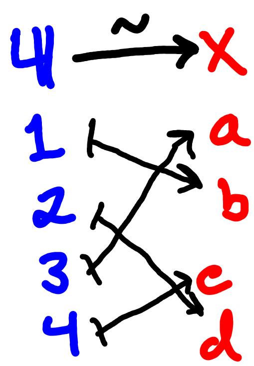
Counting Bijections

Q: How many bijections are there between the set $\mathbb{4} = \{1,2,3,4\}$ & the set $X = \{a,b,c,d\}$?

Note: A bijection $f : \mathbb{4} \xrightarrow{\sim} X$ gives us a way to rank the elements of X as "first" – "fourth"

Bijection



$f(1)$ is first
$f(2)$ is second
$f(3)$ is third
$f(4)$ is fourth

Permutation

(a way of ordering the letters in X)

bdac

Listing all permutations of elements of $\mathbb{n}$

(1) pick the first element ($\underline{n}$ possible choices)

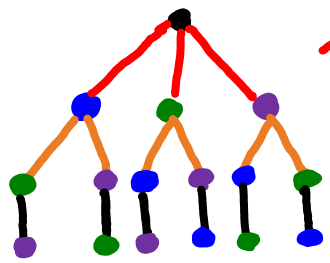
(2) pick the second element (must be different from the first $\Rightarrow$ $\underline{n-1}$ possible choices)

⋮

(n-1) pick the second to last element (must be different from the first $n-2$ choices $\Rightarrow$ exactly $\underline{2}$ possible choices)

(n) the remaining element which does not already have a ranking is forced to be last (no choice) e.g. $\{a, b, c\}$

bottom row has 6 dots
(we count them vertically)



~ choose first element

~ choose second element

~ Last element is forced

$$\boxed{3! = 3 \cdot 2 \cdot 1 = 6}$$

The above picture suggests repeated multiplication

$$\#\text{permutations of } n \text{ objects} = n! := n(n-1)\cdot \ldots \cdot 2 \cdot 1$$

Note: $n! = n \cdot (n-1)!$

So $1! = 1 \cdot (0!) \Rightarrow 0! = \frac{1!}{1} = \frac{1}{1} = 1$

Counting Injective Functions

Recall: $|X| > |Y| \Rightarrow$ **No** injective functions $f: X \hookrightarrow Y$

$|X| = |Y| \Rightarrow$ Injective functions are bijections

Case: $k = |X| < |Y| = n$

(1) choose value for $f(1)$ from $|Y| = n$

(2) choose value for $f(2)$ from $|Y - \{f(1)\}| = n-1$

$\vdots$

(k) choose value for $f(k)$ from $|Y - \{f(1), \ldots, f(k-1)\}| = n-k+1$

thus, there are

$$n\cdot(n-1)\cdot\ldots\cdot(n-k+1) = \frac{n!}{(n-k)!}$$

different injective functions from X to Y

proof of =

$$\frac{n!}{(n-k)!} = \frac{n\cdot(n-1)!}{(n-k)!} = \frac{n\cdot(n-1)\cdot(n-2)!}{(n-k)!} = \ldots$$

$$= \frac{n\cdot(n-1)\cdot\ldots\cdot(n-k+1)\cdot(n-k)!}{(n-k)!}$$

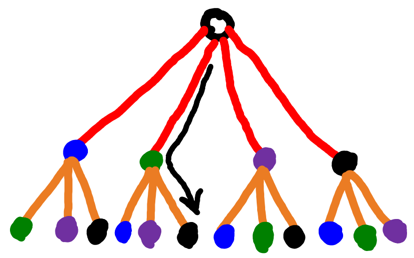
$$= n\cdot(n-1)\cdot\ldots\cdot(n-k+1)\,\underbrace{\frac{(n-k)!}{(n-k)!}}_{=1}$$

$$= n\cdot(n-1)\cdot\ldots\cdot(n-k+1) \qquad \square$$

e.g.

$$\{f : 2 \hookrightarrow 4\}$$

$$\begin{array}{ccc} & & 1 \\ 1 & \mapsto & 2 \\ 2 & \mapsto & 3 \\ & & 4 \end{array}$$



$4 \times 3 = 12$

Notation: r-permutations

$$P(n,r) := \frac{n!}{(n-r)!} = \#$$ of permutations of r things from a set of size n

$$\mapsto P(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

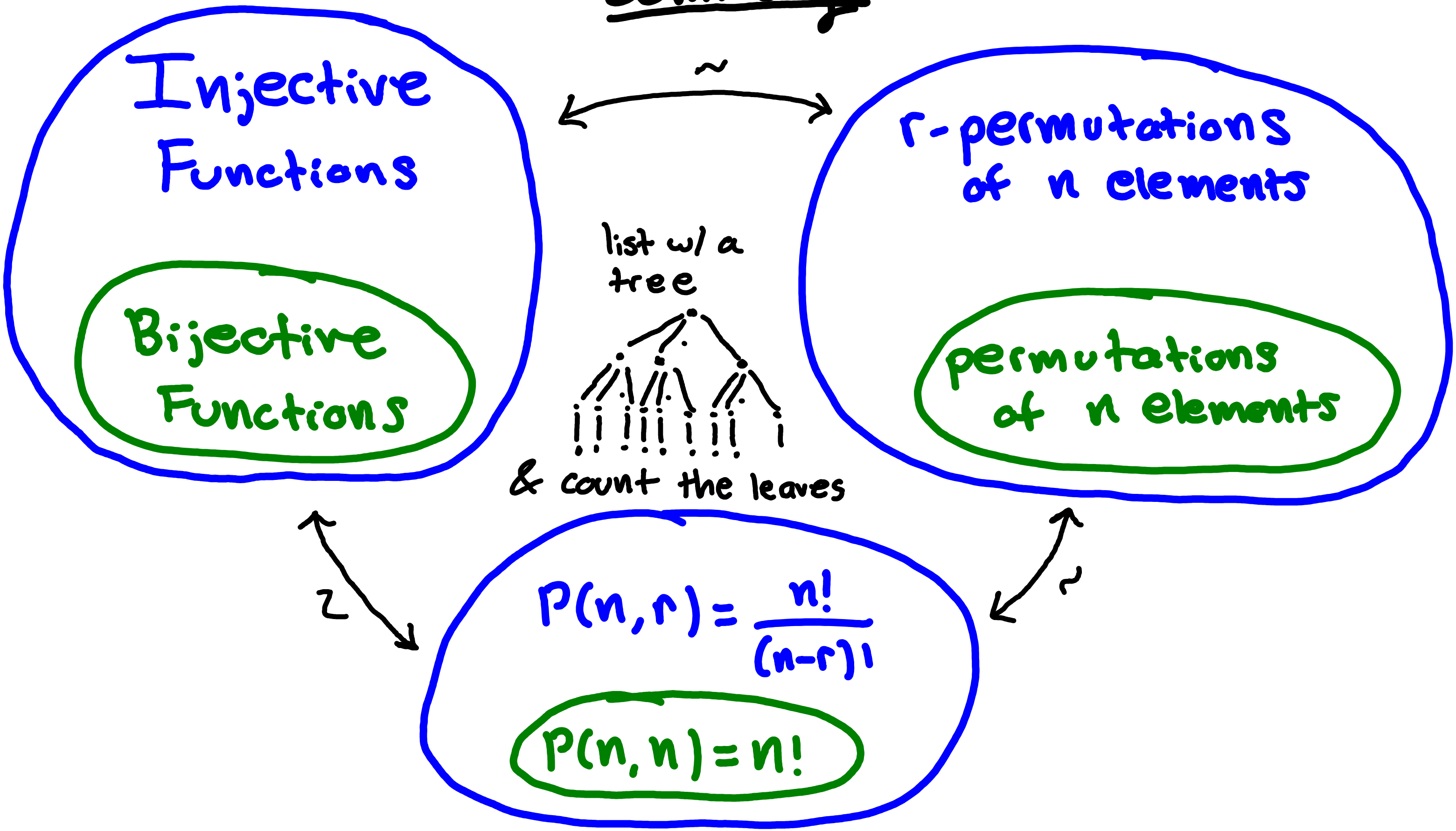
$$\mapsto P(n,0) = \frac{n!}{n!} = 1$$

Fact: $|X| = r \leq n = |Y|$

$$\Rightarrow |\{f : X \hookrightarrow Y\}| = P(n,r)$$

$= 0$ when $|X| > |Y|$ $\square$

Summary



Combinations & Subsets

|Range(f)|

$f: \mathbb{K} \hookrightarrow \mathbb{N}$

Range{f}

Range(–)

order the elements in the subset

Subsets of size k in $\mathbb{N}$

$|P_k(\mathbb{N})|$

\# injective functions & division rule

computer representations of sets

2

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

possible choices of k things from a collection of n

$\sim$

Choose k of the n positions to place a 1

Bit strings of length n with exactly k 1s

$|B(n,k)|$

$\frac{n!}{(n-k)!\,k!}$

Combinations

When order does not matter

Division Rule

forget the order

$k!$

choose an order

Multiplication Rule

$\frac{n!}{(n-k)!}$

Permutations

when order is important

$$P(n,k) = C(n,k) \cdot k!$$

\# of ways to order k things

$$\frac{P(n,k)}{k!} = C(n,k)$$

\# of times the same combination is represented as a permutation

Binomial Coefficients

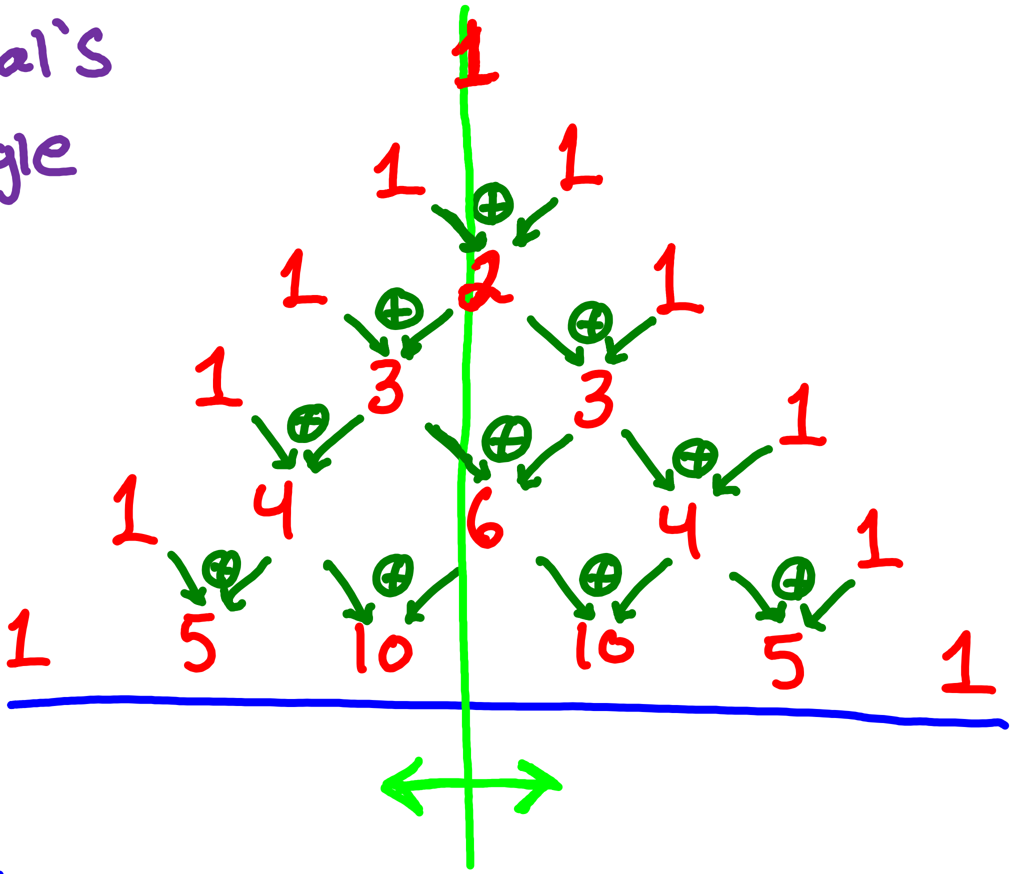
$$(x+y)^0 = 1$$

$$(x+y)^1 = 1x + 1y$$

$$(x+y)^2 = (x+y)(x+y) = x^2 + xy + yx + y^2$$
$$= 1x^2 + 2xy + 1y^2$$

$$(x+y)^3 = (x+y)(1x^2 + 2xy + 1y^2)$$
$$= x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3$$
$$= 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

Pascal's Triangle



Claim:

$$(x+y)^5 = x^5 + 5xy^4 + 10x^2y^3 + 10x^3y^2 + 5x^4y + y^5$$

Q: Why does this work?

A: Proof will be in 2 stages

(I) coeff of $x^k y^{n-k}$ in $(x+y)^n$ is equal to $\binom{n}{k}$

(II) $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

$$\binom{n}{k-1} \qquad \binom{n}{k}$$
$$\oplus$$
$$\binom{n+1}{k}$$

proof of (I):

Each term in $(x+y)^n$ is a product of xs & ys where (# of xs) + (# of ys) = n. Therefore we can write

$$(x+y)^n = \sum_{k=0}^{n} \underline{c_k}\, x^k y^{n-k}$$

goal is to find c_k

$$\overbrace{(x+y)(x+y)(x+y)\dots(x+y)}^{n \text{ different parentheticals}}$$

Each term in the final product (before combining like terms) corresponds to a choice of either x or y (to be a factor) for each parenthetical.

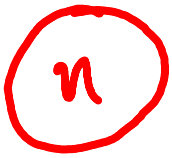
(Binomials $\sim$ Functions from n parentheticals to $\{x, y\}$)

$x^k y^{n-k}$ is possible in $\binom{n}{k}$ different ways

(Choose k of the parentheticals for which x will be the factor picked – leaving $n-k$ parentheticals where y is picked) $\square$

<u>proof of (II)</u>: (A combinatorial proof)

Begin w/ n objects



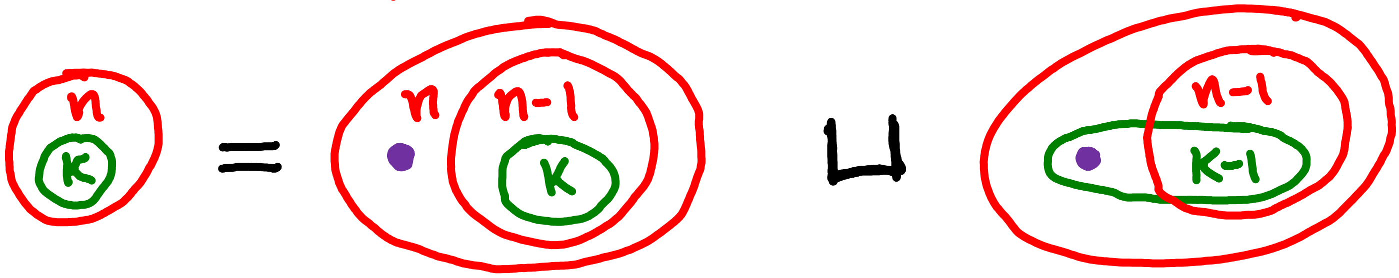
Pull one object out

When we pick k of the n objects

either...

None are the special one we pulled out

The special one <u>is</u> included



How many ways can this be done?

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$\square$

Binomial Theorem: $\forall n \in \mathbb{N}$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Corollary: (We have already seen this fact)

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \quad \left(= \sum_{i=0}^{n} |P_i([n])| = \sum_{i=0}^{n} |B(n,i)| \right)$$

proof: Set $x = y = 1$ in the binomial theorem $\square$

This formula gives us a nice way of generating relationships between binomial coefficients

e.g.

(i) Set $1 = -x = y$

$$0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$$

$$\Longleftrightarrow \underbrace{\sum \binom{n}{2k}}_{\text{even}} = \underbrace{\sum \binom{n}{2k+1}}_{\text{odd}}$$

(ii) use the fact that $x+y = y+x$

$$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x+y)^n = (y+x)^n = \sum_{k=0}^{n} \binom{n}{k} y^k x^{n-k}$$

$$\binom{n}{k}\boxed{x^k y^{n-k}}$$

&

$$\binom{n}{n-k} x^{n-k} y^{n-(n-k)}$$

$$\parallel$$

$$\binom{n}{n-k}\boxed{x^{n-k} y^{k}}$$

$$\binom{n}{k}\boxed{y^k x^{n-k}}$$

&

$$\binom{n}{n-k} y^{n-k} x^{n-(n-k)}$$

$$\parallel$$

$$\binom{n}{n-k}\boxed{y^{n-k} x^{k}}$$

binomial =

means coefficients =

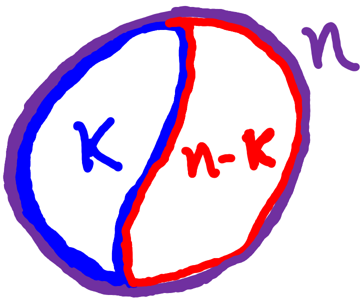
Thus $\binom{n}{k} = \binom{n}{n-k}$

Combinatorial Proof: (Fill in details yourself)

Choosing a subset of k things from n

is like choosing $n-k$ elements to leave out of the subset.

$$\binom{n}{k} = \binom{n}{n-k}$$



$\binom{n}{k} = \binom{n}{n-k}$ describes a symmetry of this triangle

1

1 1

1 2 1

1 3 3 1

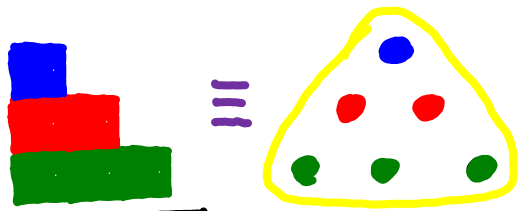
1 4 6 4 1

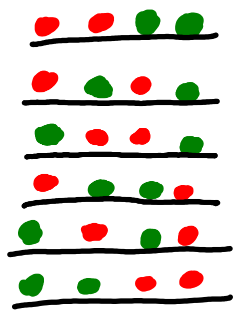
1 5 10 10 5 1

Subsets of size 2

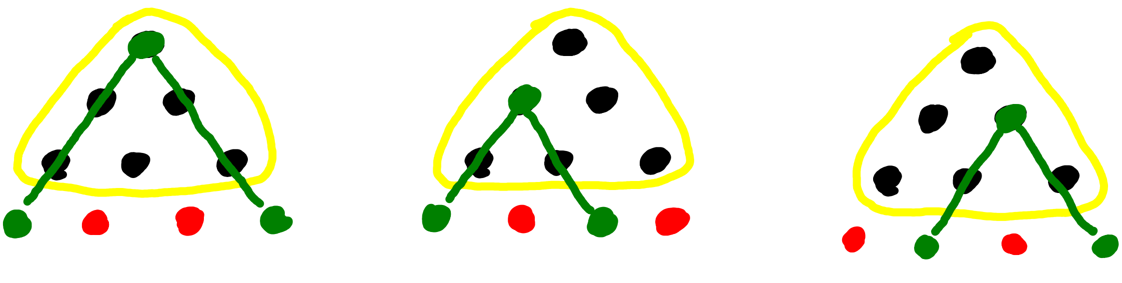
$$\binom{n+1}{2} = \sum_{k=1}^{n} k = 1+2+3+\dots+n$$

$$\begin{array}{r} 1 \\ +\,2 \\ +\,3 \\ \hline 6 \end{array}$$

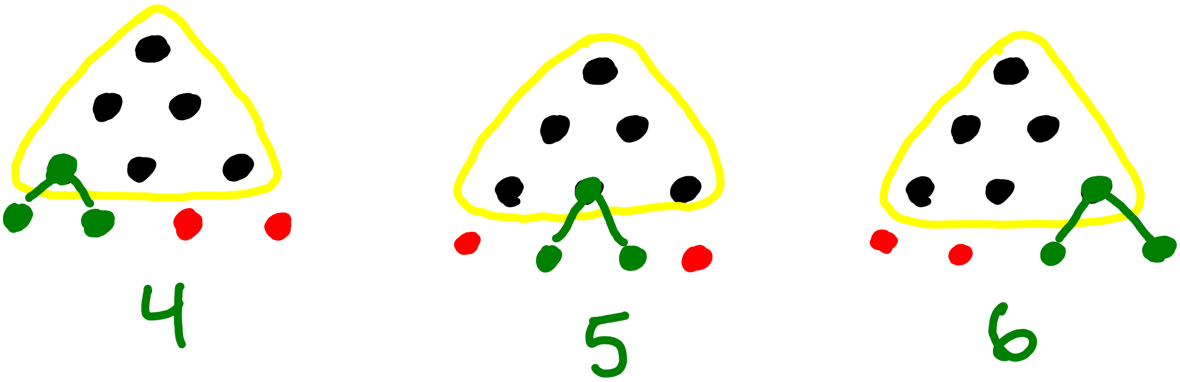




Subsets of size 2

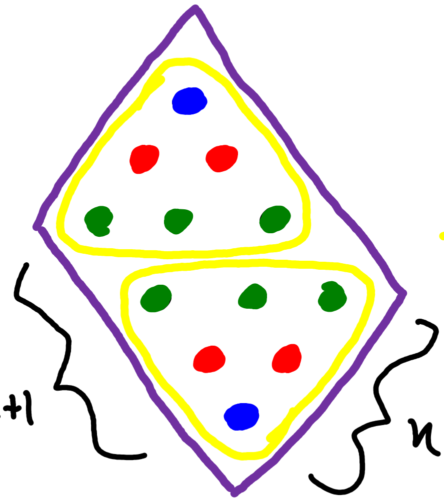


1 2 3



$$\sum_{k=1}^{n} k = 1+2+\dots+n = \frac{n(n+1)}{2}$$

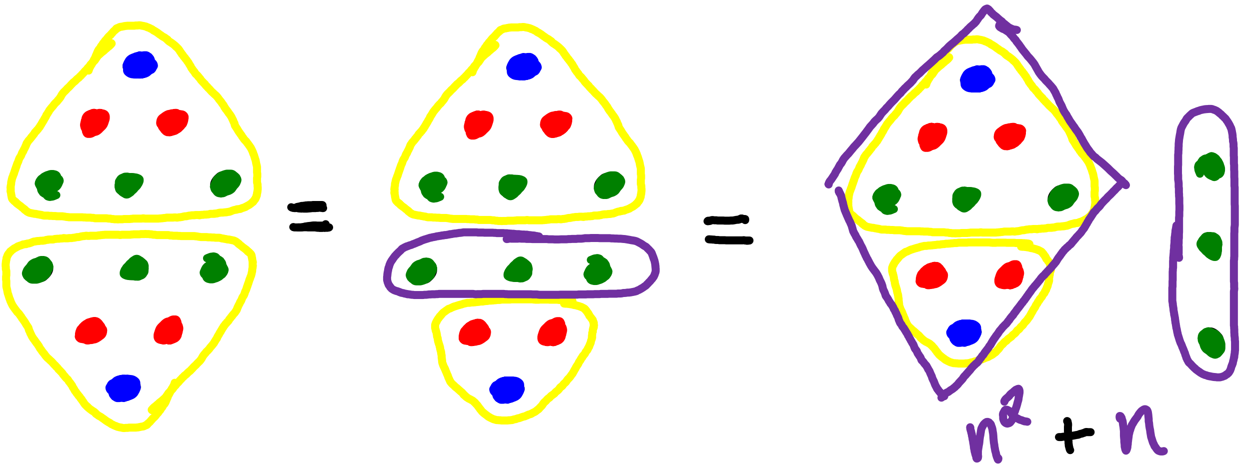
$n(n+1) =$



$= 2 \cdot \sum_{k=1}^{n} k$

$$\sum_{k=1}^{n} k = 1+2+\dots+n = \frac{n^2}{2} + \frac{n}{2}$$

x2 x2



binomial coefficients $= |P_k(\mathbf{n})| = |B(n,k)| = \binom{n}{k} = \frac{n!}{(n-k)!\,k!} =: C(n,k)$

$$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k+1}$$

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$$\binom{n+1}{2} = \sum_{k=1}^{n} k = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}$$

pascal's triangle

row #

col #

$$\begin{array}{ccccccc} & & & \binom{0}{0} & & & \\ & & \binom{1}{0} & & \binom{1}{1} & & \\ & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\ \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} \\ & & & \vdots & & & \end{array}$$

$$2^n = |\mathbf{2}^{\mathbf{n}}| = |\{f : \mathbf{n} \to \mathbf{2}\}| = |P(\mathbf{n})|$$

functions from $\mathbf{n}$ to $\mathbf{2}$

All subsets of $\mathbf{n}$

$$\binom{n}{2} = |\{\mathrm{Range}(f : \mathbf{2} \hookrightarrow \mathbf{n})\}| = |\{x \in P(\mathbf{n}) \mid |x| = 2\}|$$

ranges of injective functions from $\mathbf{2}$ to $\mathbf{n}$

Subsets of $\mathbf{n}$ with $\mathbf{2}$ elements

Riddle

Subsets of size 2 are to

Addition as

________ are to Multiplicatron