frac{n!}{(n-r)!}$
r-combinations	NO	$\frac{n!}{r!(n-r)!}$
r-permutations	YES	n^r
r-combinations	YES	$\frac{(n+r-1)!}{r!(n-1)!}$