MATH 245 NOTES: SYZYGIES

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Contents

1. Introduction

Michael Kemeny taught a course (Math 245) on Syzygies at Stanford in Spring 2017.

These are my "live-TEXed" notes from the course. Conventions are as follows: Each lecture gets its own "chapter," and appears in the table of contents with the date.

Of course, these notes are not a faithful representation of the course, either in the mathematics itself or in the quotes, jokes, and philosophical musings; in particular, the errors are my fault. By the same token, any virtues in the notes are to be credited to the lecturer and not the scribe. [1] Please email suggestions to aaronlandesman@gmail.com.

[1] This introduction has been adapted from Akhil Matthew's introduction to his notes, with his permission.

2. 4/3/17

Today, we'll discuss why people care about syzygies. Syzygies go back to mid-19th century geometric invariant theory.

A syzygy is simply a relation among the equations of a projective variety. This goes by to Sylvester in 1850.

Example 2.1 (Syzygies of the twisted cubic)**.** Consider the map

$$\begin{aligned} \nu \colon \mathbb{P}^1 &\to \mathbb{P}^3 \\ [u;v] &\mapsto \left[u^3, u^2v, uv^2, v^3\right]. \end{aligned}$$

The twisted cubic $X := \nu(\mathbb{P}^1)$, the 3-Veronese embedding of $\mathbb{P}^1$ in $\mathbb{P}^3$. We can note also that X is the scheme theoretic intersection of

$$\begin{aligned} f &:= yw - z^2 \\ g &:= yz - xw \\ h &:= xz - y^2 \end{aligned}$$

There are two syzygies:

$$\begin{aligned} xf + yg + zh &= 0 \\ yf + zg + wh &= 0. \end{aligned}$$

2.1. **Introducing notation.** For the remainder of the course, we fix the following notation. Consider $S := \mathbb{C}[x_0, \ldots, x_n]$, a graded ring in $n+1$ variables. Let S_d denote the homogeneous polynomials in S of degree d. Let M be a finitely generated graded module over S. For C a curve, we will let g denote the genus.

Example 2.2. Let the **twisted module** $S(-n)$ be the S-module defined so that $S(-n)_d := S_{d-n}$. Note that as a module (without a grading) this is isomorphic to S.

Definition 2.3. A graded S module M is **free** if one can write $M = \oplus_n S(-n)^{\oplus b_n}$.

Definition 2.4. A resolution $F_\bullet \to M$ is **minimal** if each $\delta_i : F_i \to F_{i-1}$ takes a basis of F_i to a minimal set of generators of $\mathrm{im}\,(\delta_i)$.

Theorem 2.5 (Hilbert syzygy theorem, 1890)**.** *Let M be a finitely generated graded module over $S := \mathbb{C}[x_0, \ldots, x_n]$. Then there exists a unique minimal free resolution*

(2.1)
$$0 \longleftarrow M \longleftarrow F_0 \xleftarrow[\delta_1]{} F_1 \xleftarrow[\delta_2]{} \cdots \xleftarrow[\delta_{n+1}]{} F_{n+1} \longleftarrow 0$$

of length at most $n+1$.

We'll prove this on Friday.

Definition 2.6. If M is a finitely generated graded S module, then the **Hilbert function** is the map

$$\begin{aligned} f_m \colon \mathbb{Z} &\to \mathbb{Z} \\ d &\mapsto \dim_{\mathbb{C}} M_d. \end{aligned}$$

Remark 2.7. One can verify that the Hilbert function is eventually polynomial, and one defines the **Hilbert polynomial** to be this polynomial.

Definition 2.8. Let $F_\bullet$ be the minimal free resolution of M. Then, say

$$F_i = \oplus_j S\left(-i-j\right)^{b_{ij}}.$$

These b_{ij} are the **Betti numbers** associated to f.

Define

$$\Delta_j := \sum_{i \le j} \left(-1\right)^j b_{j-i,i},$$

to be the alternating sum of the diagonal elements of the **Betti table**, which is the table containing b_{ij} in position (i,j).

Lemma 2.9. *We have*

$$f_m(d) = \sum_j \left(-1\right)^{j+1} \Delta_j \binom{n+d-j}{n}.$$

In this formula, we have $\binom{a}{n} = 0$ *if* $a < n$.

Proof. Omitted. □

Example 2.10 (Twisted cubic, revisited)**.** The homogeneous coordinate ring of the twisted cubic, S/I_X has the following minimal free resolution.

(2.2)

$$0 \longleftarrow S/I_X \longleftarrow S \xleftarrow[A]{} S(-2)^{\oplus 3} \xleftarrow[B]{} S(-3)^{\oplus 2} \longleftarrow 0.$$

Here,

$$\begin{aligned} A &= \left[yw - z^2, yz - xy, xy - z^2\right] \\ B &= \begin{pmatrix} x & y \\ y & z \\ z & w \end{pmatrix}. \end{aligned}$$

The Betti table (i.e., the table with (i,j) entry given by b_{ij} of X is Next week, we'll show that the 4 table of a rational normal curve is

j,i	0	1	2	3
0	1	0	0	0
1	0	3	2	0

TABLE 1. Betti table of twisted cubic

j,i	0	1	2	$\cdots$	$d-1$
0	1	0	0	$\cdots$	0
1	0	$\binom{d}{2}$	$2\binom{d}{3}$	$\cdots$	$(d-1)\binom{d}{d}$

TABLE 2. Betti table of twisted cubic

2.2. **Enter Brill-Noether theory.** We'd like to relate the extrinsic geometry of $C \subset \mathbb{P}^r$ (the Betti number of S/I_C to the abstract intrinsic geometry of the curve. For this, we'll use Brill Noether theory.

Remark 2.11. Recall that for C a curve of genus g, the Brill-Noether loci are

$$W^r_d(C) := \left\{ \text{line bundles } \mathcal{L} \text{ of degree } d \text{ with} h^0(\mathcal{L}) = r+1 \right\}.$$

If C is general, the Brill Noether locus

$$W^r_d(C)$$

is smooth of dimension

$$\rho(g, r, d) := g - (r+1)(g-d+r)$$
$$g - h^0(L) - h^1(L).$$

The nicest proof of this, in Michael's opinion, is Lazarsfeld's proof using K3 surfaces, which is just a couple of pages.

We'll now define some useful invariants of a curve.

Definition 2.12. Given a smooth curve C, the **gonality** of C is

$$\begin{aligned} \mathrm{Gon}(C) &:= \min_d \left\{ d : w^1_d \neq \emptyset \right\} \\ &= \min_d \left\{ d : \text{ there exists a degree } d \text{ map } C \to \mathbb{P}^1 \right\} \\ &\leq \lfloor \frac{g+3}{2} \rfloor. \end{aligned}$$

Definition 2.13. Given a smooth curve C, we let the Clifford index,

$$\mathrm{Cliff}(C) := \min_A \left\{ \deg A - 2r(A) : A \text{ is a line bundle } \deg A \leq g-1, h^0(A) \geq 2 \right\}.$$

Remark 2.14. For a generic curve, $\operatorname{Cliff} C = \operatorname{Gon} C - 1$.

We now introduce some more notation.

Definition 2.15. Let C be a curve and L a line bundle. Then,

$$\Gamma_C(L) = \oplus_n H^0(C, nL)$$

is a graded $S := \operatorname{Sym} H^0(L)$ module.
We let

$$b_{p,q} := b_{p,q}(\Gamma_C(L)) =: b_{p,q}(C, L).$$

For M a second, line bundle, let

$$\Gamma_C(L; M) := \oplus H^0(C, nL + M)$$

be the graded $H^0(L)$ module. Then,

$$b_{p,q}(C; M, L) := b_{p,q}(\Gamma_C(L; M).$$

Goal 2.16. The goal for the first part of this course is to relate $b_{p,q}(C, L)$ to Brill-Noether theory.

Theorem 2.17 (Castelnuovo-Mumford)**.** *For L a line bundle with*

$$\deg L \geq 2g + 1$$

then

$$\phi_L : C \to \mathbb{P}^r$$

defines an embedding and $\Gamma_C(L)$ coincides with S/I_C (meaning ϕ_L is projectively normal) which is equivalent to $b_{0,j} = 0$ for $j \geq 2$.

Proof. Omitted. □

Theorem 2.18 (Green, 1984)**.** *If $\deg L \geq 2g + 1 + p$, then*

$$b_{i,j} = 0 \text{ for } i \leq p, j \geq 2.$$

Lemma 2.19 (Noether)**.** *If C is not hyperelliptic, or equivalently if $\operatorname{Cliff} C \geq 1$, then $\phi_{\omega_C} : C \to \mathbb{P}^{g-1}$ is projectively normal. This means*

$$H^0(\mathbb{P}^{n-1}, \mathcal{O}(n)) \to H^0(C, \omega_C^n)$$

is surjective, or equivalently

$$b_{0,j} = 0 \text{ for } j \geq 2$$

Proof.

Exercise 2.20. Prove this!

□

Theorem 2.21 (Enriques, Petri, Babbage)**.** *Consider* $\phi_{\omega_C} : C \to \mathbb{P}^{g-1}$. *If* $\mathrm{Cliff}(C) \geq 2$ *then* I_C *is generated by quadrics. In the language of syzygies, this says*

$$b_{i,j} = 0 \text{ for } j \geq 2.$$

Conjecture 2.22 (Green's conjecture, 1984, proved by Voisin in 2002, 2005 in suitably generic cases)**.** If $p < \mathrm{Cliff}\, C$ then $b_{p,j} = 0$ for $j \geq 2$.

The main focus of this course will be to prove Green's conjecture and the secant conjecture.

3. 4/10/17

Today, we'll discuss the **Eagon-Northcott complex**.

3.1. **Constructing the Eagon-Northcott complex.**

Let R be a ring and $f : R^r \to R^s$ for $r \geq s$.

Consider the graded ring

$$S = R[x_1, \ldots, x_s].$$

Let $F = S^r(-1)$ be a graded S-module. Then, f defines a morphism

$$g : F \to S$$

of graded S-modules. We identify $S_1 \simeq R^s$ in the canonical way. Explicitly, if $e_1, \ldots, e_r$ is a basis for R^r, then

$$g\left(e_i \otimes 1\right) \mapsto f(e_i) \in R^s \simeq S_1.$$

By construction, this is indeed a homogeneous map of degree 0.

Consider the Koszul complex associated to g. That is, the complex associated to $\{f\left(e_i\right)\}$.

The Koszul complex $K_\bullet(g)$ looks like

(3.1)

$$0 \longleftarrow S \longleftarrow F \longleftarrow \wedge^2 F \longleftarrow \wedge^3 F \longleftarrow \cdots \longleftarrow \wedge^r F \longleftarrow 0$$

is a graded free complex. Now, take the degree d part. Note that the degree d part of the ith component is

$$\begin{aligned}\left(\wedge^i F\right)_d &= \left(\wedge^i_S \left(R^r \otimes_R S(-1)\right)\right)_d \\ &\simeq \left(\wedge^i_R R^r \otimes S(-i)\right)_d.\end{aligned}$$

Therefore, $K_\bullet(g)_d$ is

(3.2)

$$0 \longleftarrow S_d \xleftarrow[\delta]{} S_{d-1} \otimes_R R^r \xleftarrow[\delta]{} S_{d-2} \otimes_R \wedge^2 R^r \longleftarrow \cdots \xleftarrow[\delta]{} S_{d-r} \otimes \wedge^r R^r \longleftarrow 0$$

as a complex of R-modules.

Next, we dualize this complex. That is, we apply $\mathrm{Hom}(\bullet, R)$.

(3.3)

$$0 \longleftarrow S_{d-r}^{\vee} \longleftarrow S_{d-r+i}^{\vee} \otimes R^r \longleftarrow S_{d-r+2} \otimes \wedge^2 R^r \longleftarrow \cdots \otimes R^r \longleftarrow S_d^{\vee} \otimes \wedge^r R^r \longleftarrow 0.$$

This follows from the identification

$$\wedge^i R^r \simeq \wedge^{r-i} (R^r)^{\vee}.$$

Consider now the special case $d = r - s$. Only look at the last $r - s$ terms of $K_{\bullet}(g)_{r-s}^{\vee}$. We obtain

(3.4)

$$0 \longrightarrow S_{r-s}^{\vee} \otimes \wedge^r R^r \xrightarrow{d} S_{r-s-1}^{\vee} \otimes \wedge^{r-1} R^r \longrightarrow \cdots \longrightarrow \wedge^s R_r \longrightarrow 0$$

and we may note

$$\wedge^s R^r \simeq S_0^{\vee} \otimes \wedge^s R^r.$$

To get the Eagon-Northcott complex, we extend the length of this complex by one, via adjoining $\wedge^s R^r \xrightarrow{\wedge^s f} \wedge^s R^s \simeq R$ for the map $f : R^r \to R^s$. Altogether, we get

(3.5)

$$0 \longrightarrow S_{r-s}^{\vee} \otimes \wedge^r R^r \xrightarrow{d} S_{r-s-1}^{\vee} \otimes \wedge^{r-1} R^r \longrightarrow \cdots \longrightarrow \wedge^s R_r \xrightarrow{\wedge^s f} \wedge^s R^s \longrightarrow 0.$$

Proposition 3.1. *The composition of any two maps in Equation 3.5 is zero. That is, it is a complex.*

Proof. From the construction, this automatically holds at every term, except possibly the last one. That is, it only remains to show the composition

$$S_1^{\vee} \otimes \wedge^{s+1} R^r \xrightarrow{d} S_0^{\vee} \otimes \wedge^s R^r \xrightarrow{\wedge^s f} \wedge^s R^s \tag{3.6}$$

is zero. Dualizing, we need to show

$$R \xrightarrow{\varepsilon} \wedge^{r-s} R^r \xrightarrow{\delta} R^s \otimes \wedge^{r-s-1} R^r \tag{3.7}$$

is 0. We can use the identifications

$$\begin{tikzcd} R \arrow[r, "\varepsilon"] \arrow[d] & \wedge^{r-s} R^r \arrow[r, "\delta"] \arrow[d] & R^s \otimes \wedge^{r-s-1} R^r \arrow[d] \\ (\wedge^s R^s)^{\vee} \arrow[r] & \wedge^s (R^r)^{\vee} \arrow[r] & S_1. \end{tikzcd} \tag{3.8}$$

This composition corresponds to an element of $\mathrm{Hom}\left(\wedge^{s+1} R^r, R^s\right)$.
Let $e_1, \ldots, e_r$ be a basis for R^r. We have

$$\delta \circ \varepsilon(1)\left(e_{i_1} \wedge \cdots \wedge e_{i_{s+1}}\right) = \sum (-1)^{p+1} \varepsilon(1)\left(e_{i_1} \wedge \cdots \wedge \widehat{e_{i_p}} \wedge \cdots \wedge e_{i_{s+1}}\right) f\left(e_{i_p}\right)$$

This is some element of R^s since

$$\sum (-1)^{p+1} \varepsilon(1)\left(e_{i_1} \wedge \cdots \wedge \widehat{e_{i_p}} \wedge \cdots \wedge e_{i_{s+1}}\right)$$

is an element of R. This follows from the definition. Then,

$$\varepsilon(1) \in \mathrm{Hom}\left(\wedge^s R^r, R\right),$$

so

$$\begin{aligned} \delta \circ \varepsilon(1)\left(e_{i_1} \wedge \cdots \wedge e_{i_{s+1}}\right) &= \sum (-1)^{p+1} \varepsilon(1)\left(e_{i_1} \wedge \cdots \wedge \widehat{e_{i_p}} \wedge \cdots \wedge e_{i_{s+1}}\right) f\left(e_{i_p}\right) \\ &= \sum (-1)^{p+1} \left(f(e_{i_1}) \wedge \cdots \wedge \widehat{f(e_{i_p})} \wedge \cdots \wedge f\left(e_{i_{s+1}}\right)\right) \cdot f\left(e_{i_p}\right). \end{aligned}$$

Then, let A be the $s \times r$ matrix representing f. Let A_m be the mth column of A. Introduce the notation

$$A_{\{m_1, \ldots, m_\ell\}} := \begin{pmatrix} A_{m_1} & \cdots & A_{m_\ell} \end{pmatrix}.$$

Then, we to check the composition is zero, it suffices to verify the identity

$$\sum (-1)^{p+1} \det\left(A_{i_1, \ldots, \widehat{i_p}, \ldots, i_{s+1}}\right) A_{i_p} = 0.$$

Let's start with an example:

Example 3.2. Take

$$A := \begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

We obtain

$$\left|\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}\right| \begin{pmatrix} 4 \\ 7 \end{pmatrix} - \left|\begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix}\right| \begin{pmatrix} 5 \\ 8 \end{pmatrix} + \left|\begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}\right| \begin{pmatrix} 6 \\ 9 \end{pmatrix}.$$

The first entry being 0 is saying that

$$\left|\begin{pmatrix} 4 & 5 & 6 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\right| = 0$$

and the second entry of the vector is 0 because

$$\left|\begin{pmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\right|.$$

This example easily generalizes, showing the composition is 0. $\square$

To recap, here is our definition:

Definition 3.3. Let $f : R^r \to R^s$. The **Eagon-Northcott complex** is the complex
(3.9)
$$0 \longleftarrow R \xleftarrow[\wedge^s f]{} \wedge^s R^r \xleftarrow[d]{} S_1^\vee \otimes \wedge^{s+1} R^r \longleftarrow \cdots \longleftarrow S_{r-s}^\vee \otimes \wedge^r R^r \longleftarrow 0$$
and $S = R\left[x_1, \ldots, x_n\right]$.

Remark 3.4. Next time we will find a criterion for the Eagon Northcott complex to be exact. The key to proving exactness will be the Buchsbaum-Eisenbud criterion for exactness.

Definition 3.5. Let $\psi : R^r \to R^s$ be a map of free R-modules. We define
$$I_j(\phi) \subset R$$
to be the ideal generated by the $j \times j$ minors.

Intrinsically, $I_j(\psi)$ is the ideal given by the image of the map
$$\wedge^j R^r \otimes \wedge^j (R^s)^\vee \to R.$$
This can be thought of as an element of
$$\mathrm{Hom}\left(\wedge^j R^s, \wedge^j R^s\right)$$
given by $\wedge^j \phi$. The **rank** of ϕ, notated $\mathrm{rk}(\phi)$ is the greatest integer j so that $I_j(\phi) \neq 0$. Then,
$$I(\phi) := I_{\mathrm{rk}(\phi)}(\phi).$$

Proposition 3.6 (Proposition 20.8, Eisenbud's commutative algebra book)**.** *If $\phi : R^r \to R^s$ is a morphism, then cokerϕ is projective if and only if $I(\phi) = R$. In this case, cokerϕ has rank $s - \mathrm{rk}\,\phi$.*

Proof. Omitted. □

Next time, we'll apply following criterion for exactness of a complex to the Eagon-Northcott complex.

Theorem 3.7 (Buchsbaum-Eisenbud)**.** *Let*
$$F_0 \xleftarrow[f_1]{} F_1 \xleftarrow[f_2]{} F_2 \longleftarrow \cdots \xleftarrow[f_n]{} F_n \longleftarrow 0 \tag{3.10}$$
be a complex. Assume

(1)
$$\mathrm{rk}\left(F_k\right) = \mathrm{rk}\, f_k + \mathrm{rk}\, f_{k+1}$$
and

(2)

$$\operatorname{depth} I(f_k) \geq k$$

for $k = 1, \ldots, n$.

then $F_\bullet$ *is exact.*

Proof. Omitted. □

Definition 3.8. Recall that the depth of an ideal I is the maximal length of a regular sequence $x_i \in R$ with each $x_i \in I$.

4. 4/12/17

Today's goal is the Buchsbaum Eisenbud criterion for exactness. There are two main ingredients:

(1) Fitting ideals
(2) The Peskine-Szpiro lemma

4.1. **Fitting ideals.** We'll just state their definition and properties without proof. Given a matrix

$$\phi : R^r \to R^s,$$

let the ideal $I_j\phi \subset R$ be the ideal generated by the $j \times j$ minors of ϕ.

Definition 4.1 (Fitting ideal)**.** Let M be a finitely generated R-module for R noetherian (in the future we will assume R noetherian without comment). Choose a presentation

$$R^a \xrightarrow{\phi} R^b \longrightarrow M \longrightarrow 0. \tag{4.1}$$

Then,

$$\mathrm{Fitt}_i(M) := I_{b-i}(\phi).$$

Proposition 4.2. *Let* M *be a finitely generated* R*-module. Then,*

(1) $\mathrm{Fitt}_i(M)$ *is well defined (i.e., independent of choice of resolution)*
(2) Fitting ideals are functorial, meaning that for a maps of rings $f : R \to S$*, we have*

$$\mathrm{Fitt}_j (M \otimes_R S) = f(\mathrm{Fitt}_j(M)) \subset S.$$

(3) As a consequence of the previous point, fitting ideals commute with localization.

Proof. Omitted. □

Remark 4.3. Recall that $\operatorname{rk}\phi : F \to G$ is by definition
$$\operatorname{rk}\phi := \max_i \left\{ j : I_j(\phi) \neq 0 \right\}.$$
and
$$I(\phi) := I_{\operatorname{rk}\phi}(\phi).$$
If M is a finitely generated R-module with presentation ϕ, we have $I(M) := I(\phi)$.

Warning 4.4. $I(\phi)$ need not commute with localization because it may be that $\operatorname{rk}\phi_{\mathfrak{p}} < \operatorname{rk}\phi$.

Remark 4.5. If we assume that $I(\phi)$ contains a nonzero divisor then
$$I(\phi)_{\mathfrak{p}} \neq 0$$
for all $\mathfrak{p} \in \operatorname{Spec} R$. This implies that $\operatorname{rk}(\phi_{\mathfrak{p}}) \geq \operatorname{rk}(\phi)$, which implies $\operatorname{rk}\phi_{\mathfrak{p}} = \operatorname{rk}\phi$ and so by Proposition 4.2
$$I(\phi)_{\mathfrak{p}} = I(\phi_{\mathfrak{p}}).$$

Lemma 4.6. *Let M be a finitely generated R-module. Then, M is projective of constant rank if and only if $I(M) = R$. In this case,*
$$\operatorname{rk}(M) = b - \operatorname{rk}\phi$$
for
$$\phi : R^a \to R^b$$
a presentation of M.

Proof. See Eisenbud's commutative algebra book. □

4.2. **The Buchsbaum Eisenbud criterion.** Here is the setup for the Buchsbaum Eisenbud criterion for exactness.

We first recall some definitions:

Definition 4.7. Let M be a finitely generated R-module. A sequence $f_1, \ldots, f_r$ in R is M**-regular** if f_i is a nonzero divisor on M for $M/(f_1, \ldots, f_{i-1})M$ for $i = 1, \ldots, r$ and $M/(f_1, \ldots, f_r)M \neq 0$.

Definition 4.8. Let $I \subset R$ be an ideal. Then,
$$\operatorname{depth}_I(M) := \begin{cases} \text{maximal length of an } M\text{-regular sequence in } I & \text{if } IM \neq M \\ \infty & \text{if } IM = M. \end{cases}$$

If R is local then
$$\operatorname{depth}(M) := \operatorname{depth}_{\mathfrak{m}}(M).$$

Lemma 4.9. *Let $(R, \mathfrak{m})$ be a local ring. Suppose*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \tag{4.2}$$

is a short exact sequence of finitely generated R-modules. Then,

(1)
$$\operatorname{depth}(B) \geq \min(\operatorname{depth} A, \operatorname{depth} C).$$

(2)
$$\operatorname{depth}(C) \geq \min(\operatorname{depth} B, \operatorname{depth} A - 1)$$

(3)
$$\operatorname{depth} A \geq \min(\operatorname{depth} B, \operatorname{depth} C + 1).$$

Proof. This follows from the characterization of depth in terms of Ext. (Recall $\operatorname{depth}(M) = \min_i \operatorname{Ext}_i(k, M) \neq 0$.) □

Lemma 4.10 (Peskine-Szpiro)**.** *Let R be a local ring and let*

$$0 \longrightarrow \mathcal{F}_n \xrightarrow{f_n} F_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{F}_1 \xrightarrow{f_1} \mathcal{F}_0 \tag{4.3}$$

be a complex with $\mathcal{F}_i$ a finitely generated R-module. Suppose

(1) $\operatorname{depth}(\mathcal{F}_i) \geq i$ *and*
(2) $\operatorname{depth} H_j\mathcal{F} = 0$ *for* $j > 0$.

Then, $\mathcal{F}_\bullet$ is exact.

Proof. Suppose $\mathcal{F}_\bullet$ is not exact. Let $i > 0$ be the largest i so that $H_i\mathcal{F}$ is nonzero. If $i = n$, then

$$H_n\mathcal{F} \subset \mathcal{F}_n.$$

But, $\operatorname{depth}(\mathcal{F}_n) \geq n > 0$ by Lemma 4.9 applied to $A = H_n(\mathcal{F}), B = \mathcal{F}_n, C = \operatorname{im} f_n$.

So, we may assume $i < n$. Let $i < n$. As the complex is exact to the left of $\mathcal{F}_i$ by induction. We then have a short exact sequence

$$0 \longrightarrow \operatorname{im} f_{j+1} \longrightarrow \mathcal{F}_j \longrightarrow \operatorname{im} f_j \longrightarrow 0 \tag{4.4}$$

using that for $i < j \leq n$. From Lemma 4.9, we have

$$\begin{aligned}
\operatorname{depth} \operatorname{im} F_j &\geq \min\left(j, \operatorname{depth} \operatorname{im} f_{j+1} - 1\right) \\
&\geq \min\left(j, \operatorname{depth} \operatorname{im} f_{j+2} - 2\right) \\
&\geq \min\left(j, \operatorname{depth}\left(\operatorname{im} f_n\right) - (n-j)\right). \\
&\geq \min\left(j, \operatorname{depth}\left(\mathcal{F}_n\right) - (n-j)\right). \\
&\geq \min\left(j, n - (n-j)\right). \\
&\geq j.
\end{aligned}$$

But, we also have the exact sequence

$$0 \longrightarrow \operatorname{im} f_{i+1} \longrightarrow \ker f_i \longrightarrow H_i\mathcal{F} \longrightarrow 0. \tag{4.5}$$

By assumption, $\operatorname{depth} H_i\mathcal{F} = 0$ but $H_i\mathcal{F} \neq 0$. Therefore,

$$\begin{aligned}
\operatorname{depth} H_i\mathcal{F} &= 0 \\
&\geq \min\left(\operatorname{depth} \ker f_i, i\right)
\end{aligned}$$

This can only happen if $\operatorname{depth} \ker f_i = 0$. Note that we have $i > 0$ here. This contradicts that

$$\ker f_i \subset \mathcal{F}_i$$

so

$$\operatorname{depth} \ker f_i \geq 1.$$

since $\operatorname{depth} \mathcal{F}_i \geq i$. □

We now come to a useful criterion for the exactness of a complex.

Theorem 4.11 (Buchsbaum-Eisenbud)**.** *Let*

$$0 \longrightarrow \mathcal{F}_n \xrightarrow{f_n} F_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{F}_1 \xrightarrow{f_1} \mathcal{F}_0 \tag{4.6}$$

be a complex of free finite R*-modules. Suppose*

(1)

$$\operatorname{rk} \mathcal{F}_i = \operatorname{rk} f_i + \operatorname{rk} f_{i+1}$$

(2) if $I(f_i) \neq R$*, then* $\operatorname{depth} I(f_i) \geq i$*. for* $i \geq 1$.

Then, $\mathcal{F}_\bullet$ *is exact.*

Remark 4.12. In fact, this is an if and only if statement, but we only need one direction, so we only state and prove that direction.

Proof. The second assumption guarantees $I(f_i)$ has a nonzero divisor. So, by Remark 4.5, we know the assumptions are preserved under localization. Therefore, we may assume R is local.

Let's deal with exactness at F_i. That is, we want to show

$$\mathcal{F}_{i+1} \xrightarrow{f_{i+1}} \mathcal{F}_i \xrightarrow{f_i} \mathcal{F}_{i-1} \tag{4.7}$$

is exact.

We first consider the case $i > d := \operatorname{depth} R$, we have

$$I(f_{i+1}) = I(f_i) = R,$$

by the second assumption.

By Lemma 4.6, we have $\operatorname{coker}\mathcal{F}_i$ is free of rank

$$\operatorname{rk}\operatorname{coker}\mathcal{F}_i = \operatorname{rk}\mathcal{F}_i - \operatorname{rk} f_{i+1}.$$

Construct

$$F_i \longrightarrow \operatorname{coker}F_i \xrightarrow{\widetilde{f_i}} F_{i-1}. \tag{4.8}$$

To proving exactness at F_i, it suffices to show $\widetilde{f_i}$ is injective. We see that

$$\begin{aligned}\operatorname{rk}\widetilde{f_i} &= \operatorname{rk} f_i\\ &= \operatorname{rk}\mathcal{F}_i - \operatorname{rk} f_{i+1}\\ &= \operatorname{rk}\operatorname{coker}\mathcal{F}_i\end{aligned}$$

We hence have

$$I(\widetilde{f_i}) = I(f_i) = R.$$

Then, dualizing the sequence

$$I\left(\widetilde{f_i}^{\vee}\right) = R.$$

By Lemma 4.6 we have $\operatorname{coker}\widetilde{f_i}^{\vee}$ is free of rank

$$\begin{aligned}\operatorname{rk}\operatorname{coker}\mathcal{F}_i - \operatorname{rk}\widetilde{f_i}^{\vee} &= \operatorname{rk}\widetilde{f_i}\\ &= 0.\end{aligned}$$

This implies $\widetilde{f_i}^{\vee}$ is surjective so $\widetilde{f_i}$ is injective.

To conclude, we only need prove the case that $i \le d$. In this case, we will apply Lemma 4.10. By truncating $\mathcal{F}_\bullet$, and replacing $\mathcal{F}_d$ with $\operatorname{coker} f_{d+1}$, we may assume that $\mathcal{F}_\bullet$ has length at most d. That is, we may assume $n \le d$.

Without generality, we have that R is local.

Lemma 4.13. *Suppose $\mathcal{F}_\bullet$ satisfies the following condition: $(\mathcal{F}_\bullet)_\mathfrak{p}$ is exact for every $\mathfrak{p} \neq \mathfrak{m}$ in the local ring $(R, \mathfrak{m})$.*

Proof. By this hypothesis, we have

$$\operatorname{Supp}(H_i \mathcal{F}_\bullet) \subset \{\mathfrak{m}\}.$$

Therefore, there is some ℓ for which $\mathfrak{m}^\ell H_i \mathcal{F}_\bullet = 0$. That is,

$$\operatorname{depth} H_i \mathcal{F} = 0.$$

Note that each $\mathcal{F}_i$ is free, so $\operatorname{depth} \mathcal{F}_i = d \geq i$. Therefore, by Lemma 4.10, we know $\mathcal{F}_\bullet$ is exact. □

To complete the proof, it suffices to reduce to the case that $(\mathcal{F}_\bullet)_\mathfrak{p}$ is exact for every $\mathfrak{p} \neq \mathfrak{m}$.

In the general case, we induct on $\dim R$. If $\dim R = 0$, then we are done as there are no primes other than $\mathfrak{m}$. If $\dim R = n + 1$ then $\dim R_\mathfrak{p} < \dim R$. Therefore, by the induction hypothesis for the ring $R_\mathfrak{p}$, which is of lower dimension, we know that $(\mathcal{F}_\bullet)_\mathfrak{p}$ has vanishing cohomology, and therefore the same follows for $\mathcal{F}_\bullet$ by Lemma 4.13. □

5. 4/14/17

We'll now discuss some applications of the Buchsbaum-Eisenbud criterion for exactness of a complex in a geometric setting. In particular, we'll examine the relation to the Eagon-Northcott complex. Let $f : R^r \to R^s$ for $r \geq s$. Recall we constructed an Eagon-Northcott complex associated to f Eagon-Northcott(f) from the complex

(5.1)

$$0 \longleftarrow R \underset{\wedge^s f}{\longleftarrow} \wedge^s R^r \underset{d}{\longleftarrow} S^\vee \otimes \wedge^{s+1} R^r \longleftarrow S_2^\vee \otimes \wedge^{s+2} R^r \longleftarrow \cdots \longleftarrow S_{r-s}^\vee \wedge^r R^r \longleftarrow 0$$

where we are using that $R \simeq \wedge^s R^s$ and $S = R[x_1, \ldots, x_s]$.

We state the following theorem without proof, though we will come back to it in Corollary 6.3.

Theorem 5.1 (Eagon-Northcott)**.** *Assume* $\operatorname{depth} I_s(f) \geq r + 1 - s$. *Then,* Eagon-Northcott(f) *is exact.*

5.1. **Rational normal curves.** Recall a rational normal curve $C \subset \mathbb{P}^d$ is the embedding from $\mathbb{P}^1 \xrightarrow{\phi_L} \mathbb{P}^d$ for $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(d)$. Recall that C is smooth, rational, degree d, nondegenerate, The ideal of C is generated by the equations $x_i x_j - x_{i-1} x_{j+1}$. It's clear that the rational normal curves lies in the intersection of these equations, and you can check it actually is the vanishing of these equations by computing the Hilbert polynomial (in fact, these equations already form a Grobner basis, so you can just use the leading terms).

Let

$$A = \begin{pmatrix} x_0 & \cdots & x_{d-1} \\ x_1 & \cdots & x_d \end{pmatrix}$$

be a $2 \times d$ matrix of linear form. Let I_C be the ideal generated by the 2×2 minors of A. Let $R = \mathbb{C}[x_0, \ldots, x_d]$ (be the graded ring). View A as a map $f : R^d(-1) \to R^2$. Let $I_C = I_2(f)$ as C has codimension $d-1$.

As C has codimension $d-1$, we have $\operatorname{depth} I_2(f) = d-1$. Therefore, by Theorem 5.1, Eagon-Northcott(f) is exact.

Let $S = R[y_1, y_2]$. Recall that Eagon-Northcott(f) is the complex

(5.2)
$$\begin{aligned} R \xleftarrow[\wedge^2 f]{} \wedge^2 R^d(-2) \longleftarrow \left(R^2\right)^\vee \otimes \wedge^3 R^d(-3) \\ \longleftarrow (\operatorname{Sym}_2 R^2)^\vee \otimes \wedge^4 R^d(-4) \longleftarrow \cdots \longleftarrow (\operatorname{Sym}_{d-2} R^2)^\vee \otimes \wedge^d R^d(-d) \longleftarrow 0 \end{aligned}$$

using that $S_1^\vee \simeq R^2$ by identifying y_1, y_2 with the basis of R^2 and identifying $S_2 \simeq \operatorname{Sym}_2 R^2$. Note that we have an exact complex

$$0 \longleftarrow R/I_C \longleftarrow \text{Eagon-Northcott}(f) \tag{5.3}$$

Let $S = k[x_0, \ldots, x_n]$. Recall that a graded free exact S-complex $(F_\bullet, d_\bullet)$ is minimal if each d_i takes a basis of F_i to a minimal set of generators of $\operatorname{im} d_i$.

Proposition 5.2. *Let $F_\bullet$ be a graded free exact complex. Then $F_\bullet$ is minimal if and only if im d_i is contained in $(x_0, \ldots, x_n) F_{i-1}$.*

Proof. The proof uses the graded Nakayama lemma. We recall it now:

Lemma 5.3 (Graded Nakayama lemma)**.** *Suppose M is a finitely generated graded S-module. Let $\mathfrak{m} = (x_0, \ldots, x_n)$. If $a_1, \ldots, a_r$ generate $M/\mathfrak{m}M$. Then $a_1, \ldots, a_r \in \mathfrak{m}$ generate M.*

Proof. The idea is to look for elements of least degree in $M/(a_1, \ldots, a_r)$.

Exercise 5.4. Flush this idea out to give a proof.

□

By the graded Nakayama lemma, we know that $F_\bullet$ is minimal, if and only if

$$F_i/\mathfrak{m}F_i \xrightarrow{d_i} \text{im } d_i/\mathfrak{m}\text{im } d_i.$$

is an isomorphism. Note that we have an exact sequence

$$F_{i+1} \longrightarrow F_i \longrightarrow \text{im } d_i \longrightarrow 0 \tag{5.4}$$

The statement that d_i above defines an isomorphism is equivalent to the map d_{i+1}

$$F_{i+1}/\mathfrak{m}F_{i+1} \xrightarrow{d_{i+1}} F_i/\mathfrak{m}F_i$$

is 0. That is, im $d_{i+1} \subset (x_0, \ldots, x_d) F_i$. □

If $S^a(c) \xrightarrow{f} S^b(c+1)$ is a morphism then f is represented by a matrix of linear forms.

Corollary 5.5. *We have that* Eagon-Northcott(f) *is minimal.*

Proof. This is just because at each step we are multiplying by linear forms, so the result follows from Proposition 5.2. □

6. 4/19/17

We want to find the syzygies of a k gonal curve, meaning a curve C with a map $f : C \to \mathbb{P}^1$. Let $A := f^*\mathcal{O}_{\mathbb{P}^1}(1)$. We have a scroll X_A with $C \subset X_A \subset \mathbb{P}^{g-1}$ and a map

$$H^0(A) \otimes H^0(\omega_C \otimes A^\vee) \to H^0(\omega_C).$$

We also have a map $\mathbb{P}\mathcal{E}_f(-2) \to \mathbb{P}^1$. We have an exact sequence

$$\mathcal{E}_f \longrightarrow f_*\omega_f \longrightarrow \mathcal{O}^1_{\mathbb{P}}. \tag{6.1}$$

Let H be a hyperplane class and let R be the ruling in the scroll. We have

$$h^0(C, A) \simeq h^0 \left(\mathbb{P}\left(\mathcal{E}_f(-2)\right), R\right)$$

and

$$H^0\left(C, \omega_C \otimes A^\vee\right) \simeq H^0\left(\mathbb{P}\left(\mathcal{E}_f\left(-2\right)\right), H - R\right).$$

We also have a map

$$\begin{tikzcd} C \arrow[rr] \arrow[dr] & & \mathbb{P}\left(\mathcal{E}_f(-2)\right) \arrow[dl] \\ & \mathbb{P}^{g-1}. & \end{tikzcd} \tag{6.2}$$

We have that

$$j\left(\mathbb{P}\left(\mathcal{E}_f(-2)\right)\right) \subset X_A.$$

Recall that $H^0\left(C, A^2\right) = 3$, meaning j is an embedding.

Proposition 6.1. *We have* $\mathbb{P}\left(\mathcal{E}_f(-2)\right) \simeq X_A$.

Proof. Recall that X_A was defined so that $\{u, v\}$ is a basis of $H^0(C, A)$ and $t_1, \ldots, t_\ell$ are a basis of $H^0(\omega_C - A)$, with $\ell = g + 1 - k$.

We have that I_{X_A} is generated by the 2×2 minors of

$$M := \begin{pmatrix} \psi_{ut_1} & \cdots & \psi_{ut_\ell} \\ \psi_{rt_1} & \cdots & \psi_{rt_\ell} \end{pmatrix}$$

with $\psi_{ut_i} \in H^0(\mathbb{P}^{g-1}, \mathcal{O}(1)) \simeq H^0(C, \omega_C)$. It suffices to show X_A is irreducible of dimension $b - 1 = \dim \mathbb{P}\left(\mathcal{E}_f(-2)\right)$.

Let

$$V := \operatorname{im}\left(H^0(A) \otimes H^0(\omega_C - A) \to H^0(C, \omega_C)\right).$$

We have $\mathbb{P}V^\vee \subset \mathbb{P}^{g-1}$ with $H^0(\mathbb{P}V^\vee, \mathcal{O}(1)) \simeq V$. Then, X_A is a cone over the projective variety $\mathbb{P}V^\vee \simeq \mathbb{P}^r$ defined by the 2×2 minors of $M_{\mathbb{P}V^\vee}$.

To prove X_A is irreducible of codimension $g - k = \ell - 1$, we may assume the multiplication map

$$V \simeq H^0(\mathbb{P}^{g-1}, \mathcal{O}(1)),$$

since a cone over a variety is irreducible of codimension d if and only if the variety is irreducible of codimension d.

Then, M defines a morphism of projective bundles

$$\psi : \mathcal{O}^{g+1-k}(-1) \to \mathcal{O}^2,$$

and X_A is the degeneracy locus of ψ. That is, X_A is the locus where ψ does not have full rank. Restricting

$$\psi|_{X_A} : \mathcal{O}^{g+1-k}(-1) \to \mathcal{O}^2$$

has rank at most 1. Since the multiplication map

$$H^0(A) \otimes H^0(\omega_C - A) \to H^0(C, \omega_C)$$

is surjective, so $V \simeq H^0(\mathbb{P}^{g-1}, \mathcal{O}(1))$, it is not possible for all entries of M to vanish.

One can verify that $\operatorname{coker}\psi$ is a line bundle, call it $\mathcal{L}$.

We have

$$\mathcal{O}(-1)^{g+1-k} \longrightarrow \mathcal{O}^2 \longrightarrow L \longrightarrow 0. \tag{6.3}$$

The second map corresponds to two sections of a line bundle L, which defines a morphism $X_A \to \mathbb{P}^1$. The fiber over

$$[s:t] \in \mathbb{P}^1$$

is those points $p \in X_A$ so that there exists f so that the diagram

$$\begin{tikzcd} \mathcal{O}^2 \arrow[rr, "{(s,t)}"] \arrow[dr] & & \mathcal{O} \arrow[dl, "f"] \\ & L & \end{tikzcd} \tag{6.4}$$

commutes. This, in turn, is true if and only if the composition

$$\mathcal{O}(-1)^{g+1-k} \longrightarrow \mathcal{O}^2 \xrightarrow{(s,t)} \mathcal{O} \tag{6.5}$$

If

$$z := su + tv \in H^0(C, A)$$

then the fiber over $[s:t]$ is the locus where the composition is 0, which is the same as saying that $\psi_{zt_i} = 0$ for all t_i. Recall we have $g+1-k$ independent linear forms. The fiber is thus a linear space of codimension ℓ. This implies that X_A has codimension $\ell - 1$ (because these linear spaces are distinct). Since each fiber of $X_A \to \mathbb{P}^1$ is irreducible. □

Remark 6.2. If we drop the assumption that $H^0(C, A^2) = 3$, then j is no longer an isomorphism, but

$$j(\mathbb{P}(\mathcal{E}(-2))) = X_A,$$

where j may no longer be an embedding, but X_A is a cone over a smooth scroll.

In fact, X_A has only rational singularities, meaning loosely that you can compute the cohomology of line bundles on X_A from line bundles on $\mathcal{E}_f(-2)$.

Corollary 6.3. *The complex* Eagon-Northcott(X_A) *is exact.*

We have

$$b_{i,j}\left(\mathcal{O}_{X_A}\right) = \begin{cases} 1 & \text{if } i = j = 0 \\ i \cdot \binom{g+1-k}{i+1} & \text{if } j = 1, i > 0 \\ 0 & \text{else} \end{cases}$$

For a projective variety Z, define

$$\ell(Z) := \min\left\{m : b_{p,1}(\mathcal{O}_Z) = 0, p > m\right\}.$$

called the length of the 2**-linear strand**. For brevity, we will just call this the length of the linear strand.

Conjecture 6.4 (Green)**.** Suppose C has gonality k and $\mathrm{Cliff}(C) = k-2$. Then,

$$\phi_{\omega_C} : C \to \mathbb{P}^{g-1}$$

for $A \in W^1_k$. Then,

$$\ell\left(\Gamma_C(\omega_C)\right) = \ell\left(\mathcal{O}_{X_A}\right) = g-k.$$

There is a refinement of this due to Schreyer.

Conjecture 6.5 (Schreyer)**.** If W^1_k is a reduced point A and A is the unique line bundle line bundle of degree at most $k-1$ with $\mathrm{Cliff}\, C = k-2$, then

$$b_{g-k,1}\left(C, \omega_C\right) = b_{g-k,1}\left(X_A, \mathcal{O}(1)\right) = g-k.$$

6.1. **Picard group of $\mathcal{M}_g$.**

Definition 6.6. A line bundle on $\mathcal{M}_g$ consists of the following data:

(1) For each $\pi : \mathcal{C} \to S$, a line bundle $\ell(\pi)$
(2) For each morphism $f : S_1 \to S_2$,

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, "F"] \arrow[d, "\pi_1"] & \mathcal{C}_2 \arrow[d, "\pi_2"] \\ S_1 \arrow[r, "f"] & S_2 \end{tikzcd} \tag{6.6}$$

we are given an isomorphism

$$L(F) : L(\pi_1) \simeq F^* L(\pi_1)$$

satisfying the cocycle condition. In more detail, if we have a composition

$$
\begin{array}{ccccc}
C_1 & \xrightarrow{F} & C_2 & \xrightarrow{G} & C_3 \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_3} \\
S_1 & \xrightarrow{f} & S_2 & \xrightarrow{g} & S_3
\end{array} \tag{6.7}
$$

then the diagram

$$
\begin{array}{ccc}
L(\pi_1) & \longrightarrow & f^* L(\pi_2) \\
\downarrow & & \downarrow \\
(g \circ f)^* \left(L(\pi_3) \right) & \longrightarrow & f^* g^* \left(L(\pi_3) \right).
\end{array} \tag{6.8}
$$

Remark 6.7. There is a natural notion of isomorphism of line bundles. There is also a notion of tensor product. Therefore, Pic $(\mathcal{M}_g)$ is an abelian group.

Example 6.8. One interesting example of a vector bundle is the Hodge bundle which for a family $\pi : \mathcal{C} \to S$, we have

$$L(\pi) := \det(\pi_* \omega_\pi)$$

is a vector bundle of rank g (if S is reduced this follows by Grauert's theorem). And further of ω_π commutes with base change (see Liu, Ch. 6, Thm 4.9).

7. 4/21/17

Today, we'll have an introduction to divisor calculations on $\mathcal{M}_g$.

7.1. **Grothendieck Riemann Roch.** To start, we'll have a brief introduction to Grothendieck-Riemann Roch. See for example, Hartshorne, appendix A. Assume X and Y are smooth and quasi-projective. Suppose we have a proper morphism $\pi : X \to Y$. Then, $K(X)$ is the free group generated by coherent sheaves modulo the relations that on X if we have a short exact sequence

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0 \tag{7.1}$$

we impose the relation $[\mathcal{F}_2] = [\mathcal{F}_1] + [\mathcal{F}_3]$. Note that this is a ring, and we can replace all coherent sheaves by locally free sheaves if we'd like.

For $\mathcal{F} \in \mathrm{Coh}(X)$. We have

$$\pi_! \mathcal{F} := \sum (-1)^i R^i \pi_* \mathcal{F} \in K(Y).$$

Definition 7.1. For E a vector bundle, we define the **total Chern class**

$$c_t(E) := \sum_i c_i(E) t^i$$

with

$$c_i(E) \in A^*(X, \mathbb{Q}).$$

the Chern class.

Remark 7.2. By the splitting principle, we can write

$$\begin{aligned} c_t(E) &:= \sum_i c_i(E) t^i \\ &= \prod_i (1 + \alpha_i t). \end{aligned}$$

Definition 7.3. If E is a vector bundle with $c_t(E) = \prod (1 + \alpha_i t)$, then the **Chern character**

$$\mathrm{Ch}(E) = \sum_i e^{\alpha_i}$$

Example 7.4. Viewing $\mathrm{Ch}(E)$ as an element of the graded ring $\mathrm{Ch}_i(E)$ is the ith graded piece of $\mathrm{Ch}(E)$. We have

$$\begin{aligned} \mathrm{Ch}_0(E) &= \mathrm{rk}\, E \\ \mathrm{Ch}_1(E) &= c_1(E). \end{aligned}$$

We have

$$\mathrm{Ch}_2(E) = \frac{c_1^2 - 2c_2}{2}.$$

Lemma 7.5. *The Chern class defines a ring homomorphism*

$$\begin{aligned} K(X) &\to A^*(X) \\ F &\mapsto \mathrm{Ch}(F). \end{aligned}$$

Proof.

Exercise 7.6. Verify this.

□

Definition 7.7. Let E be a vector bundle with $c_t(E) = \prod_i (1 + \alpha_i t)$. We define the **Todd class**

$$\mathrm{Td}(E) := \prod_i \frac{\alpha_i}{1 - e^{-\alpha_i}}.$$

Remark 7.8. The Todd class is multiplicative, so that if we have

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0 \tag{7.2}$$

then

$$\mathrm{Td}\left(\mathcal{F}_2\right) = \mathrm{Td}\left(\mathcal{F}_1\right)\mathrm{Td}\left(\mathcal{F}_3\right)$$

Theorem 7.9 (Grothendieck Riemann Roch)**.** *Suppose $\pi : X \to Y$ is proper. Then,*

$$\mathrm{Ch}\,\pi_! \mathcal{F} = \pi_*\left(c\,\mathcal{F}\,\mathrm{Td}(T_\pi)\right)$$

Let $\mathcal{F} \in K(X)$. where $T_\pi = T_X - \pi^* T_Y$.

Theorem 7.10 (Mumford's formula)**.** *Let $\pi : \mathcal{C} \to \mathcal{M}_g$ be the universal curve. Let*

$$\lambda := c_1\left(\pi_* \omega_\pi\right)$$

be the hodge class and

$$\kappa := \pi_*\left(c_1 \omega_\pi \cdot c_1 \omega_\pi\right)$$

be the kappa class. Then,

$$\kappa = 12\lambda.$$

Proof. We have

$$\begin{aligned} \pi_! \omega_\pi &= \pi_* \omega_\pi - R^1 \pi_* \omega_\pi \\ &= \pi_* \omega_\pi - \left(\pi_* \mathcal{O}_C\right)^\vee \\ &= \pi_* \omega_\pi - \mathcal{O}_{\mathcal{M}_g}^\vee \\ &= \pi_* \omega_\pi + \mathcal{O}_{\mathcal{M}_g} \end{aligned}$$

This implies

$$\begin{aligned} \mathrm{Ch}\left(\pi_! \omega_\pi\right) &= \mathrm{Ch}\left(\pi_* \omega_\pi\right) + \mathrm{Ch}\left(\mathcal{O}_{\mathcal{M}_g}\right) \\ &= \mathrm{Ch}\left(\pi_* \omega_\pi\right) + 1 \\ &= g + 1 + c_1(\pi_* \omega_\pi) + \frac{1}{2}\left(c_1^2\left(\pi_* \omega_\pi\right) - c_2\left(\pi_* \omega_\pi\right)\right) + \cdots \end{aligned}$$

Using that $T_\pi = -\omega_\pi$, we have

$$\pi_*\left(\mathrm{Ch}\left(\omega_\pi\right)\mathrm{Td}\,T_\pi\right)$$

$$= \pi_*\left(1 + c_1\left(\omega_\pi\right) + \frac{1}{2}\left(c_1^2 - c_2\right) + \cdots\right)\left(1 - \frac{1}{2}c_1\left(\omega_\pi\right) + \frac{1}{12}c_1\left(\omega_\pi\right)^2 + \cdots\right).$$

We then have

$$\lambda = [\mathrm{Ch}(\pi_! \omega_\pi)]_1,$$

so

$$\pi_* \left(\mathrm{Ch}(\omega_\pi)\,\mathrm{Td}(-\omega_\pi)\right) = \pi_* \left(1 + \frac{1}{2}c_1(\omega_\pi) + \frac{1}{2}c_1^2 - \frac{1}{2}c_1^2 + \frac{1}{12}c_1^2\right).$$

We obtain

$$[\pi_* \left(\mathrm{Ch}(\omega_\pi)\,\mathrm{Td}(-\omega_\pi)\right)]_1 = \frac{1}{12}\kappa.$$

□

7.2. **Hurwitz Divisor.**

Definition 7.11. Assume that $g = 2k - 1$ is odd. Define the **Hurwitz divisor** $\mathrm{Hur} \subset \mathcal{M}_g$ to be

$$\left\{ C : \exists f : C \to \mathbb{P}^1 \text{ of degree at most } k \right\}.$$

Remark 7.12. We can also define $\mathrm{Hur} \subset \mathcal{M}_g$ determinantally. Consider $\pi : \mathcal{C} \to \mathcal{M}_g$. Define

$$\mathcal{C}^n := \mathcal{C} \times_{\mathcal{M}_g} \cdots \times_{\mathcal{M}_g} \mathcal{C}$$

as an n-fold fiber product. Let $p_i : \mathcal{C}^n \to \mathcal{C}$ be the ith projection. Consider $\pi_n : \mathcal{C}^n \to \mathcal{M}_g$. Consider

$$Z = \left\{ (p_1, \ldots, p_k) \in \mathcal{C}^k : h^0\left(C, \sum_i p_i\right) \geq 2 \right\}.$$

Then, $\pi(Z)$ is Hur.

Remark 7.13. Note that π is not finite on Z, since the fibers are at least 1 dimensional as the divisor moves by the assumption $h^0(C, \sum_i p_i) > 1$.

Warning 7.14. Therefore, $\pi|_Z$ has fibers of dimension at least 1, meaning $\pi_*(Z) = 0$. Here is a fix to the issue that π is not generically finite, so the pushforward would be 0 by definition.

We can fix this by demanding that the first point p_1 lies in some fixed canonical divisor. That is, we define

$$K_i \in \mathrm{Pic}\left(\mathcal{C}^k\right) := p_1^*(\omega_\pi).$$

Then, we can define

$$\mathrm{Hur} := \frac{1}{(2g-2)(k-1)!} \cdot (\pi_k)_* ([K_1] \cdot [Z]).$$

We are almost ready to define $[Z]$ determinantally.

Definition 7.15. Define the diagonal

$$\Delta_{i,j} \subset \mathcal{C}^n$$

to be the (i,j)th diagonal, i.e., the image of $\mathcal{C}^{n-1} \to \mathcal{C}^n$ under the map sending $p_1, \ldots, p_{n-1}$ to $p_1, \ldots, p_i, p_{i+1}, \ldots, p_{j-1}, p_i, p_{j+1}, \ldots, p_{n-1}$.

Given the short exact sequence
(7.3)

$$0 \longrightarrow \mathcal{O}_{\mathcal{C}^{k+1}} \longrightarrow \mathcal{O}_{\mathcal{C}^{k+1}}\left(\sum_j \Delta_{j,k+1}\right) \longrightarrow \mathcal{O}_{\sum_i \Delta_{j,k+1}} \longrightarrow 0$$

and let

$$p : \mathcal{C}^{k+1} \to \mathcal{C}^k$$

be the projection away from the last factor.

Remark 7.16. Note that

$$p_*\left(\mathcal{O}_{\mathcal{C}^{k+1}}\left(\sum_{j=1}^{k} \Delta_{j,k+1})\right)\right) \simeq \mathcal{O}_{\mathcal{C}^{k+1}}.$$

If $(p_1, \ldots, p_k) \in \mathcal{C}^k$ are general, then $h^0(C, \sum_i p_i) = 1$.
This is the same as saying the map

$$H^0(C, \mathcal{O}_C) \simeq H^0\left(\mathcal{O}_C\left(\sum_i p_i\right)\right)$$

for $p_1, \ldots, p_k$ general on C. We have a natural map

$$p_*\left(\mathcal{O}_{\mathcal{C}^{k+1}}\right) \to p_*\left(\mathcal{O}_{\mathcal{C}^{k+1}}\left(\sum \Delta_{j,k+1}\right)\right).$$

The above computation implies that for any sufficiently small open U, the map

$$p_*\mathcal{O}_{\mathcal{C}^{k+1}}(U) \to p_*\left(\mathcal{O}_{\mathcal{C}^{k+1}}\left(\sum_j \Delta_{j,k+1}\right)\right)(U).$$

Taking derived pushforwards of the exact sequence
(7.4)

$$0 \longrightarrow \mathcal{O}_{\mathcal{C}^{k+1}} \longrightarrow \mathcal{O}_{\mathcal{C}^{k+1}}\left(\sum_j \Delta_{j,k+1}\right) \longrightarrow \mathcal{O}_{\sum_i \Delta_{j,k+1}} \longrightarrow 0$$

on 0th degree, we get

$$p_* \mathcal{O}_{\mathcal{C}^{k+1}} \simeq p_* \mathcal{O}_{\mathcal{C}^{k+1}} \left(\sum_j \Delta_{j,k+1} \right)$$

and continuing we get

(7.5)

$$0 \longrightarrow p_* \mathcal{O}_{\sum_i \Delta_{j,k+1}} \xrightarrow{\alpha} R^1 p_* \mathcal{O}_{\mathcal{C}^{k+1}} \longrightarrow R^1 p_* \mathcal{O}_{\mathcal{C}^{k+1}} \left(\sum_j \Delta_{j,k+1} \right) \longrightarrow 0$$

Note that the first term $p_* \mathcal{O}_{\sum_i \Delta_{j,k+1}}$ is locally free of rank 1, and then Z is the locus where the map α is not injective.

8. 4/24/17

Last time, we were discussing the divisor $\mathrm{Hur} \subset \mathcal{M}_g$ for $g = 2k - 1$. We were discussing the construction of the Hurwitz divisor as $\pi_k : \mathcal{C}_k \to \mathcal{M}_g$, with

$$\mathrm{Hur} := \frac{1}{(2g-2)(k-1)!} \pi_{k*} ([Z] K_1)$$

where K_1 is the pullback of the canonical divisor of $\mathcal{C} \to \mathcal{M}_g$ along the first projection $\mathcal{C}^k \to \mathcal{C}$. Let's review how this went. We can view

$$[Z] = \left\{ (p_1, \ldots, p_k) : h^0(C, \sum p_i) \geq 2 \right\}.$$

We have an exact sequence

(8.1)

$$0 \longrightarrow \mathcal{O}_{\mathcal{C}^{k+1}} \longrightarrow \mathcal{O}\left(\sum \Delta_{i,k+1}\right) \longrightarrow \mathcal{O}_{\sum \Delta_{i,k+1}} \longrightarrow 0$$

Pushing this forward along $p : \mathcal{C}^{k+1} \to \mathcal{C}^k$, which is projection onto the first k factors, we get an isomorphism between the pushforward of the first two sheaves, we get a short exact sequence

(8.2)

$$0 \longrightarrow p_* \mathcal{O}_{\sum \Delta_{i,k+1}} \xrightarrow{\alpha} R^1 p_* \mathcal{O}_{\mathcal{C}^{k+1}} \longrightarrow R^1 p_* \left(\mathcal{O} \sum \Delta_{i,k+1}\right) \longrightarrow 0$$

The fibers α_q (for q corresponding to a curve) are given by

(8.3)

$$0 \longrightarrow H^0(\mathcal{O}_C) \longrightarrow H^0(\mathcal{O}_C\left(\sum p_i\right)) \longrightarrow H^0\left(\mathcal{O}_{\sum_i p_i}\right) \xrightarrow{\alpha_q} H^1(\mathcal{O}_C)$$

and α_q fails to be injective if and only if $h^0\left(\mathcal{O}_C\left(\sum p_i\right)\right) \geq 2$. We will define Z as the locus of points q with α_q not injective.

Theorem 8.1 (Porteous)**.** *Suppose X is a smooth scheme over $\mathbb{C}$ and $\phi : E \to F$ is a map of vector bundles with E of rank n and F a vector bundle of rank m. Let*

$$X_k(\phi) := \{p \in X : \phi_p \text{ has rank at most } k\}.$$

Assume $X_k(\phi)$ has the expected dimension $(m-k)(n-k)$. Then,

$$[X_k(\phi)] = \Delta_{m-k,n-k}\left(c_+(F)/\, c_t(E)\right).,$$

where $\Delta(\bullet)$ is defined as follows. If

$$a(t) = \sum_k a_k t^k$$

is a formal power series, we have

$$\Delta_{p,q}(a(t)) := \det \begin{pmatrix} q_p & \cdots & a_{p+q-1} \\ \vdots & \ddots & \vdots \\ a_{p-q+1} & \cdots & a_p. \end{pmatrix}$$

To apply Porteous' theorem, we must know that $Z \subset \mathcal{C}^k$ has codimension $g-(k-1) = k$. For this, we need the following:

Clebsch The Hurwitz stack

$$\mathcal{H}_{d,g} := \left\{C \to \mathbb{P}^1 : \deg d, C \text{ smooth of genus } g\right\} / \operatorname{Aut}\left(\mathbb{P}^1\right).$$

has dimension $2g-5-2d$.

Segre The map $\mathcal{H}_{d,g} \to \mathcal{M}_g$ is generically finite for (with some assumption on g and d that Michael wasn't exactly sure about, he thought it was $g \geq 7$).

Exercise 8.2. Show that the above results mean we can apply Porteous' theorem to show Hur is indeed a divisor.

Theorem 8.3 (Harris-Mumford, Harer, Kempf)**.** *We have*

(1)

$$[\mathrm{Hur}] = c\lambda$$

(2) In fact,

$$c = \frac{(2k-4)\,!}{k!\,(k-2)\,!}.$$

Remark 8.4. Harris-Mumford used test curves, explicit curves in $\mathcal{M}_g$ that they understood well for which they could compute things explicitly.

Proof. We have

$$[Z] = \Delta_{g-k+1,1}\left(c_t\left(R^1p_*\left(\mathcal{O}_{\mathcal{C}}\left(\sum \Delta_{j,k+1}\right)\right)\right)\right) = c_{g-k+1}\left(R^1p_*\left(\mathcal{O}_{\mathcal{C}}\left(\sum \Delta_{j,k+1}\right)\right)\right)$$
$$= 1 - p_!\left(\mathcal{O}_{\mathcal{C}}\left(\sum \Delta_{j,k+1}\right)\right).$$

The Chern classes are expressible as polynomials in Ch_d. That is, $[Z]$ is polynomial in

$$\mathrm{Ch}\left(p_!\left(\mathcal{O}_{\mathcal{C}}\left(\sum \Delta_{j,k+1}\right)\right)\right) = p_*\left(\mathrm{Ch}\left(\mathcal{O}_{\mathcal{C}}\left(\sum \Delta_{i,k+1}\right)\mathrm{Td}\left(\omega_p\right)\right)\right),$$

using, Grothendieck Riemann-Roch. Therefore, $[Z]$ is a polynomial in

$$p_*(\left[\Delta_{j,k+1}\right])$$

and

$$p_*\left[\omega_p\right] = p_*(K_{k+1}).$$

where $K_i = p_i^*\omega_{\mathcal{C}/\mathcal{M}_g}$ with p_i the ith projection $\mathcal{C}^n \to \mathcal{C}$. To simplify this, we can use the push-pull formula. We have

$$\begin{aligned} p_*\left(\left[\Delta_{j,k+1} \cdot p^*\left(\zeta\right)\right]\right) &= p_*\left(\left[\Delta_{j,k+1}\right]\right) \cdot \zeta \\ &= \zeta \end{aligned}$$

Note here $p^*\zeta \in A^*\left(\mathcal{C}^k\right)$. We want to express as much of the above as possible as the pullback of cycles on $\mathcal{C}^k$. We have

$$[\Delta_{i,k+1}] \cdots \left[\Delta_{j,k+1}\right] = [\Delta_{i,k+1}] \cdot p^*\left[\Delta_{i,j}\right].$$

Loosely this is saying that if $p_1, \ldots, p_{k+1}$ with $p_i = p_{k+1}$ and $p_j = p_{j+1}$ this is equivalent to saying $p_i = p_{k+1}$ and $p_i = p_j$. We also have the relation

$$\left[\Delta_{j,k+1}\right] \cdot K_{k+1} = [\Delta_{i,k+1}] \cdot p^*K_j.$$

The above relations let us deal with any monomial in which no repeated $\left[\Delta_{j,k+1}\right]$ appears. For example,

$$\begin{aligned} p_*\left(\Delta_{1,k+1} \cdot \Delta_{2,k+1} \cdot \Delta_{3,k+1} \cdot K_{k+1}\right) &= p_*\left(\Delta_{1,k+1}p^*\Delta_{1,2}p^*\Delta_{1,3}p^*K_1\right) \\ &= \Delta_{1,2} \cdot \Delta_{1,3} \cdots K_1. \end{aligned}$$

Next, we have to deal with powers of these diagonals. For this, we use the self intersection formula (see Hartshorne, p. 431).

Lemma 8.5 (Self Intersection Formula)**.** *Suppose we have a closed immersion of schemes (or stacks)* $i : Y \to X$ *with both smooth of codimension* r. *We have* $Y \cdot Y = i_*\left(c_r\left(N_{Y/X}\right)\right)$.

Using this self intersection formula, and letting $q : \mathcal{C}^{k+1} \to \mathcal{C}^k$ be projection away from the jth factor, we get

$$\begin{aligned}\left[\Delta_{j,k+1}\right]^2 &= i_*\left(c_1\left(N_{\Delta_{j,k+1}/\mathcal{C}^{k+1}}\right)\right)\\ &= -i_* c_1\left(\Omega_q\right)\\ &= -\left[\Delta_{j,k+1}\right] p^*\left(K_j\right).\end{aligned}$$

We then conclude that $[Z]$ can be written as a polynomial in the cycles $\Delta_{i,j}$ and K_i. We also may include $p_*\left(K_{k+1}^{\ell}\right)$ for some positive integer e. To simplify this, we use flat pullback of cycles

Lemma 8.6 (Flat pullback, Fulton proposition 1.7). *Suppose we have a Cartesian square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"] & B \arrow[d, "g"] \\ C \arrow[r, "h"] & D \end{tikzcd} \tag{8.4}$$

with f *flat and* g *proper, we have*

$$c_* f^* \alpha = h^* g_* \alpha$$

for $\alpha \in A^*(B)$.

In our situation, we can apply this to the diagram

$$\begin{tikzcd} \mathcal{C}^{k+1} \arrow[r, "p_{k+1}"] \arrow[d, "p"] & \mathcal{C} \arrow[d, "\pi"] \\ \mathcal{C}^k \arrow[r, "\pi_k"] & \mathcal{M}_g \end{tikzcd} \tag{8.5}$$

we obtain

$$p_*\left(K_{k+1}^{\ell}\right) = \pi_k^* \pi_*\left(K^{\ell}\right)$$

Therefore, Hur is a polynomial in $\pi_{k,*}$ times polynomials in $\Delta_{k,j}$, K_j's and $\pi_k^*\pi_*\left(K^{\ell}\right)$. To conclude, one can factor π_k as a composition

$$\mathcal{C}^k \to \mathcal{C}^{k-1} \to \cdots \to \mathcal{C}.$$

Repeating the above, things remain polynomials in the analogous classes, except that certain $\Delta_{i,j}$ classes become 1. When one factors this, one obtains that [Hur] is a polynomial in $\pi_*\left(K^{\ell}\right)$. Note that Hur is a divisor, and the only divisor in $\pi_*(K^\ell)$ is $\pi_*(K^2)$ (meaning $\ell = 1$). This implies by dimension reasons that

$$\text{Hur} = c\pi_*\left(K^2\right) = \kappa,$$

so Hur $= c'\lambda$, which is Mumford's formula. □

9. 4/26/17

9.1. **Graded Tor.** Today, we'll discuss graded tor. Let R be a graded k-algebra. Let M and N be two graded R-modules. Note that $M \otimes_k N$ is graded. We have

$$(M \otimes_R N)_d = \left\{ \sum_i m_i \otimes n_i : \deg(m_i) + \deg(n_i) = d \right\}.$$

Note that $\mathrm{Tor}_R(M, N)$ is a bigraded module gotten by taking a projective resolution

$$0 \longleftarrow M \xleftarrow[f^0]{} P^0 \xleftarrow[f^1]{} P^1 \longleftarrow \cdots \tag{9.1}$$

Then, tensoring up, we obtain a complex of graded R-modules $P_\bullet \otimes_R N$. Then,

$$\mathrm{Tor}_k^p(M, N)_q$$

is the pth homology of $(P_\bullet \otimes_R N)_q$. Then,

$$\mathrm{Tor}_R^p(M, N) = \oplus_{q \geq 0} \mathrm{Tor}^p(M, N)_q.$$

We have

$$\mathrm{Tor}^p(M, N)_q \simeq \mathrm{Tor}^p(N, M)_q.$$

9.2. **The Koszul complex, revisited.** Let $S = k[x_0, \ldots, x_n]$ and M be a finitely generated S-module. We have a minimal free resolution

$$0 \longleftarrow M \longleftarrow \mathcal{F}_0 \longleftarrow \cdots \longleftarrow \mathcal{F}_{n+1} \tag{9.2}$$

We know that for a minimal free resolution, if we tensor by $k := S/(x_0, \ldots, x_n)$, all differentials are 0, and therefore, all differentials are 0 by Nakayama's lemma. That is, if we have $\mathcal{F}_i = \oplus_j S(-i-j)^{b_{i,j}}$. Therefore,

$$\mathcal{F}_i \times_S k = \oplus_j k(-i-j)^{b_{i,j}} \tag{9.3}$$

as a graded k-module. We then have

$$\dim\left(\mathrm{Tor}_R^i(M, k)_{i+j}\right) = b_{i,j}(M).$$

We can treat $\mathrm{Tor}^i_R(M,k)_{i+j}$ as the "canonical" invariant of interest, since the isomorphism Equation 9.3 is canonical.

Now, considering k as an S-module, we have a minimal free resolution given by the Koszul complex

(9.4)
$$0 \longleftarrow k \longleftarrow S \longleftarrow E(-1) \longleftarrow \wedge^2 E(-2) \longleftarrow \wedge^{n+1} E(-n-1) \longleftarrow 0$$

Then, $\mathrm{Tor}^i_S(M,k)_{i+j}$ is the homology of

(9.5)
$$\wedge^{i+1} E(-i-1) \otimes_S M)_{i+j} \longrightarrow \wedge^i E(-i) \otimes_S M \longrightarrow \wedge^{i-1} E(-i+1) \otimes_S M$$

with $V := S_1 \simeq k^{\oplus n+1}$. Then,
$$\wedge^i E(-i) \otimes M \simeq \left(\wedge^i E \otimes M(-i)\right) \qquad \simeq \left(\wedge^i V \otimes M(-i)\right)$$

So,
$$\left(\wedge^i E(-i) \otimes M\right)_{i+j} \simeq \wedge^i V \otimes_k M_j,$$

and $\mathrm{Tor}^i_S(M,N)_{i+j}$ is the homology of

$$\wedge^{i+1} V \otimes_k M_{j-1} \longrightarrow \wedge^i V \otimes_k M_j \longrightarrow \wedge^{i-1} V \otimes M_{j+1} \tag{9.6}$$

9.3. **Kernel Bundles.** Let X be a projective variety and $\mathcal{L}$ a line bundle. Define
$$\Gamma_X(L) := \oplus_q H^0(qL).$$

as a graded $\mathrm{Sym}\, H^0(L)$ module. Assume $\mathcal{L}$ is base point free so that the evaluation map
$$H^0(X,\mathcal{L}) \otimes \mathcal{O}_X \to \mathcal{L}$$

is surjective. Define M_L as the kernel

$$0 \longrightarrow M_L \longrightarrow H^0(X,L) \otimes \mathcal{O}_X \longrightarrow L \longrightarrow 0. \tag{9.7}$$

More geometrically, if we have a map $\phi_L : X \to \mathbb{P}^r$ with $r = h^0(L) - 1$. By the Euler exact sequence, we have

(9.8)
$$0 \longrightarrow \omega_{\mathbb{P}^r}(1) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^r(1)}) \otimes \mathcal{O}_{\mathbb{P}^r} \longrightarrow \mathcal{O}_{\mathbb{P}^r}(1) \longrightarrow 0$$

Then, $M_L = \phi_L^*(\Omega_{\mathbb{P}^r}(1))$ is a vector bundle.

Recall the notation
$$b_{p,q}(X,L) := b_{p,q}(\Gamma_X(L)).$$

We have a short exact sequence
(9.9)
$$0 \longrightarrow \wedge^{p+1} M_L \otimes (q-1)L \longrightarrow \wedge^{p+1} H^0(L) \otimes (q-1)L \longrightarrow \wedge^p M_L \otimes qL \longrightarrow 0$$
(see Hartshorne, II-5).

Taking global sections, we get a map
$$\wedge^{p+1} H^0(L) \otimes H^0((q-1)L) \xrightarrow{\alpha} H^0(\wedge^p M_L \otimes qL).$$

Theorem 9.1 (Lazarsfeld). *We have $b_{p,q}(X, L)$ is the dimension of coker α.*

Proof. Let $V := H^0(X, L)$.
(9.10)
$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & \wedge^{p+1} M \otimes (q-1)L \arrow[d] & & & \\
 & \wedge^{p+1} V \otimes (q-1)L \arrow[d] \arrow[dr, "\delta"] & & & \\
0 \arrow[r] & \wedge^p M_L \otimes qL \arrow[r] & \wedge^p V \otimes qL \arrow[r] \arrow[dr] & \wedge^{p-1} M_L \otimes (q+1)L \arrow[r] \arrow[d] & 0 \\
 & & & \wedge^{p-1} V \otimes (q+1)L \arrow[r] & 0
\end{tikzcd}$$

Now, $H^0(\delta)$ is the Koszul differential
$$d^p : \wedge^p V \otimes H^0(qL) \to \wedge^{p-1} V \otimes H^0((q+1)L).$$
Then,
$$\ker d^p \simeq H^0(\wedge^p M_L \otimes qL)$$
and
$$\operatorname{im} d^{p+1} \simeq \operatorname{im}(\alpha).$$
This is exactly the claim that the tors are given by cokernel of α. $\square$

Corollary 9.2 (Serre-duality). *If C is a smooth curve, we have*
$$b_{p,q}(C, \omega_C) = b_{g-2-p,3-q}(C, \omega_C).$$
In particular, for $q > 3$, we have $b_{p,q}(C, \omega_C) = 0$.

Proof. Consider

$$\wedge^{p+1}V \otimes H^0((q-1)L) \xrightarrow{\alpha} H^0(\wedge^p M_L \otimes qL).$$

We get an exact sequence
(9.11)

$$0 \longrightarrow \wedge^{p+1}M_L \otimes (q-1)L \longrightarrow \wedge^{p+1}V \otimes (q-1)L \longrightarrow \wedge^p M_L \otimes qL \longrightarrow 0.$$

Observe $\wedge^p \mathcal{V}^\vee \simeq \wedge^{r-p}V \otimes \det \mathcal{V}^\vee$. Then,

$$\wedge^{p+1}V \otimes H^0((q-1)\omega_C)^\vee \simeq \wedge^{g-p-1}V \otimes H^1\left((2-q)\omega_C\right).$$

We also have

$$H^0(\wedge^p M_L \otimes q\omega_C)^\vee \simeq H^1(\wedge^p M_L^\vee \otimes (1-q)\omega_C).$$

We have $\operatorname{rk} M_L = k-1$, so $\det M_L \simeq L$, as comes from the exact sequence. Therefore,

$$\begin{aligned} H^0(\wedge^p M_L \otimes q\omega_C)^\vee &\simeq H^1(\wedge^p M_L^\vee \otimes (1-q)\omega_C) \\ &\simeq H^1(\wedge^{g-1-p}M_{\omega_C} \otimes (2-q)\omega_C). \end{aligned}$$

Then,

$$\operatorname{coker}\alpha^\vee = \ker\left(H^1\left(\wedge^{g-1-p}M_{\omega_C} \otimes (2-q)\omega_C\right) \to \wedge^{g-p-1}V \otimes H^1\left((2-q)\omega_C\right)\right).$$

Using the long exact sequence Equation 9.11, we have

$$\begin{aligned} \operatorname{coker}\alpha^\vee &= \ker\left(H^1\left(\wedge^{g-1-p}M_{\omega_C} \otimes (2-q)\omega_C\right) \to \wedge^{g-p-1}V \otimes H^1\left((2-q)\omega_C\right)\right). \\ &\simeq \operatorname{coker}\left(\wedge^{g-1-p}H^0(\omega_C) \otimes H^0\left((2-q)\omega_C\right) \to H^0\left(\wedge^{g-2-p}M_{\wedge_C} \otimes (3-q)\omega_C\right)\right). \end{aligned}$$

The first term has dimension $b_{g-2-p,3-q}(C,\omega_C)$. This finishes the proof by Theorem 9.1. $\square$

10. 4/28/17

Let $S = k[x_1, \ldots, x_n]$. Given an exact sequence of graded S-modules

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0 \tag{10.1}$$

we get a long exact sequence on tor
(10.2)

$$\cdots \longrightarrow \operatorname{Tor}_S^{i+1}(M_3,k) \longrightarrow \operatorname{Tor}_S^i(M_1,k) \longrightarrow \operatorname{Tor}^i(M_2,k) \longrightarrow \cdots$$

Taking the $i+j$th strand of this complex, we get
(10.3)

$$\cdots \longrightarrow \operatorname{Tor}_S^{i+1}(M_3,k)_{i+j} \longrightarrow \operatorname{Tor}_S^i(M_1,k)_{i+j} \longrightarrow \operatorname{Tor}^i(M_2,k)_{i+j} \longrightarrow \cdots$$

We define

$$K_{i,j}(M) := \mathrm{Tor}_S^i(M, k)_{i+j}$$

so

$$b_{i,j}(M) = \dim_k K_{i,j}(M).$$

Definition 10.1. The **long exact sequence of Koszul cohomology** is the exact sequence obtained above, which can be rewritten as
(10.4)

$$\begin{tikzcd} \cdots \arrow[r] & K_{i+1,j-1}(M_3) \arrow[r] & K_{i,j}(M_1) \arrow[r] & K_{i,j}(M_2) \arrow[dll] \\ & K_{i,j}(M_3) \arrow[r] & K_{i-1,j+1}(M_3) \arrow[r] & \cdots \end{tikzcd}$$

Lemma 10.2 (Semicontinuity). *Suppose $\pi : X \to S$ is a flat map of finite type schemes over Spec $\mathbb{C}$ (where S is not a polynomial ring). Suppose S is integral and $\mathcal{L} \in \mathrm{Pic}(X)$. Assume*

$$h^0(X_s, \mathcal{L}_s), h^0(X_s, (q-1)L_s), h^0(X_s, qL_s), h^0(X_s, (q+1)L_s)$$

are all constant for closed points s of S. Then, the function

$$\begin{aligned} \psi \colon S &\to \mathbb{Z} \\ s &\mapsto b_{p,q}(X_s, L_s) = b_{p,q}(\Gamma_{L_s}(X)) \end{aligned}$$

is upper semicontinuous.

Proof. Without loss of generality, we can take $S = \mathrm{Spec}\, R$ to be affine. Let

$$\begin{aligned} \mathcal{E} &:= \pi_* L \\ \mathcal{F}_1 &:= \pi_* L^{q-1} \\ \mathcal{F}_2 &:= \pi_* L^q \\ \mathcal{F}_3 &:= \pi_* L^{q+1}. \end{aligned}$$

There is a morphism

$$\pi_* L \otimes \pi_* L^q \to \pi_* L^{q+1}.$$

This is the same as given an element of

$$\mathrm{Hom}\left(\pi_* L \otimes \pi_* L^q, \pi_* L^{q+1}\right) = \mathrm{Hom}\left(\pi^* \pi_* L \otimes \pi^* \pi_* L^q, \pi * \pi_* L^{q+1}\right)$$

So, we have adjunction maps

$$\pi^*\pi_* L \to L$$
$$\pi^*\pi_* L^q \to L^q$$

We then obtain

$$\bigwedge^{p+1}\mathcal{E} \otimes_R \mathcal{F}_1 \longrightarrow \bigwedge^p \mathcal{E} \otimes \mathcal{F}_2 \xrightarrow{\delta} \bigwedge^{p-1}\mathcal{E} \otimes \mathcal{F}_3 \tag{10.5}$$

with δ given by

$$(s_1 \wedge \cdots \wedge s_p \otimes t) \mapsto \sum_j (-1)^j s_1 \wedge \cdots \wedge \widehat{s_j} \wedge \cdots \wedge s_p \otimes s_j t.$$

For all closed points s in Spec R,
(10.6)

$$\bigwedge^{p+1}\mathcal{E} \otimes_R \mathcal{F}_1 \otimes \kappa(s) \xrightarrow{\delta_1 \otimes \kappa(s)} \bigwedge^p \mathcal{E} \otimes \mathcal{F}_2 \otimes \kappa(s) \xrightarrow{\delta_2 \otimes \kappa(s)} \bigwedge^{p-1}\mathcal{E} \otimes \mathcal{F}_3 \otimes \kappa(s)$$

The middle cohomology is

$$K_{p,q}(X_s, \mathcal{L}_s)$$

and

$$\begin{aligned} b_{p,q}(X_s, \mathcal{L}_s) &= \dim\ker(\delta_2 \otimes \kappa(s)) - \dim\operatorname{im}\ (\delta_1 \otimes \kappa(p)) \\ &= \operatorname{rk}(\bigwedge^p \mathcal{E} \otimes \mathcal{F}_2) - \dim\operatorname{im}\ (\delta_2 \otimes \kappa(s)) - \dim\operatorname{im}\ (\delta_1 \otimes \kappa(s)). \end{aligned}$$

Note that we are assume $\operatorname{rk}(\bigwedge^p \mathcal{E} \otimes \mathcal{F}_2)$ is constant. Therefore, as we can work locally, we have reduced to verifying that for $\psi : A \to B$ a morphism of finitely generated free R modules, we have

$$s \mapsto \operatorname{rk}(\psi \otimes \kappa(s))$$

is lower semicontinuous. But, for any $r \in \mathbb{N}$,

$$\{p \in \operatorname{Spec} R : \operatorname{rk}(\psi \otimes \kappa(s)) < r\}$$

is closed with ideal given by entries of $\bigwedge^r \psi$. □

Let X be a projective variety L and line bundle and F a coherent sheaf on X. We have

$$\Gamma_X(F, L) = \oplus_q H^0(qL \otimes F)$$

is a graded Sym $H^0(L)$ module.

Definition 10.3. Define

$$K_{p,q}(X, F; L) = K_{p,q}(\Gamma_X(F, L))$$

and

$$b_{p,q}(X, F; L) = \dim K_{p,q}(X, F; L).$$

We now want another way for computing $K_{p,1}$.

Proposition 10.4. *Let* X *be a projective variety and* L *be a line bundle on* X. *Assume* L *is very ample and the embedding*

$$\phi_L : X \to \mathbb{P}^r,$$

for $r = h^0(L) - 1$ *is projectively normal, meaning*

$$H^0(\mathbb{P}^r, \mathcal{O}(n)) \to H^0(X, nL)$$

is surjective for $n \geq 1$.

Then,

$$K_{p,1}(X, L) = H^0\left(\mathbb{P}^r, (\wedge^{p-1}\Omega_{\mathbb{P}^r})(p+1) \otimes \mathcal{I}_X\right).$$

Proof. We have a short exact sequence of sheaves defining X given by

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_{\mathbb{P}^r} \longrightarrow \mathcal{O}_X \longrightarrow 0 \tag{10.7}$$

we get an exact sequence of graded $\operatorname{Sym} H^0(\mathbb{P}^r, \mathcal{O}(1)) \simeq \operatorname{Sym} H^0(X, \mathcal{L})$ modules

(10.8)

$$0 \longrightarrow \oplus_{q \geq 0} H^0(\mathbb{P}^r, \mathcal{I}_X(q)) \longrightarrow \oplus_q H^0(\mathbb{P}^r, \mathcal{O}(q)) \longrightarrow \oplus_q H^0(X, qL) \longrightarrow 0$$

We get a short exact sequence

(10.9)

$$\cdots \longrightarrow K_{p,1}(\mathbb{P}^r, \mathcal{O}(1)) \longrightarrow K_{p,1}(X, \mathcal{L}) \longrightarrow K_{p-1,2}(\mathbb{P}^r, \mathcal{I}_X; \mathcal{O}(1)) \longrightarrow K_{p-1,2}(\mathbb{P}^r, \mathcal{O}(1))$$

By the same proof as last time,

$$\begin{aligned} &K_{p-1,2}(\mathbb{P}^r, I_X; \mathcal{O}(1)) \\ &\simeq \operatorname{coker}\left(\wedge^p H^0(\mathbb{P}^r, \mathcal{O}(1)) \otimes H^0(\mathcal{I}_X \otimes O(1)) \to H^0\left(\wedge^{p-1}\Omega(1) \otimes I_X \otimes O(2)\right)\right) \\ &\simeq H^0\left(\wedge^{p-1}(\Omega(1)) \otimes I_X \otimes O(2)\right) \end{aligned}$$

using that X is linearly normal to say $H^0(X, \mathcal{I}_X \otimes O(1)) = 0$. So, it suffices to show

$$K_{p,q}(\mathbb{P}^r, \mathcal{O}(1)) = 0$$

if $(p, q) \neq (0, 0)$.

Lemma 10.5. *We have*

$$K_{p,q}(\mathbb{P}^r, \mathcal{O}(1)) = 0$$

for all $(p, q) \neq (0, 0)$.

Proof. Take $V := H^0(\mathbb{P}^r, \mathcal{O}(1))$ which is an $r+1$ dimensional vector space. Then, we get
(10.10)
$$\wedge^{p+1} V \otimes H^0\left(O(q-1)\right) \longrightarrow \wedge^p V \otimes H^0(O(q)) \longrightarrow \wedge^{p-1} V \otimes H^0\left(O(q+1)\right),$$

which we want to show is exact. We know that the Koszul complex
(10.11)
$$0 \longleftarrow k \longleftarrow S \longleftarrow V \otimes S(-1) \longleftarrow \wedge^2 V \otimes S(-2)$$

with $S = \operatorname{Sym} V$. Taking the $p+q$ graded part, we get
(10.12)
$$0 \longleftarrow k \longleftarrow S_{p+q} \longleftarrow V \otimes S_{p+q-1} \longleftarrow \wedge^2 V \otimes S_{p+q-2}$$

The complex we want to show is exact, is precisely this complex at step p. □

□

Recall now Green's conjecture:

Conjecture 10.6. We have $b_{p,2}(C, \omega_C) = 0$ if and only if $p < \operatorname{Cliff} C$ or $p \geq g$.

Remark 10.7. This can be equivalently rephrased as $b_{p,1}(C, \omega_C) = 0$ if and only if $p > g - 2 - \operatorname{Cliff}(C)$.

10.1. **Hirschowitz-Ramanan.** Taking $g = 2k-1$, consider $\operatorname{Hur} \mathcal{M}_g = \{C : \operatorname{Gon} C \leq k\}$. Green's conjecture this is the same as the set

$$\operatorname{Kos} := \left\{C : b_{g-k,1}(C, \omega_C) \neq 0\right\}$$

We have a universal canonical embedding

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "j"] \arrow[dr, "\pi"'] & & \mathbb{P}\left(\pi_* \omega_\pi\right) \arrow[dl, "p"] \\ & \mathcal{M}_g & \end{tikzcd} \tag{10.13}$$

Then,

$$0 \longrightarrow \mathcal{M} \longrightarrow p^* p_* \mathcal{O}(1) \longrightarrow \mathcal{O}(1) \longrightarrow 0 \tag{10.14}$$

where $\mathcal{M}$ has fibers $\Omega_{\mathbb{P}^r}(1)$. The degeneracy locus of

$$p_* \left(\wedge^{k-2} \mathcal{M}(2)\right) \to \pi_* j^* \wedge^{k-2} \mathcal{M}(2)$$

is Kos.

11. 5/8/17

Recall the Koszul divisor Kos $\subset \mathcal{M}_g$ for $g = 2k-1$ is set theoretically $\{C : b_{g-k,1}(C, \omega_C) \neq 0\}$.

Consider the universal cover $\mathcal{C} \to \mathcal{M}_g$ given by

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "j"] \arrow[dr, "\pi"'] & & \mathbb{P}\pi_*\omega_\pi \arrow[dl, "p"] \\ & \mathcal{M}_g & \end{tikzcd} \tag{11.1}$$

Recall we have the relative Euler sequence for p given by

$$0 \longrightarrow \Omega_p(1) \longrightarrow p^*p_*\mathcal{O}(1) \longrightarrow \mathcal{O}(1) \longrightarrow 0. \tag{11.2}$$

Lemma 11.1. *The two sheaves*

$$p_*\left(\Omega^{k-1}(k)\right) \text{ and } \pi_*j^*\Omega^{k-1}(k)$$

are vector bundles of rank $(k-1)\binom{2k}{k}$.

Proof. Recall the Bott formula implies

$$h^0(\mathbb{P}^n, \Omega^p(j) = \begin{cases} \binom{j+n-p}{j}\binom{j-1}{p} & \text{if } 0 \leq p \leq n, j > p \\ 1 & \text{if } j = p = 0 \\ 0 & \text{else} \end{cases}$$

By the Bott formula we want to compute

$$h^0(\mathbb{P}^{g-1}, \Omega^{k-2}(k)) = \binom{k-1}{k-2}\binom{2k}{k} = (k-1)\binom{2k}{k}.$$

We will show

$$p_*\left(\Omega^{k-1}(k)\right)$$

is a vector bundle of rank $\chi\left(C, \wedge^{k-2}M_{\omega_C}\right)$.

We have an exact sequence

$$0 \longrightarrow M_{\omega_C} \longrightarrow H^0(\omega_C) \otimes \mathcal{O}_C \longrightarrow \omega_C \longrightarrow 0 \tag{11.3}$$

with $M_{\omega_C} = j^*\Omega_{\mathbb{P}^{g-1}}(1)$.

First, we compute $\chi\left(C, \wedge^{k-2}M_{\omega_C}\right)$. We have by Riemann Roch and the splitting principle

$$\chi\left(C, \wedge^{k-2}M_{\omega_C}\right) = \deg\left(\wedge^{k-2}M_{\omega_C}(2\omega_C)\right) + \mathrm{rk}\left(\wedge^{k-2}M_{\omega_C}(2\omega_C)\right)(1-g).$$

Using

$$\det \wedge^j E = \binom{\mathrm{rk}-1}{j-1} \det E.$$

(where multiplication means tensor power) we obtain

$$\begin{aligned}
\chi\left(C, \wedge^{k-2} M_{\omega_C}\right) &= \deg\left(\wedge^{k-2} M_{\omega_C}(2\omega_C)\right) + \mathrm{rk}\left(\wedge^{k-2} M_{\omega_C}(2\omega_C)\right)(1-g) \\
&= 2(2g-2)\binom{g-1}{k-2} + \deg\left(\wedge^{k-2} M_{\omega_C}\right) + \binom{g-1}{k-2}(1-g) \\
&= 2(2g-2)\binom{g-1}{k-2} - \binom{g-2}{k-3}(2g-2) - (g-1)\binom{g-1}{k-2} \\
&= 3(g-1)\binom{g-1}{k-2} - \binom{g-2}{k-3}(2g-2) \\
&= 3(g-1)\binom{2k-2}{k-2} - \binom{2k-3}{k-3}(2g-2) \\
&= 3(2k-2)\binom{2k-2}{k} - \frac{(k-2)2(2k-2)}{2k-2}\binom{2k-2}{k} \\
&= (4k-2)\binom{2k-2}{k} \\
&= (k-1)\binom{2k}{k}.
\end{aligned}$$

Above we used

$$\begin{aligned}
\frac{\binom{2k-3}{k}}{\binom{2k-2}{k}} &= \frac{(2k-3)!}{k!(k-3)!}\frac{k!(k-2)!}{(2k-2)!} \\
&= \frac{k-2}{2k-2}.
\end{aligned}$$

So, we have computed $\chi\left(C, \wedge^{k-2} M_{\omega_C}\right)$. To complete our proof, we only need verify

$$h^1(C, \wedge^{k-2} M_{\omega_C}(\omega_C^{\otimes 2}) = 0.$$

We now know

$$\begin{aligned}
K_{p,1}(C, \omega_C) &= \operatorname{coker}\left(\wedge^{p+1} H^0(\omega_C) \otimes H^0((q-1)\omega_C) \to H^0(\wedge^p M_{\omega_C}(q-1)\omega_C)\right) \\
&= \ker\left(H^1\left(\wedge^{p+1} M_{\omega_C} \otimes (q-1)\omega_C\right) \to \wedge^{p+1} H^0(\omega_C) \otimes H^1((q-1)\omega_C)\right).
\end{aligned}$$

This tells us

$$K_{k-3,3}(C, \omega_C) \simeq H^1\left(\wedge^{k-2} M_{\omega_C}\left(\omega_C^{\otimes 2}\right)\right)$$

as $H^1\left(\omega_C^{\otimes 2}\right) = 0$. Serre duality tells us that

$$b_{k-3,3}(C, \omega_C) = b_{k,0}(C, \omega_C)$$

It suffices to show the following:

Lemma 11.2. *We have* $b_{j,0}(C, \omega_C)$ *for* $j > 0$.

Proof. We have

$$0 \longrightarrow \wedge^j H^0(\omega_C) \otimes H^0(\mathcal{O}_C) \xrightarrow{d} \wedge^{j-1} H^0(\omega_C) \otimes H^0(\omega_C) \tag{11.4}$$

Note that the first term is 0 because $\wedge^{j+1} H^0(\omega_C) \otimes H^0(-\omega_C) = 0$. We want to show d is injective. We have a wedge map

$$\begin{aligned} \wedge \colon \wedge^{j-1} H^0(\omega_C) \otimes H^0(\omega_C) &\to \wedge^j H^0(\omega_C) \\ s_1 \wedge \cdots \wedge s_{j-1} \otimes t &\mapsto s_1 \wedge \cdots \wedge s_{j-1} \wedge t \end{aligned}$$

Then,

$$d\left(s_1 \wedge \cdots \wedge s_j\right) \mapsto \sum_j (-1)^i s_1 \wedge \cdots \wedge \hat{s_i} \wedge \cdots \wedge s_j$$

so $\wedge \circ d = \pm j \cdot \mathrm{id}$ implying d is injective. □

□

We have a natural map

$$F : p_*\left(\Omega_p^{k-2}(k)\right) \pi_* j^*\left(\Omega^{k-2}(k)\right)$$

given by adjunction via

$$\mathrm{Hom}\left(p_*\Omega_p^{k-2}(k), \pi_* j^* \Omega^{k-2}(k)\right) = \mathrm{Hom}\left(p_*\Omega_p^{k-2}(k), p_* j_* j^* \Omega^{k-2}(k)\right)$$

and we obtain the desired map by pushing forward along p the map adjoint to the identity.

The map F on fibers is given by

$$F \otimes \kappa(p) : H^0\left(\mathbb{P}^{g-1}, \wedge^{k-2}(\Omega(k)) \otimes O(2)\right) \to H^0(C, \wedge^{k-2} M_{\omega_C} \otimes \wedge_C^2).$$

We are looking for when this map drops rank, which is either on a divisor or everywhere. So, if Kos is not all of $\mathcal{M}_g$, it is a divisor. Then, $H^0(I_C \otimes \Omega^k(k)) \neq 0$ if and only if $b_{k-1,1}(C, \omega_C) \neq 0$.

Definition 11.3. We define $\mathrm{Kos} = c_1\left(\pi_* j^* \Omega^{k-2}(k)\right) - c_1\left(p_*\left(\Omega_p^{k-2}(k)\right)\right)$.

We now introduce some notation. Define
$$\mathcal{M} := \Omega_p(1)$$
and define
$$G_{a,b} := p_* \wedge^a M(b).$$
We have an Euler sequence
$$0 \longrightarrow \mathcal{M} \longrightarrow p^* p_* \mathcal{O}(1) \longrightarrow \mathcal{O}(1) \longrightarrow 0. \tag{11.5}$$
This yields an exact sequence
(11.6)
$$0 \longrightarrow \wedge^w \mathcal{M} \longrightarrow \wedge^w p^* p_* \mathcal{O}(1) \longrightarrow \wedge^{w-1} \mathcal{M} \otimes \mathcal{O}(1) \longrightarrow 0$$
we obtain
$$R^1 p_* \wedge^w \mathcal{M}(q) = 0$$
for $w \geq 0, q > 0$ as $b_{w-1,q+1}\left(\mathbb{P}^{g-1}, \mathcal{O}(1)\right) = 0$. Pushing forward the above sequence via p yields
(11.7)
$$0 \longrightarrow p_* \left(\wedge^w \mathcal{M}(q)\right) \longrightarrow \wedge^w p_* \mathcal{O}(1) \otimes p_* \mathcal{O}(q) \longrightarrow p_* \left(\wedge^{w-1} \mathcal{M}\left(q-1\right)\right) \longrightarrow 0$$
using the projection formula. We can rewrite this as
(11.8)
$$0 \longrightarrow G_{a,b} \longrightarrow \wedge^a G_{0,1} \otimes G_{0,b} \longrightarrow G_{a-1,b-1} \longrightarrow 0$$
Using this we can determine Chern classes by induction on a.

12. 5/10/17

Recall our setup: We have
$$\begin{tikzcd} \mathcal{C} \arrow[rr, "j"] \arrow[dr, "\pi"'] & & \mathbb{P}\left(\pi_* \omega_\pi\right) \arrow[dl, "p"] \\ & \mathcal{M}_g & \end{tikzcd} \tag{12.1}$$
Recall we have defined the Koszul divisor
$$[\mathrm{Kos}] := c_1\left(\pi_* j^* \Omega^{k-2}(k) - p_* \Omega^{k-2}(k)\right).$$
We want to now compute the classes of the two bundles in this difference.

12.1. **Computing the second class.** We have that

$$[\mathrm{Kos}] = \{[C] : b_{k-1,1}(C, \omega_C) = 0\}.$$

Recall $G_{a,b} = p_* \wedge^a \mathcal{M}(b)$. with $\mathcal{M} := \Omega(1)$. We have an exact sequence

(12.2)

$$0 \longrightarrow G_{a,b} \longrightarrow \wedge^a G_{0,1} \otimes G_{0,b} \longrightarrow G_{a-1,b+1} \longrightarrow 0$$

as we saw yesterday. Then, using this short exact sequence iteratively and then the splitting principle, we obtain

$$\begin{aligned} c_1\left(p_*\Omega^{k-2}(k)\right) &= c_1\left(G_{k-2,2}\right) \\ &= c_1\left(\wedge^{k-2}G_{0,1} \otimes G_{0,2}\right) - c_1\left(G_{k-3,3}\right) \\ &= c_1\left(\sum_{\ell=0}^{k-2} \wedge^{k-2-\ell}G_{0,1} \otimes G_{0,2+\ell}\right) \\ &= \sum_{\ell=0}^{k-2}(-1)^\ell\left(c_1\left(\wedge^{k-2-\ell}G_{0,1}\right)\operatorname{rk}\left(G_{0,2+\ell}\right) + \operatorname{rk}\left(\wedge^{k-2-\ell}G_{0,1}\right)c_1\left(G_{0,2+\ell}\right)\right) \end{aligned}$$

Again, above we used the splitting principle to deduce

$$c_1\left(\mathcal{E} \otimes \mathcal{F}\right) = \operatorname{rk}\mathcal{E}\, c_1(\mathcal{F}) + c_1(\mathcal{E})\operatorname{rk}\mathcal{F}.$$

To compute the terms at the end, we have

$$\begin{aligned} G_{0,1} &= p_*\mathcal{O}(1) \\ &\simeq \pi_* j^*\mathcal{O}(1) \\ &\simeq \pi_*\omega_\pi \end{aligned}$$

using the isomorphism $H^0(\mathbb{P}^{g-1}, \mathcal{O}(1)) \simeq H^0(C, \omega_C)$ on fibers, which implies we have an isomorphism globally. So, we have $c_1\left(G_{0,1}\right) = \lambda$, the hodge class.

Next, note that

$$c_1\left(\wedge^n E\right) = \binom{\operatorname{rk}E - 1}{n-1} c_1(E)$$

This implies

$$c_1(G_{k-2,2}) = \sum(-1)^\ell\left(\binom{g}{k-2-\ell}c_1(G_{0,2+\ell}) + \binom{g-1}{k-3-\ell}\operatorname{rk}\left(G_{0,2+\ell}\right)\lambda\right)$$

So, we still need to compute

(1) $c_1\left(G_{0,2+\ell}\right)$
(2) $\operatorname{rk} G_{0,2+\ell}$.

For this, observe

$$\begin{aligned}\operatorname{Sym}_b G_{0,1} &= \operatorname{Sym}^b p_* \mathcal{O}(1) \\ &\simeq p_* \operatorname{Sym}^b p^* p_* \mathcal{O}(1) \\ &\to p_* \operatorname{Sym}^b \mathcal{O}(1) \\ &\simeq p_* \mathcal{O}(b).\end{aligned}$$

Therefore, $G_{0,b} \simeq \operatorname{Sym}^b G_{0,1}$. Then,

$$c_1 \left(\operatorname{Sym}_n E\right) = \binom{\operatorname{rk}(E) + n - 1}{\operatorname{rk}(E)} c_1(E).$$

Plugging this into our expression, we can work out that $c_1(G_{k-2,2})$ is some constant multiple of λ.

Remark 12.1. One can also determine this constant with Porteous' formula as Kempf did in his proof. But that approach would be much more difficult.

12.2. **Computing the first class.** Next, we want to compute

$$c_1 \left(\pi_* j^* \wedge^{k-2} \mathcal{M}(2)\right).$$

Define

$$\mathcal{H}_{a,b} := \pi_* j^* \wedge^a \mathcal{M}(b)$$

If $b \geq 2$, using cohomology and base change, we see that

$$H^1 \left(C, \wedge^a \mathcal{M}_{\omega_C} \otimes \omega_C^{\otimes b}\right) = 0$$

(as $K_{a-1,3}(C, \omega_C) = 0$). This implies that $H_{a,b}$ are vector bundles for $b \geq 2$.

We then have an exact sequence

(12.3)
$$0 \longrightarrow \mathcal{H}_{a,b} \longrightarrow \wedge^a \mathcal{H}_{0,1} \otimes \mathcal{H}_{0,b} \longrightarrow \mathcal{H}_{a-1,b+1} \longrightarrow 0$$

We can use Grothendieck Riemann Roch to compute $\mathcal{H}_{0,b}$ and we know $\lambda = \mathcal{H}_{0,1}$.

Using that $j^* \mathcal{O}(1) = \omega_\pi$, for $b \geq 2$, we have

$$\begin{aligned}
c_1 (\mathcal{H}_{a,b}) &= c_1 \left(\pi_! \omega_\pi^b\right) \\
&= \pi_* \left(\left[\operatorname{Ch} \omega_\pi^b \operatorname{Td}\left(\omega_\pi^\vee\right)\right]_2\right) \\
&= \pi_* \left(1 + b\, c_1(\omega_\pi) + \frac{b^2}{2} c_1^2(\omega_\pi) + \cdots\right)\left(1 - \frac{c_1(\omega_\pi)}{2} + \frac{c_1^2(\omega_\pi)}{12} + \cdots\right) \\
&= \left(\frac{1}{12} + \frac{b^2}{2} - \frac{b}{2}\right) \pi_* c_1^2 (\omega_\pi) \\
&= \left(1 + 6b^2 - 6b\right) \lambda
\end{aligned}$$

where the last step uses Mumford's formula.

Either using a computer, or doing it by hand, one obtains the following expression for the Koszul class:

Theorem 12.2 (Hirschowitz-Ramanan)**.** *We have*

$$\begin{aligned}
[\mathrm{Kos}] &= \frac{6(k+1)(\lambda-1)(2k-4)!}{(k-2)!k!}\lambda \\
&= (k-1)\,[\mathrm{Hur}]\,.
\end{aligned}$$

Proof. The first equality follows from what we have seen above, and the second follows from the Harris-Mumford calculation (which was originally done over $\overline{\mathcal{M}_g}$, but later Kempf was able to carry out Porteous just over $\mathcal{M}_g$). □

12.3. **The factor of** $k-1$**.**

Question 12.3. How do we interpret the mysterious factor of $k-1$.

Here is an answer: Let X be a complex manifold. Let $M(x) = \left[a_{i,j}(x)\right]$ be a matrix of holomorphic functions on X.

Lemma 12.4. *Suppose* $p \in X$ *and* $M(p)$ *has rank at most* r*. Then the holomorphic function* $\det(M(x))$ *vanishes to order at least* $n-r$*.*

Proof. After replacing $M(x)$ with $P^{-1}MP$ for some scalar rank d matrix, with P chosen so $M(p)$ is in Jordan normal form. This then tells us the determinant has at least $n-r$ zeros.

There are now two cases

Case 1: The first row of $M(p)$ **is nonzero.** In this case, let $M_{1,c}$ be the $1, c$ minor. As $M(p)$ is in Jordan normal form, we have $\operatorname{rk} M_{1,c}(p) \leq r-1$. By induction on n, $\det(M_{1,c}(x)$ vanishes to order at least $(n-1)-$

$(r-1) = n-r$. This implies $\det(M(x))$ vanishes to order at least $n-r$.

Case 2: The first row of $M(p)$ is 0. In this case, the result is straightforward. Here we have $\det(M_{1,c}(p))$ vanishes to order at least $n-r-1$, the whole determinant vanishes to order at least $n-r$. $\square$

Proposition 12.5 (Hirschowitz-Ramanan)**.** *Let $[C] \in \mathcal{M}_g$ be a smooth point of the Hurwitz divisor. Then, $b_{k-1,1}(C, \omega_C) \le k-1$.*

Remark 12.6. In fact, the reverse inequality follows easily from the eagon northcott complex. That is, we have

$$b_{k-1,1}(C, \omega_C) \ge k-1.$$

13. 5/12/17

We'll start with explaining why scrolls come up. We'll describe the relation between syzygies of curves and syzygies of scrolls.

Last time, we saw the Hirschowitz Ramanan computation implies that for a curve $C \in \mathrm{Hur}$ which is a smooth point of Hur, then $b_{k-1,1}(C, \omega_C) \le k - 1 \frac{[\mathrm{Kos}]}{[\mathrm{Hur}]}$. In fact, the reverse inequality holds as well.

We will show that for any $[C] \in \mathrm{Hur}$ we also have the reverse inequality.

Lemma 13.1. *For every curve $C \in \mathrm{Hur}$, we have $b_{k-1,1}(C, \omega_C) \ge k-1$.*

Proof. This follows by using the scroll in which the curve lies. By upper semicontinuity, it suffices to show that for a general curve $C \in \mathrm{Hur}$ we have $b_{k-1,1}(C, \omega_C) \ge k-1$. Let $f : C \to \mathbb{P}^1$ be a base point free pencil of degree k. Let X be the associated scroll with $C \subset X \subset \mathbb{P}^{g-1}$ so that the rulings of the scroll cut out the basepoint free pencil on the curve. Thanks to the eagon Northcott complex, we know $\Gamma_X(H)$ has a minimal free resolution of length $k-1$, with $H = \mathcal{O}_X(1)$, and $b_{k-1,1}(X, H) = k-1$. The claim then follows immediately from the following proposition. $\square$

Proposition 13.2. *The restriction $\mathcal{O}_X \to \mathcal{O}_C$ induces an inclusion $b_{p,1}(X, H) \to b_{p,1}(C, \omega_C)$.*

Remark 13.3. This says, geometrically, that some of the syzygies of the curve come from syzygies of the scroll.

Proof. Recall the definition of the kernel bundle, given by

$$0 \longrightarrow M_H \longrightarrow H^0(\mathcal{O}_X(H)) \longrightarrow \mathcal{O}_X(1) \longrightarrow 0 \tag{13.1}$$

Writing out the kernel bundle description of Koszul cohomology, we have the exact sequence
(13.2)

$$\begin{tikzcd} 0 \arrow[r] & \wedge^{p+1} H^0(X,H) \arrow[r] \arrow[d, "\simeq"] & H^0(\wedge^p M_H \otimes H) \arrow[r] \arrow[d, "r_C"] & K_{p,1}(X,H) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \wedge^{p+1} H^0(C,\omega_C) \arrow[r] & H^0(\wedge^p M_{\omega_C} \otimes \omega_C) \arrow[r] & K_{p,1}(C,\omega_C) \arrow[r] & 0 \end{tikzcd}$$

To show the natural map $K_{p,1}(X,H) \to K_{p,1}(C,\omega_C)$ is injective, it is equivalent to show r_C is injective. We have

$$\ker(r_C) = H^0\left(X, \wedge^p M_H \otimes H \otimes I_{C/X}\right)$$

coming from the following exact sequence

$$0 \longrightarrow I_{C/X} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0. \tag{13.3}$$

To conclude, is suffices to show $H^0(X, \wedge^p M_H \otimes H \otimes I_{C/X}) = 0$. Now recall that we have

$$K_{p+1,1}(X, I_{C/X}; H) = H^0(X, \wedge^p M_H \otimes H \otimes I_{C/X}),$$

so it suffices to show

$$K_{p+1,1}(X, I_{C/X}; H) = 0.$$

For this, the exact sequence

$$0 \longrightarrow \mathcal{O}_X(H) \longrightarrow I_{C/X}(H) \longrightarrow \omega_C \longrightarrow 0 \tag{13.4}$$

yields an exact sequence on Koszul cohomology
(13.5)

$$\begin{tikzcd} K_{p+2,0}(X, I_{C/X}; H) \arrow[r] & K_{p+2,0}(X,H) \arrow[r, "\phi"] & K_{p+2,0}(C,\omega_C) \arrow[dll] \\ K_{p+1,1}(X, I_{C/X}; H) \arrow[r] & K_{p+1,1}(X;H). & \end{tikzcd}$$

Observe that ϕ is an isomorphism because the map on global sections $H^0(X, nH) \to H^0(C, \omega_C^{\otimes n})$ is an isomorphism, using that both X and C are linearly normal, so the two groups above are both identified with $H^0(\mathbb{P}^{g-1}, nH)$. To conclude, it suffices to verify that $K_{p+1,1}(X;H) = 0$. Indeed, from the definition of Koszul cohomology, we have that this is the middle cohomology of the complex

$$\begin{array}{c} \wedge^{p+2} H^0(X, H) \otimes H^0(X, I_{C/X}) \\ \downarrow \\ \wedge^{p+1} H^0(X, H) \otimes H^0(X, H \otimes I_{C/X}) \\ \downarrow \\ \wedge^{p} H^0(X, H) \otimes H^0(X, 2H \otimes I_{C/X}). \end{array} \tag{13.6}$$

So, it suffices to show

$$\wedge^{p+1} H^0(X, H) \otimes H^0(X, H \otimes I_{C/X}) = 0.$$

In turn, it suffices to verify

$$H^0(X, H \otimes I_{C/X}) = 0,$$

which holds by the crucial assumption that C is linearly normal. □

13.1. **K3 Surfaces.**

Definition 13.4. A K3 surface is a smooth projective surface X with $K_X \simeq \mathcal{O}_X$ and $H^1(\mathcal{O}_X) = 0$.

Lazarsfeld and Voisin studied Brill Noether theory and syzygies using K3 surfaces. Let $C \subset X$ be a smooth curve. We want to relate $\mathcal{L}$, a line bundle on a curve C to the study of vector bundles on a K3 surface. The basic construction is due to Mukai:

Construction 13.5. Let $\mathcal{L}$ be a line bundle on a curve $C \subset X$ and suppose $\mathcal{L}$ is base point free. We have an evaluation map

$$\text{ev} : H^0(C, \mathcal{L}) \otimes \mathcal{O}_C \to \mathcal{L}.$$

We studied $M_C = \ker \text{ev}$. We next construct a related line bundle on X.

Let $i : C \to X$ be a closed immersion. We define the **Lazarsfeld-Mukai bundle** $F_{\mathcal{L}}$ as the kernel

$$0 \longrightarrow F_{\mathcal{L}} \longrightarrow H^0(C, \mathcal{L}) \otimes \mathcal{O}_X \longrightarrow i_*\mathcal{L} \longrightarrow 0 \tag{13.7}$$

Proposition 13.6. *The Lazarsfeld-Mukai bundle $F_{\mathcal{L}}$ constructed in Construction 13.5 is locally free.*

Proof. The claim is local, so we may assume L is trivial. Then, we have an exact sequence

$$0 \longrightarrow F_{\mathcal{L}} \longrightarrow H^0(C, \mathcal{L}) \otimes \mathcal{O}_X \longrightarrow i_*\mathcal{O}_C \longrightarrow 0 \tag{13.8}$$

Remark 13.7. Recall that the **homological dimension** of an R-module M is the minimal length of a projective resolution. By Auslander-Buchsbaum formula relates the homological dimension to depth. That is,

$$\mathrm{dh}(M) + \mathrm{depth}(M) = \dim R.$$

For E a coherent sheaf, we have

$$\mathrm{dh}(E) = \max\{E_x : x \in X\}.$$

As follows from the Auslander-Buchsbaum formula and inequalities on depth, if we have a short exact sequence of coherent sheaves

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0 \tag{13.9}$$

for F free, we have

$$\mathrm{dh}(E) = \max\{0, \mathrm{dh}(G) - 1\}.$$

Using the above, it is equivalent to show that $\mathrm{dh}(\mathcal{O}_C) = 1$. But indeed this follows because homological dimension is the minimal length of a projective resolution, and we have a length 1 resolution

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0 \tag{13.10}$$

so $\mathrm{dh}(\mathcal{O}_C) = 1$. □

14. 5/15/17

Today, we'll discuss using K3 surfaces to understand curves.

14.1. **Interlude on the Picard group of a K3 surface.**

Definition 14.1. Let X be a smooth projective variety. A **cycle of codimension** r is an element Z of the free abelian group over $\mathbb{Z}$, $Z^r(X)$, generated by closed irreducible subschemes of codimension r.

We have the following three notions of equivalence for cycles:

Definition 14.2. Two cycles $Z_1, Z_2 \in Z^r(X)$ are **rationally equivalent** denoted $Z_1 \sim_{\mathrm{rat}} Z_2$ if there exists a closed subscheme $V \subset X \times \mathbb{P}^1$, flat over $\mathbb{P}^1$ so that

$$V \cap X \times \{0\} = Z_1$$
$$V \cap X \times \{\infty\} = Z_2$$

We define the **Chow group**

$$A^r(X) := Z^r(X)/\sim_{\text{rat}}$$

Further, $\text{Pic}(X) = A^1(X)$.

Definition 14.3. Two cycles $Z_1, Z_2 \in Z^r(X)$ are **algebraically equivalent** denoted $Z_1 \sim_{\text{alg}} Z_2$ if there exists smooth curve C and a closed subscheme $V \subset X \times C$, flat over C with two points $p, q \in C$ so that

$$V \cap X \times \{p\} = Z_1$$
$$V \cap X \times \{q\} = Z_2$$

The **Neron-Severi** group is

$$NS(X) := Z^1(X)/\sim_{\text{alg}}$$

Remark 14.4. It turns out the Neron-Severi group is the set of connected components of the Picard group.

Theorem 14.5 (Neron-Severi)**.** $NS(X)$ *is a finitely generated abelian group.*

Definition 14.6. Let X be a surface. Two invertible sheaves L_1, L_2 are **numerically equivalent**, denoted $L_1 \sim_{\text{num}} L_2$ if for all $M \in \text{Pic}(X)$ we have

$$(L_1 \cdot M) = (L_2 \cdot M)$$

Remark 14.7. Note that $L_1 \sim_{\text{alg}} L_2$ implies $L_1 \sim_{\text{num}} L_2$. We further define $\text{num}(X) = \text{Pic}(X)/\sim_{\text{num}}$.

Remark 14.8. $\text{num}(X)$ is a quotient of $NS(X)$, hence $\text{num}(X)$ is finitely generated. Further, $\text{num}(X)$ turns out to be free.

This is because if $L^n = \mathcal{O}_X$, we obtain $\mathcal{L} \sim_{\text{num}} \mathcal{O}_X$ because $0 = \mathcal{L}^n \cdot M = n\mathcal{L} \cdot M$, which implies $\mathcal{L} \cdot M = 0$.

Proposition 14.9. *Let X be a K3 surface over the complex numbers. Then the natural maps*

$$\text{Pic}(X) \to NS(X) \to \text{num}(X)$$

are all isomorphisms. In particular, $\text{Pic}(X) \simeq \mathbb{Z}^{r(X)}$ *for some integer* $r(x)$.

Proof. Assume L is numerically trivial. Then, for any ample line bundle H with $\mathcal{L} \cdot H = 0$. Assume $\mathcal{L} \neq \mathcal{O}_X$. This implies that neither $\mathcal{L}$ nor $\mathcal{L}^\vee$ are effective, since the intersection of an ample with any curve is positive (as we can take a high enough power of $\mathcal{L}$ which is very ample and moves in a pencil, and then has positive intersection).

We then have by Serre duality that

$$h^0(L) = 0 = h^2(L) = h^0(L^\vee).$$

Then,

$$-h^1(L) = \chi(L) = \frac{1}{2}(L \cdot L + \omega_X) + \chi(\mathcal{O}_X) = \frac{1}{2}L^2 + 2$$

Therefore,

$$\frac{1}{2}(L)^2 + 2 \leq 0$$

and so

$$L^2 < 0$$

which contradicts numerical triviality of L. □

Definition 14.10. Let X be a K3 surface and fix an ample line bundle H. For a coherent sheaf $\mathcal{F} \in \mathrm{coh}(X)$, define the **rank** of $\mathcal{F}$ to be its rank as a sheaf over the generic point (i.e., the rank of the corresponding module over $K(X)$).

Define the **slope** of $\mathcal{F}$ to be

$$\mu(\mathcal{F}) := \frac{\deg_H \mathcal{F}}{\mathrm{rk}(\mathcal{F})}.$$

Definition 14.11. Let $\mathcal{F}$ be. Then $\mathcal{F}$ is said to be H**-stable** if for all subsheaves $0 \subsetneq E \subsetneq F$ with $0 < \mathrm{rk}\, E < \mathrm{rk}\, F$, we have

$$\mu(E) < \mu(F).$$

Example 14.12. The sheaf $H \oplus \mathcal{O}_X$ is not stable. To see why, note that $c_1(H + \mathcal{O}_X) = H$. Therefore, $\mu(H \oplus \mathcal{O}_X) = H^2/2 > 0$. But,

$$\mu(H) = H^2 > H^2/2 = \mu(H \oplus \mathcal{O}_X)$$

Remark 14.13. Slope stability is a useful condition for controlling the automorphisms groups. This yields an analogy between stable vector bundles and stable curves. We now explain this further.

Proposition 14.14. *Let F be H-stable. Then, any nonzero endomorphism $\phi : \mathcal{F} \to \mathcal{F}$ is an isomorphism.*

Proof. Suppose ψ were not an isomorphism. Then, we claim $\ker \psi \neq 0$. To see this, if ψ were injective, we would have

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0 \tag{14.1}$$

which implies that $c_i(G) = 0$ for all i and the rank is 0 (by additivity of rank in short exact sequences). This implies that in fact $\mathcal{G} = 0$.

Therefore, ψ is not injective. Hence, we have some kernel

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F} \tag{14.2}$$

Thus, $\mathcal{G} \subset \mathcal{F}$ is some nonzero subbundle. Note that $\mathcal{G}$ is also torsion free because it is a subsheaf of $\mathcal{F}$, which is torsion free by assumption. Hence, $\mathrm{rk}(\mathcal{F}) > \mathrm{rk}(\mathcal{G}) \geq 1$. Which implies $\mathrm{rk}\,\mathrm{im}\,\phi < \mathrm{rk}\,\mathcal{F}$. We now apply stability to $\mathrm{im}\,\phi \subsetneq \mathcal{F}$.

Using stability, we have

$$\mu\left(\mathrm{im}\,\phi\right) < \mu\mathcal{F}.$$

We have a short exact sequence

$$0 \longrightarrow \ker\phi \longrightarrow \mathcal{F} \longrightarrow \mathrm{im}\,\phi \longrightarrow 0 \tag{14.3}$$

Since degree and rank are additive, we have

$$\deg\mathcal{F} = \deg\ker\phi + \deg\mathrm{im}\,\phi\,\mathrm{rk}\,\mathcal{F} \qquad = \mathrm{rk}\ker\phi + \mathrm{rk}\,\mathrm{im}\,\phi$$

We then obtain

$$\mu(\mathcal{F}) - \mu(\ker\phi) = \frac{\mathrm{rk}\,\mathrm{im}\,\phi}{\mathrm{rk}\ker\phi}\left(\mu(\mathrm{im}\,\phi) - \mu\mathcal{F}\right).$$

We know the right hand side is negative. But, by stability of $\mathcal{F}$, the left hand side is positive, a contradiction. $\square$

14.2. **Stability of Lazarsfeld-Mukai bundles.** Recall the setup from last time. Let $i : C \subset X$ be a curve and let $A \in \mathrm{Pic}\,C$ be a basepoint free line bundle. Then, we construct F_A as the kernel

$$0 \longrightarrow F_A \longrightarrow H^0(A) \otimes \mathcal{O}_X \longrightarrow i_*A \longrightarrow 0. \tag{14.4}$$

Proposition 14.15. *Assume the Picard rank* $\rho(X) = 1$ *and* $\mathrm{Pic}\,X = \mathbb{Z}\,[C]$ *and* $H := \mathcal{O}_X(C)$. *Then,* F_A *is stable.*

Proof. For any vector bundle which is a subsheaf of a free sheaf $V \subset \mathcal{O}_X^{\oplus a}$ and any $1 \leq s \leq \mathrm{rk}(V)$, consider $\wedge^s V^\vee$ as a vector bundle. Let $b = \binom{a}{s}$. We then have $\wedge^s V \subset \mathcal{O}_X^{\oplus b}$. Therefore,

$$\mathrm{End}_{\mathcal{O}_X}(\wedge^s V) \simeq \wedge^s V \otimes \wedge^s V^\vee \subset \left(\wedge^s V^\vee\right)^{\oplus b}.$$

It follows

$$0 \neq \mathrm{id} \in H^0\left(\mathrm{End}\left(\wedge^s V\right)\right)$$

Therefore, $H^0\left(\wedge^s V^\vee\right) \geq 1$.

Now, let $E \subset F_A$ be a subsheaf with $\operatorname{rk} E < \operatorname{rk} F_A$. By the above, taking $s = \operatorname{rk} E$, and noting that $E \subset F_A \subset \mathcal{O}_X \otimes H^0(A)$, we see

$$h^0(\det E^\vee) \geq 1$$
$$h^0(\wedge^{e-1} E^\vee) \geq 1.$$

Then, since $\det E$ has a section, we conclude $\det E = kH^\vee$ for $k \geq 0$.

Lemma 14.16. *We have* $\det E \neq 0$*. That is,* $k \neq 0$ *above.*

Proof. Note that

$$E \simeq \wedge^{e-1} E^\vee \otimes \det E.$$

Since $k = 0$, we have $E \simeq \wedge^{e-1} E^\vee$, and so E would then have a section. Taking cohomology for the exact sequence, we get

$$F_A \longrightarrow H^0(A) \otimes \mathcal{O}_X \longrightarrow i_* A, \tag{14.5}$$

we see

$$0 \longrightarrow H^0(F_A) \longrightarrow H^0(A) \xrightarrow{\simeq} H^0(A) \tag{14.6}$$

which implies $H^0(F_A) = 0$ and so $H^0(E) = 0$. □

So, we know $k > 0$. Then,

$$\det F_A = -c_1(\text{im } A) = -c_1(\mathcal{O}_C).$$

This implies $i_* A \simeq \mathcal{O}_C$ outside of codimension 2. Hence, from the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0 \tag{14.7}$$

we obtain $c_1(\mathcal{O}_C) = c_1(\mathcal{O}_X(C))$ It follows that $\det F_A = -[C] = -H$. Since these degrees are negative, and the rank of E is less than that of A, we obtain $\operatorname{rk}(E) < \operatorname{rk}(F)$, using that $k > 0$. □

15. 5/17/17

15.1. **Hirzebruch-Riemann-Roch for K3 surfaces.** Let us start by recalling Hirzebruch-Riemann-Roch: For E a vector bundle and X smooth and projective then

$$\chi(E) = \int_X c(E) \cdot \mathrm{Td}(X)$$

where integration means taking the top degree piece. Here, $\mathrm{Td}(X) = \mathrm{Td}(T_X)$.

Lemma 15.1. *For* X *a K3 surface, we have* $\deg c_2(T_X) = 24$.

Proof. If X is a K3 surface, then $c_1(T_X) = 0$. Therefore,

$$\begin{aligned} 2 &= \chi(\mathcal{O}_X) \\ &= \deg\left[1 + \frac{c_1}{2} + \frac{1}{12}(c_2 + c_2)\right]_2 \end{aligned}$$

This implies $12 \cdot 2 = \deg c_2(T_X)$. Hence, $\deg c_2(T_X) = 24$. $\square$

Recall that

$$\mathrm{Ch}(E) = \mathrm{rk}\, E + c_1(E) + \frac{1}{2}\left(c_1^2(E) - 2\,c_2(E)\right).$$

Corollary 15.2. *For X a K3 surface and E any vector bundle, we have*

$$\chi(E) = \deg\left(\mathrm{Ch}_2(E) + 2\,\mathrm{rk}(E)\right).$$

Proof. This follows by plugging in the result of the previous lemma to Hirzebruch-Riemann-Roch. $\square$

15.2. **Recollection of Brill-Noether theory.** Let C be a curve and $A \in \mathrm{Pic}(C)$. Recall the Brill-Noether number

$$\rho(A) := g - h^0(A)h^1(A).$$

This is the "expected dimension" of the locus of line bundles of degree $\deg A$ and $h^0(A)$ sections. We let $W^{h^0(A)-1}_{\deg A}$ denote this locus.

We'll give a simple argument due to Lazarsfeld for that a general curve the dimension of W^r_d is the expected dimension.

Assume $C \subset X$ is a K3 surface. Let A be basepoint free. Recall

$$0 \longrightarrow F_A \longrightarrow H^0(C, A) \otimes \mathcal{O}_X \longrightarrow i_* A \longrightarrow 0 \tag{15.1}$$

Proposition 15.3. *For A a basepoint free invertible sheaf on a curve C with $C \subset X$ for X a K3 surface, we have*

$$\chi\left(F_A \otimes F_A^\vee\right) = 2 - 2\rho(A).$$

Proof. Ch gives a ring homomorphism from the K group to the chow ring. That is,

$$\mathrm{Ch}(F_A \otimes F^\vee) = \mathrm{Ch}(F_A) \cdot \mathrm{Ch}(F_A^\vee).$$

Exercise 15.4. Let

$$\begin{aligned} c_t(E) &= 1 + c_1(E)t + c_2(E)t^2 + \cdots \\ &= (1 + a_1(E)t) \cdots (1 + a_s(E)t) \end{aligned}$$

and comparing $c_t(E)$ to

$$\begin{aligned} c_t(E^\vee) &= 1 + c_1(E^\vee)t + c_2(E^\vee)t^2 + \cdots \\ &= (1 - a_1(E)t) \cdots (1 - a_s(E)t) . \end{aligned}$$

Show using the above and the splitting principle that

$$c_i(E^\vee) = (-1)^i c_i(E).$$

By the above, we have

$$\begin{aligned} \mathrm{Ch}(F_A) &= \mathrm{rk}(F_A) + c_1(F_A) + \frac{1}{2}\left(c_1^2(F_A) - 2\,c_2(F_A)\right) \\ \mathrm{Ch}(F_A^\vee) &= \mathrm{rk}(F_A) - c_1(F_A) + \frac{1}{2}\left(c_1^2(F_A) - 2\,c_2(F_A)\right). \end{aligned}$$

Letting $e = h^0(C, A)$ and $L = [C]$, we have

$$\begin{aligned} \mathrm{rk}\, F_A &= e \\ c_1\, F_A &= -L \end{aligned}$$

We using the exact sequence

$$0 \longrightarrow F_A \longrightarrow H^0(C, A) \otimes \mathcal{O}_X \longrightarrow i_*A \longrightarrow 0 \tag{15.2}$$

and the facts that $\chi i_* A = \chi(C, A)$ (we saw this last time, but essentially it follows because the two agree outside a codimension 2 set) and $\chi(\mathcal{O}_X) = 2$, we have

$$\begin{aligned} 2e &= \chi(F_A) + \chi(C, A) \\ &= \chi(F_A) + (d + 1 - g). \end{aligned}$$

Hence, using Hirzebruch Riemann Roch as above, we get $\chi(F_A) = \mathrm{Ch}_2(F_A) + 2\,\mathrm{rk}(F_A)$, and so

$$2e + \mathrm{Ch}_2(F_A) + d + 1 - g = 2e$$

It follows

$$\frac{1}{2}L^2 - c_2(F_A) = g - 1 - d. \tag{15.3}$$

We have an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow L \longrightarrow \mathcal{O}_C(C) \longrightarrow 0 \tag{15.4}$$

Note that $\mathcal{O}_C(C) = \omega_C$ by adjunction because $K_X = \mathcal{O}_X$. Therefore,

$$\chi(L) - 2 = g - 1.$$

Using this and Riemann-Roch (on surfaces) which says $\chi(L) = \frac{1}{2} L \cdot (L + K_X) + 2$, we obtain

$$\frac{1}{2} L^2 = g - 1.$$

We then obtain, by plugging in the previous line to Equation 15.3

$$c_2(F_A) = d,$$

Collating this, and using Hirzebruch-Riemann-Roch, we have

$$\begin{aligned} \chi\left(F_A \otimes F_A^\vee\right) &= \left[2 \operatorname{rk}\left(F_A \otimes F_A^\vee\right) + \mathrm{Ch}_2\left(F_A \otimes F_A^\vee\right)\right] \\ &= 2e^2 + \mathrm{Ch}_2(F_A) \end{aligned}$$

Therefore,

$$\chi\left(F_A \otimes F_A^\vee\right) = 2e^2 + e\,(2g - 2 - 2d) - (2g - 2)\,.$$

Therefore, since

$$\rho(A) = g - e\,(e + g - d - 1)\,,$$

with $e + g - d - 1 = h^1(A)$. It follows that

$$\chi\left(F_A \otimes F_A^\vee\right) = 2 - 2\rho(A).$$

□

15.3. **Brill Noether on K3 surfaces.**

Definition 15.5. Let k be a field with $\operatorname{ch} k = 0$, X a variety over k and E a vector bundle on X. We say a vector bundle E is **simple** if $\mathrm{Hom}(E, E) = k$.

Lemma 15.6. *Assume* $\operatorname{Pic} X = \mathbb{Z}[C]$ *and* F_A *as above the Lazarsfeld-Mukai bundle associated to* A *on* $C \subset X$*. Then,* F_A *is simple.*

Proof. We know F_A is stable, so any nonzero morphism $\phi : F_A \to F_A$ is an isomorphism. We claim ϕ is constant. Assume otherwise. Pick any $x \in X$. Then, $\phi \otimes k(x)$ is a matrix. Pick any eigenvalue λ. Then $\phi - \lambda$idis not an isomorphism and is nonzero, a contradiction. □

We deduce the following result in Brill-Noether theory.

Proposition 15.7. *Let* $C \subset X$ *be a smooth curve and* $\operatorname{Pic}(X) \simeq \mathbb{Z}[C]$*. Then, for any* $A \in \operatorname{Pic}(C)$ *we have* $\rho(A) \geq 0$*.*

Proof. Assume A is basepoint free. Then,

$$\begin{aligned}\chi\left(F_A \otimes F_A^\vee\right) &= \left(F_A \otimes F_A^\vee\right) - h^1\left(F_A \otimes F_A^\vee\right) + h^2(F_A \otimes F_A^\vee) \\ &= 2 - h^1\left(F_A \otimes F_A^\vee.\right).\end{aligned}$$

Therefore,

$$2 - 2\rho(A) = 2g - h^1(F_A \otimes F_A^\vee),$$

which implies $\rho(A) \geq 0$.

If A is not basepoint free, let Z be the base locus. Then, $A(-Z)$ is basepoint free. Then,

$$\begin{aligned}\deg(A(-Z)) &< \deg(A) \\ h^0(A(-Z)) &= h^0(A)\end{aligned}$$

This implies

$$0 < \rho\left(A(-Z)\right) < \rho(A)$$

$\square$

15.4. **Deformation theory of Hilbert schemes.** One reference for this section is Sernesi's book on deformation theory. We now state the main results on properties of the Hilbert scheme.

Let $X \subset \mathbb{P}^r$ be a projective variety and fix a polynomial $p(t) \in \mathbb{Q}[t]$. The Hilbert functor $\mathcal{H}^Y_{p(t)}$ assigns to any scheme locally noetherian S-scheme all flat morphisms

$$\begin{tikzcd} \mathcal{X} \arrow[rr] \arrow[dr] & & X \times S \arrow[dl] \\ & S & \end{tikzcd} \tag{15.5}$$

so that each fiber has Hilbert polynomial $p(t)$.

Theorem 15.8. *The functor $\mathcal{H}^Y_{p(t)}$ is representable by a scheme H. Further, there is a universal family $Z \subset Y \times H$, flat over H which is universal, meaning that for any S and $\mathcal{X} \subset Y \times S$ in $\mathcal{H}^Y_{p(t)}(S)$ so that there is a fiber square*

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d] & Z \arrow[d] \\ S \arrow[r] & H \end{tikzcd} \tag{15.6}$$

We have the following result on deformation theory:

Theorem 15.9. *Assume* Y *is smooth and let* $[X \subset Y] \in H(\mathit{Spec}\, k)$ *where* X *is smooth. Then,*

(1) $T_X(H) \simeq H^0(X, N_{X/Y})$
(2) *If* $H^1(X, N_{X/Y}) = 0$ *then* H *is smooth at* $[X]$.

16. 5/22/17

Today's objective is to prove Lazarsfeld's Brill Noether theorem. We've already seen that on a K3 surface of Picard rank 1, the corresponding W^r_d is empty. Lazarsfeld's theorem says that when $\rho(g, r, d) > 0$, all W^r_d's are smooth of the expected dimension ρ.

Definition 16.1. Fix $r, d > 0$ and let X be a K3 surface with $\operatorname{Pic} X \simeq \mathbb{Z}L$. Fix an isomorphism $V \simeq \mathbb{C}^{r+1}$. Define P^r_d to be the scheme parameterizing tuples (C, A, λ) so that

(1) $C \in |\mathcal{L}|$ is smooth and irreducible
(2) $A \in W^r_d(C)$ is base point free
(3) λ is a surjection
$$V \otimes \mathcal{O}_X \twoheadrightarrow i_* A$$
so that
$$H^0(\lambda) : V \simeq H^0(C, A).$$

modulo the equivalence relation that
$$(C, A, \lambda) \sim (C, A, a\lambda)$$
with $a \in \mathbb{C} - \{0\}$.

Lemma 16.2. *The functor* P^r_d *is a open subscheme of the Hilbert scheme* $\mathcal{H}(X \times \mathbb{P}(V))$*, contained in a single component of the Hilbert scheme (i.e., there is a single Hilbert polynomial associated to the Hilbert scheme in which* P^r_d *lies.*

Proof. The triple (C, A, λ) determines
$$\lambda|_C : V \otimes \mathcal{O}_C \to A$$
which yields a morphism $C \to \mathbb{P}(V)$. Since we started with a closed immersion $C \to X$ we hence have a closed immersion $C \subset X \times \mathbb{P}V$. Conversely, given such a closed embedding, we obtain (C, A, λ), and these two constructions are clearly mutually inverse. We are just using that $\lambda|_C$ being a surjection is an open condition. $\square$

There is a morphism

$$\begin{aligned}\pi\colon P^r_d &\to \mathbb{P}|\mathcal{L}| \\ (C, A, \lambda) &\mapsto C \in |\mathcal{L}|\end{aligned}$$

where $|\mathcal{L}| := \mathbb{P}H^0(\mathcal{L})$. We want to study the differential of π. A triple $(C, A, \lambda) \in P^r_d$ corresponds to $C \subset X \times \mathbb{P}(V)$. We obtain $g : C \to \mathbb{P}V$. Here $g = \phi_A$. We then have a short exact sequence

$$0 \longrightarrow g^* T_{\mathbb{P}V} \longrightarrow N_{C/X\times\mathbb{P}(V)} \xrightarrow{\alpha} N_{C/X} \longrightarrow 0 \tag{16.1}$$

Hence,

$$T_{(C,A,\lambda)} P^r_d = H^0(C, N_{C/X\times\mathbb{P}V}).$$

We have

$$\begin{aligned}T_{C,\lambda}|\mathcal{L}| &= H^0(C, N_{C/X}) \\ &= H^0(C, \mathcal{O}_C(C)) \\ &= H^0(C, \omega_C).\end{aligned}$$

We can identify $d\pi$ at (C, A, λ) with $H^0(\alpha)$.

Theorem 16.3 (Lazarsfeld). *Assume* $\operatorname{Pic} X \simeq \mathbb{Z}\mathcal{L}$. *Then,* $d\pi_{(C,A,\lambda)}$ *is surjective if and only if Petri's multiplication map*

$$\mu : H^0(C, A) \otimes H^0(C, \omega_C - A) \to H^0(C, \omega_C)$$

is injective.

Proof. To start, recall the short exact sequence

$$0 \longrightarrow g^* T_{\mathbb{P}V} \longrightarrow N_{C/X\times\mathbb{P}V} \xrightarrow{\alpha} N_{C/X} \longrightarrow 0 \tag{16.2}$$

We have the following claim:

Lemma 16.4. *The map* $H^1(\alpha)$ *is an isomorphism.*

Proof. We know $H^2(C, g^* T_{\mathbb{P}V}) = 0$ so $H^1(\alpha)$ is surjective. Second, we know $h^1(N_{C/X}) = h^1(\omega_C) = 1$. So, it suffices to show $h^1(N_{C/X\times\mathbb{P}V}) = 1$. For this, let us describe the embedding $C \subset X \times \mathbb{P}V$ with projections $pr_1 : X \times \mathbb{P}V \to X, pr_2 : X \times \mathbb{P}V \to \mathbb{P}V$. We can canonically identify

$$V \simeq H^0\left(\mathbb{P}V, \mathcal{O}(1)\right).$$

We have an exact sequence

$$0 \longrightarrow F_A \xrightarrow{\phi} V \otimes \mathcal{O}_X^\lambda \longrightarrow i_* A \longrightarrow 0 \tag{16.3}$$

Then ϕ induces a morphism

$$\mathrm{pr}_1^*(F_A) \otimes \mathrm{pr}_2^* \mathcal{O}(-1) \to \mathcal{O}_{X\times\mathbb{P}V}.$$

This map sends $S \otimes T \mapsto T\left(\phi(S)\right)$. We obtain a global section z of

$$\mathrm{pr}_1^*\left(F_A^\vee\right) \otimes \mathrm{pr}_2^* \mathcal{O}(1).$$

We can check from the exact sequence (16.3) $Z(z) = C \subset X \times \mathbb{P}V$. It follows that

$$N_{C/X\times\mathbb{P}V} \simeq F_A^\vee|_C \otimes A|_C$$

We need $h^1\left(X, F^\vee \otimes i_*A\right) = 1$. This is the same as the restriction to C because i_*A is supported on C.

Tensoring our definition of F_A by $F_A^\vee$ we get

$$0 \longrightarrow F_A \otimes F_A^\vee \longrightarrow V \otimes F_A^\vee \longrightarrow A \otimes F_A^\vee \longrightarrow 0. \tag{16.4}$$

Since F_A is simple, using Serre duality, we have

$$h^0(F_A \otimes F_A^\vee) = h^2(F_A \otimes F_A^\vee) = 1.$$

Then,

$$h^2\left(V \otimes F_A^\vee\right) = \dim V \cdot h^2(F_A^\vee) = \dim V \cdot h^0(F_A) = 0,$$

as follows from the sequence

$$0 \longrightarrow F_A \longrightarrow V \otimes \mathcal{O}_X \xrightarrow{\eta} i_*A \longrightarrow 0 \tag{16.5}$$

and the fact that $H^0(\eta)$ is an isomorphism. To show the map

$$H^1(A \otimes F^\vee) \to H^2(F_A \otimes F_A^\vee) \simeq \mathbb{C}$$

is an isomorphism, which would complete the proof, it only remains to check $H^1(V \otimes F_A^\vee)$.

Indeed, this follows because $H^1(\mathcal{O}_X) = 0$ and so from the sequence

$$\begin{tikzcd} 0 \arrow[r] & h^0(F_A) \arrow[r] & V \arrow[r, "H^0(\lambda)"] & H^0(i_*A \arrow[dll] \\ & & h^1(F_A) \arrow[r] & 0 \end{tikzcd} \tag{16.6}$$

This implies $H^1(F_A) = 0$ since $H^0(\lambda)$ is an isomorphism. Hence, $H^1(F_A^\vee) = 0$, as we wanted to show. $\square$

So, we have seen $H^1(\alpha)$ is an isomorphism. We want to show the petri map is injective if and only if $d\pi$ is surjective.

We have an exact sequence

(16.7)
$$\begin{tikzcd} 0 \arrow[r] & H^0(C, g^*T_{\mathbb{P}V}) \arrow[r] & H^0(N_{C/X\times\mathbb{P}V}) \arrow[r, "d\pi"] & H^0(N_{C/X}) \arrow[dll] \\ & H^1(g^*T_{\mathbb{P}V}) \arrow[r] & 0 & \end{tikzcd}$$

Then, $d\pi$ is surjective if and only if $H^1(g^*T_{\mathbb{P}V}) = 0$.

To complete the proof, it suffices to check $H^1(C, g^*T_{\mathbb{P}V}) = 0$ if and only if the Petri map is injective.

Here, $g = \phi_A : C \to \mathbb{P}V$, up to a choice of basis. Then, the result follows from Petri's theorem, which we'll talk about next time.

That is, Petri's theorem says:

Theorem 16.5 (Petri)**.**
$$H^1(C, g^*T_{\mathbb{P}V})^\vee = \ker\left(\mu : H^0(\omega_C - A) \otimes H^0(A) \to H^0(\omega_C)\right).$$

This tells us $d\pi$ is surjective if and only if μ is injective. □

17. 5/24/17

Today, we'll discuss the Petri map.

Definition 17.1. Given a curve C and a line bundle L on C, the **petri map** is
$$\mu : H^0(L) \otimes H^0(\omega_C - L) \to H^0(\wedge_C).$$

Late time, we studied the map $p : P^r_d \to |\mathcal{L}|$. sending $(C, A, \lambda) \mapsto C$. We saw that generic smoothness of the map p implies that the differential is surjective, which means μ_A is injective.

Today's goal will be to relate this to Brill-Noether theory.

17.1. The construction of $W^r_d(C)$.

Let $\mathcal{L}$ be a Poincare bundle on $C \times \mathrm{Pic}^d(C)$, meaning that $\mathcal{L}|_{C\times[M]} = M$ on C. These are determined up to pullback of a bundle from $\mathrm{Pic}^d(C)$. Let $q : C \times \mathrm{Pic}^d(C) \to \mathrm{Pic}^d(C)$ be the projection.

Fix a divisor E on C of degree $2g - d - 1$. Here, $g = g(C)$. Let
$$F := E \times \mathrm{Pic}^d(C) \subset C \times \mathrm{Pic}^d(C).$$

We get an exact sequence

$$0 \longrightarrow \mathcal{L}(-F) \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}|_F \longrightarrow 0 \tag{17.1}$$

and twisting by F we get

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}(F) \longrightarrow \mathcal{L}|_F(F) \longrightarrow 0 \tag{17.2}$$

For any $L \in \mathrm{Pic}^d(C)$ we have that $\deg L(E) \geq 2g-1$ so

$$h^1(C, L(E)) = 0.$$

From Grauert's formula,

$$R^1 q_* \mathcal{L}\,(F) = 0.$$

Pushing forward the exact sequence above by q, we get the exact sequence
(17.3)

$$0 \longrightarrow q_*\mathcal{L} \longrightarrow q_*\mathcal{L}(F) \xrightarrow{f} q_*\left(\mathcal{L}|_F(F)\right) \longrightarrow R^1 q_*\mathcal{L} \longrightarrow 0$$

Let

$$A := q_*\mathcal{L}(F)$$
$$B := q_*\mathcal{L}|_F(F).$$

Both A and B are vector bundles by Grauert's theorem. Further, pushforward commute with base change, so we obtain

$$f \otimes k(p) : H^0(C, L(E)) \to H^0(C, L|_E(E)).$$

Choose $[L] = P \in \mathrm{Pic}^d(C)$. Pick $H^0(C, L) \geq r+1$. Then the rank of $f \otimes \kappa(p)$ is at most $2g - d - 1 + m - (r+1)$. We define $W^r_d(C)$ as the locus of points p so that $f : A \to B$ has rank at most $2g - d - 1 + m - (r+1)$.

We saw long ago how to construct such loci in a functorial way via fitting ideals:

Set

$$\mathcal{O}_{W^r_d(C)} := \mathcal{O}_{\mathrm{Pic}^d(C)}/I_j(f),$$

where $I_j(f)$ is the fitting ideal. Recall the definition: We have a map

$$\begin{aligned}\wedge^j f : \wedge^j A \to \wedge^j B \in \mathrm{Hom}(\wedge^j A, \wedge^j B) \\ \simeq \wedge^j A^\vee \otimes \wedge^j B \\ \simeq \mathrm{Hom}\left(\wedge^j A \otimes \wedge^j B^\vee, \mathcal{O}_{\mathrm{Pic}^d(C)}\right).\end{aligned}$$

Under this correspondence, we define $I_j(f)$ as the image of $\bigwedge^j f$ in $\mathcal{O}_{\mathrm{Pic}^d(C)}$. The key property that we talked about in lecture 6 or so is that $I_j(f)$ is functorial. That is, it commutes with base change. That is, $I_j(f \otimes \operatorname{Spec} B) = I_j(f) \otimes \operatorname{Spec} B$. Loosely speaking "fitting ideals are independent of presentation."

Let's try to understand

$$T_{[L]} W_d^r(C) \subset T_{[L]} \operatorname{Pic}^d(C) \simeq H^1(C, \mathcal{O}_C) \simeq H^0(\omega_C)^\vee.$$

We wish to describe those $\mathcal{L}' \in \operatorname{Pic}^d(C \times \operatorname{Spec} k[\varepsilon])$ reducing to $\mathcal{L}$ which come from elements of $T_{[L]} W_d^r(C)$. That is we want to characterize first order deformations of L so that the sections $s_1, \ldots, s_{r+1}$ of $H^0(C, L)$ also deform.

We can describe a line bundle L via transition functions. That is, L is the same as the data $\{U_\alpha\}$ together with transition functions

$$\left\{ g_{\alpha\beta} \in \mathcal{O}^\times \left(U_\alpha \cap U_\beta \right) \right\}.$$

Let

$$\widetilde{U}_\alpha := \{U_\alpha \times \operatorname{Spec} k[\varepsilon]\}, \left\{\widetilde{g}_{\alpha\beta}\right\}$$

be the transition data for L'. As L' reduces to L, we have

$$\widetilde{g}_{\alpha\beta} = g_{\alpha\beta}(1 + \varepsilon h_{\alpha\beta}).$$

Then,

$$h_{\alpha\beta} \in \mathcal{O}^\times \left(U_\alpha \cap U_\beta \right).$$

Then, these $g_{\alpha\beta}$ satisfy the cocycle relation, which means

$$\widetilde{g}_{\alpha\beta} \widetilde{g}_{\beta\gamma} = \widetilde{g}_{\alpha\gamma}.$$

We obtain

$$g_{\alpha\beta} \left(1 + \varepsilon h_{\alpha\beta}\right) g_{\beta\gamma} \left(1 + \varepsilon h_{\beta\gamma}\right) = g_{\alpha\gamma} \left(1 + \varepsilon h_{\alpha\gamma}\right)$$

so

$$g_{\alpha\beta} g_{\beta\gamma} \left(1 + \varepsilon \left(h_{\alpha\beta} + h_{\beta\gamma}\right)\right). = g_{\alpha\gamma} \left(1 + \varepsilon h_{\alpha\gamma}\right).$$

Comparing the ε coefficient, we obtain

$$g_{\alpha\gamma} \left(h_{\alpha\beta} + h_{\beta\gamma}\right) = g_{\alpha\gamma}(h_{\alpha\gamma}).$$

It follows that

$$h_{\alpha\beta} + h_{\beta\gamma} = h_{\alpha\gamma}.$$

We want to understand when a section $s \in H^0(C, L)$ lifts to a section of $\mathcal{L}'$.

In terms of transition data, a section is a collection

$$\{s_\alpha\}, s_\alpha \in \Gamma(\mathcal{O}_{U_\alpha}) \text{ such that } s_\alpha = g_{\alpha\beta} s_\beta.$$

Then, an extension $\widetilde{s}$ of s to a global section of L' is

$$\{\widetilde{s}_\alpha\}, \widetilde{s}_\alpha \in \Gamma\left(\widetilde{\mathcal{O}}_{U_\alpha}\right)$$

satisfying the two conditions that

(1)
$$\widetilde{s}_\alpha = s_\alpha + \varepsilon s'_\alpha$$

(2)
$$\widetilde{s}_\alpha = \widetilde{g}_{\alpha\beta} \widetilde{s}_\beta$$

Expanding the second equation, we get

$$\begin{aligned} s_\alpha + \varepsilon s'_\alpha &= \widetilde{s}_\alpha \\ &= \widetilde{g}_{\alpha\beta} \cdot \widetilde{s}_\beta \\ &= g_{\alpha\beta}\left(1 + \varepsilon h_{\alpha\beta}\right)\left(s_\beta + \varepsilon s'_\beta\right). \end{aligned}$$

Expanding this and comparing coefficients of ε, we see

$$s'_\alpha = g_{\alpha\beta} s'_\beta + g_{\alpha\beta} h_{\alpha\beta} s_\beta$$

This can be simplified slightly to

$$h_{\alpha\beta} s_\alpha = s'_\alpha - g_{\alpha\beta} s'_\beta.$$

By the property that

$$h_{\alpha\beta} + h_{\beta\gamma} = h_{\alpha\gamma},$$

we have that h is a 1-cocycle in $\mathcal{C}^1\left(\{U_\alpha\}, \{\mathcal{O}_{U_\alpha}\}\right)$. That is, it determines an element $z \in H^1(C, \mathcal{O}_C)$. In other words, the tangent space to the Picard group is $H^1(C, \mathcal{O}_C) = T_{[L]} \operatorname{Pic}^d(C)$. Here, z corresponds to L'. Further, $z \otimes s = \{h_{\alpha\beta} s_\alpha\} \in H^1(C, L)$. The condition for s to extend is saying $z \otimes s = 0$ because

$$h_{\alpha\beta} s_\alpha = s'_\alpha - g_{\alpha\beta} s'_\beta = \delta\left(s'_\alpha\right) \in \mathcal{C}^0\left(\{U_\alpha\}, \{L_\alpha\}\right).$$

Theorem 17.2. *So, assume that* $L \in W^r_d(C)$ *has* $r+1$ *sections. Then,*

$$T_{[L]} W^r_d(C) = \left\{ z \in H^1\left(\mathcal{O}_C\right) : z \otimes H^0\left(C, L\right) = 0 \right\} \in H^1(C, L).$$

Equivalently, using Serre duality,

$$T_{[L]} W^r_d(C) = (\mathit{im}\ \mu)^\perp$$

For μ *the petri map.*

18. 5/26/17

18.1. **Wrapping up the Petri map.** Assume $L \in W^r_d(C)$ with $h^0(C) = r+1$. We saw

$$\nu \in H^1(C, \mathcal{O}_C) \simeq \mathrm{Pic}^d(C)$$

lies in

$$T_{[L]} W^r_d(C)$$

if and only if

$$\begin{aligned} \nu \otimes H^0(C, L) &\to H^1(C, L) \\ s &\mapsto s \otimes \nu \end{aligned}$$

is the zero map.

Lemma 18.1. *The above is equivalent to saying*

$$T_{[L]} W^r_d(C) = (\mathit{im}\ \mu)^{\perp}$$

with

$$\mu : H^0(C, L) \otimes H^0(C, \omega_C - L) \to H^0(C, \omega_C).$$

Proof. Indeed, for $s \in H^0(C, L)$ and $\nu \in H^1(C, \mathcal{O}_C) \simeq H^0(\omega_C)^\vee$ we have

$$s \otimes \nu = 0 \in H^1(C, L) \simeq H^0(\omega_C - L)^\vee$$

if and only if

$$(s \otimes \nu, t) = 0 \text{ for all } t \in H^0(\omega_C - L).$$

In turn, this is equivalent to

$$(\nu, \mu(st)) = 0 \text{ for all } t \in H^0(\omega_C - L).$$

Above, $\nu \in H^0(C, \omega_C)^\vee, \mu(st) \in H^0(C, \omega_C)$. This is the condition we wanted. Namely, if the above holds for all s and t as above, then

$$\nu \in (\mathrm{im}\ \mu)^{\perp}.$$

$\square$

Corollary 18.2. *If the Petri map* μ *is injective, then*

$$\dim T_{[L]} W^r_d(C) = \rho = g - h^0(L) h^1(L).$$

Theorem 18.3 (Brill-Noether theorem)**.** *We have* $\dim W^r_d(C) \geq \rho$ *at each point. Further, if* μ *is injective then* $W^r_d(C)$ *is smooth of dimension* ρ.

18.2. **Return to syzygies.** Suppose we have a smooth curve in a K3 surface, $C \subset X$, with $\operatorname{Pic} X \simeq \mathbb{Z}[C]$.

Then, provided C is general in its linear system $|C|$, then μ is injective, so C is Brill-Noether general.

We suspect there is a close relation between syzygies and Brill Noether theory. Therefore, it makes sense to study syzygies of curves on a K3 surface, since they behave like Brill Noether general curves.

The reason K3 surfaces are so useful is because of the Lefschetz theorem. One definition of a K3 surface is that it is a surface $X \subset \mathbb{P}^g$ so that each hyperplane section of X is a canonical curve of genus g in $\mathbb{P}^{g-1}$.

On a K3 surface, let H be the hyperplane section. Then, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X(-H) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0 \tag{18.1}$$

for $D \in |H|$. Then, twisting up by H, we get

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(H) \longrightarrow \omega_D \longrightarrow 0 \tag{18.2}$$

We then obtain a surjection on cohomology

$$H^0\left(\mathcal{O}_X(H)\right) \to H^0(\omega_D),$$

using $H^1(X, \mathcal{O}_X) = 0$. We further obtain

$$h^0(\omega_D) + 1 = h^0(\mathcal{O}_X(H)).$$

This means the curve is embedded by a complete linear system, so it is a canonical embedding.

Choosing a non-canonical splitting, we obtain an isomorphism

$$H^0(\mathcal{O}_X(H)) \simeq H^0(D, \omega_D) \oplus \mathbb{C}\{s\}$$

corresponding to choosing an embedding of $H^0(D, \omega_D) \to H^0(X, \mathcal{O}_X)$.

Now, recall some notation. Let L and $H \in \operatorname{Pic}(X)$ be two line bundles on a K3 surface X. Recall

$$K_{p,q}(X, -L; H)$$

was defined as the (p, q)th syzygy space of the $\operatorname{Sym} H^0(X, H)$ module $\oplus_{q \in \mathbb{Z}} H^0(X, qH - L)$. Let us also assume that the graded pieces of $H^0(X, qH - L)$ are 0 for $q \ll 0$. Recall $K_{p,q}(X, -L; H)$ was the cohomology of

$$\begin{aligned} \wedge^{p+1} H^0(X, H) \otimes H^0\left((q-1)H - L\right) &\to \wedge^p H^0(H) \otimes H^0(qH - L) \\ &\to \wedge^{p-1} H^0(H) \otimes H^0\left((q+1)H - L\right). \end{aligned}$$

The following lemma explains the power of K3 surfaces for studying syzygies.

Lemma 18.4 (Lefschetz theorem). *Suppose X is a K3 surface with line bundles L and $H \in \operatorname{Pic}(X)$. Suppose H is base point free and let $D \in |H|$ be a smooth curve. Assume either*

(1) $L \simeq \mathcal{O}_X$ and $H^1(X, qH) = 0$ for $q \geq 0$ (this may hold often by Kodaira vanishing, if H is ample)
(2) $(H \cdot L) > 0$ and $H^1(X, qH - L) = 0$ for $q \geq 0$.

Then,

$$K_{p,q}(X, -L; H) \simeq K_{p,q}(D, -L|_D, \omega_D).$$

Proof. By the assumption we have the following short exact sequence of $\operatorname{Sym} H^0(X, H)$ modules:
(18.3)

$$\begin{tikzcd} 0 \arrow[r] & \oplus_{q \in \mathbb{Z}} H^0(X, (q-1)H - L) \arrow[r, "s"] & \oplus_{q \in \mathbb{Z}} H^0(X, qH - L) \arrow[dl] \\ & \oplus_{q \in \mathbb{Z}} H^0(D, q\omega_D - L) \arrow[r] & 0 \end{tikzcd}$$

using that H^1 of the first term is 0. Rename the third term above by

$$B := \oplus_{q \in \mathbb{Z}} H^0(D, q\omega_D - L)$$

thought of as a $\operatorname{Sym} H^0(X, H)$ module.

Take the long exact sequence of Koszul cohomology of Equation 18.3. We obtain
(18.4)

$$K_{p,q-1}(X, -L; H) \longrightarrow K_{p,q}(X, -L; H) \longrightarrow K_{p,q}(B, H^0(X, H)) \longrightarrow K_{p-1,q}(X, -L; H)$$

We now require the following lemma due to green:

Lemma 18.5 (Green). *Let M be a graded $\operatorname{Sym}(V)$ module and $s \in V$. Then,*

$$K_{p,q}(M, V) \xrightarrow{\otimes s} K_{p,q+1}(M, V)$$

is zero.

Proof. The Koszul cohomology $K_{p,q+1}(M, V)$ is associated to the complex

$$\to \wedge^p V \otimes M_q \xrightarrow{d} \cdots .$$

Take $t_1 \wedge \cdots \wedge t_p \otimes m$ a representative of ker d. Now, consider

$$t_1 \wedge \cdots \wedge t_p \otimes sm \in K_{p,q+1}(M, V).$$

We want to show this is 0. To do this, we need to construct an element $w \in \wedge^{p+1} V \otimes M_q$ with $d(w) = t_1 \wedge \cdots \wedge t_p \otimes sm$.

For this, consider

$$s \wedge t_1 \wedge \cdots \wedge t_p \otimes m \in \wedge^{p+1} V \otimes M_q.$$

Then,

$$\begin{aligned} d(s \wedge t_1 \wedge \cdots \wedge t_p \otimes m) &= t_1 \wedge \cdots \wedge t_p \otimes sm - s \wedge d(t_1 \wedge \cdots \wedge t_p \otimes m) \\ &= t_1 \wedge \cdots \wedge t_p \otimes sm. \end{aligned}$$

Hence, $t_1 \wedge \cdots \wedge t_p \otimes sm = 0$. $\square$

Coming back to the proof of the Lefschetz hyperplane theorem, observe that we have

(18.5)

$$K_{p,q-1}(X, -L; H) \xrightarrow{\otimes s} K_{p,q}(X, -L; H) \longrightarrow K_{p,q}(B, H^0(X, H)) \longrightarrow K_{p-1,q}(X, -L; H)$$

Since the $\otimes s$ map is 0, we obtain

$$K_{p,q}(B, H^0(X < H)) \simeq K_{p,q}(X, -L; H) \oplus K_{p-1,q}(X - L, H).$$

Recall that $B = \oplus_q H^0(q\omega_D - L)$. It only remains to relate syzygies of B with respect to $\operatorname{Sym} H^0(X, H)$ to syzygies when viewed as an $\operatorname{Sym} H^0(D, \omega_D)$ module. Due to the surjective map

$$H^0(X, H) \to H^0(D, \omega_D)$$

we can consider B as a $\operatorname{Sym} H^0(D, \omega_D)$ module. We want to compare the resulting $K_{p,q}$ groups. Indeed, we obtain

$$\wedge^p H^0(X, H) \simeq \wedge^p H^0(D, \omega_D) \oplus \wedge^{p-1} H^0(D, \omega_D).$$

using the non-canonical identification

$$H^0(X, H) \simeq H^0(D, \omega_D) \oplus \mathbb{C}\langle s \rangle.$$

Exercise 18.6. Using the fact that $s|_D = 0$, show that the isomorphism

$$\wedge^p H^0(X, H) \simeq \wedge^p H^0(D, \omega_D) \oplus \wedge^{p-1} H^0(D, \omega_D).$$

is compatible with the differentials of the Koszul complex associated to B.

From this, we obtain

$$K_{p,q}(B, H^0(X, H))$$

is computed by the direct sum of the two complexes
(18.6)

$$\begin{array}{c} \wedge^p H^0(K_D) \otimes H^0((q-1)K_D - L) \oplus \wedge^{p+1} H^0(K_D) \otimes H^0((q-1)K_D - L) \\ \downarrow \\ \wedge^p H^0(K_D) \otimes H^0(qK_D - L) \oplus \wedge^p H^0(K_D) \otimes H^0(qK_D - L) \\ \downarrow \\ \vdots \end{array}$$

Therefore,

$$\begin{aligned} K_{p,q}(B, H^0(X, H)) &\simeq K_{p,q}(D, -L_D, \omega_D) \oplus K_{p-1,q}(D, -L_D, \omega_D) \\ &\simeq K_{p,q}(X, -L; H) \oplus K_{p-1,q}(X, -L; H). \end{aligned}$$

So, for $p = 0$, we get

$$K_{0,q}(D, -L, \omega_D) \simeq K_{0,q}(X, -L; H)$$

and by induction, we get

$$K_{p,q}(D, -L, \omega_D) \simeq K_{p,q}(X, -L; H)$$

□

19. 5/31/17

Hilbert scheme of points. Voisin rephrases the Koszul cohomology of a line bundle on a surface in terms of the Hilbert scheme of points.

There's a lot of techniques using chow theory and hodge theory to attack Koszul cohomology.

So, today, we'll prove some things about Hilbert schemes of points. Let X be a surface and L be a very ample line bundle. Consider $\phi_L : X \to \mathbb{P}^g$. If X is a K3, then $L^2 = 2g - 2$, as follows from Riemann Roch for surfaces.

Let $Z \subset X$ be a zero dimensional subscheme of a surface. Let $n := h^0(\mathcal{O}_Z)$. Then, $p(t) = p_Z(t) = n$.

We have

$$\mathcal{H}_X^n := \{Z \subset X : p_Z(t) = n\}.$$

Then,

$$\begin{aligned} T_{[Z]}\mathcal{H}^n &\simeq H^0(Z, N_{Z/X}) \\ &\simeq H^0(Z, \mathcal{H}om_{\mathcal{O}_Z}(I_Z/I_Z^2, \mathcal{O}_Z). \end{aligned}$$

Let's try and work out this normal bundle. We can assume Z is supported at one point by analyzing the support at just one of the points. Further, we can assume $X = \operatorname{Spec} A$ is affine, where A is local, and regular of dimension 2.

Then, $I_Z \subset A$. The condition that Z has length n means A/I_Z has length n as an A-module. We have

$$\operatorname{Hom}_{A/I}\left(I/I^2, A/I\right) \simeq \operatorname{Hom}_A(I, A/I).$$

Proposition 19.1. *Let A be a regular local ring of dimension 2. Let I be an ideal with A/I of length n. Then, $\operatorname{Hom}_A(I, A/I)$ is an A module of length at most $2n$.*

Proof. Note that depth $I \geq 1$. By Auslander Buchsbaum,

$$\operatorname{depth}(I) + \operatorname{Projdim}(I) = \dim A = 2.$$

Since the projective dimension of I is nonzero. Since it is also not free, it must have depth 1.

Localizing away from the maximal ideal, we see I is free of rank 1. Let

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow I \longrightarrow 0 \tag{19.1}$$

be a resolution of I. We obtain $P_0 = A^{k+1}, P_1 = A^k$. Applying the functor

$$\operatorname{Hom}(\bullet, A/I),$$

we get

$$0 \to \operatorname{Hom}_A(I, A/I) \to \operatorname{Hom}_A(A^{k+1}, A/I) \to \operatorname{Hom}_A(A^k, A/I) \to \operatorname{Ext}^1(I, A/I) \to 0. \tag{19.2}$$

Then,

$$\begin{aligned} \ell_A(\operatorname{Hom}_A(A^k, A/I)) &= k\left(\operatorname{Hom}_A(A, A/I)\right) \\ &= k\ell_A(A/I) \\ &= kn. \end{aligned}$$

So,

$$\ell_A(I, A/I) - (k+1)\ell_A(A/I) + k\ell_A(A/I) - \ell(\operatorname{Ext}^(I, A/I)) = 0.$$

To conclude, we only have to show

$$\ell_A(\mathrm{Ext}^1(I, A/I)) \geq n.$$

We only need

$$\ell_A(\mathrm{Ext}^1(I, A/I)) \leq n.$$

We then use

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0 \tag{19.3}$$

and apply $\mathrm{Hom}(\bullet, A/I)$. We obtain

$$0 = \mathrm{Ext}^1(A, A/I) \longrightarrow \mathrm{Ext}^1(I, A/I) \longrightarrow \mathrm{Ext}^2(A/I, A/I) \longrightarrow 0 = \mathrm{Ext}^2(A, A/I). \tag{19.4}$$

It then suffices to show

$$\ell_A\left(\mathrm{Ext}^2(A/I, A/I)\right) \leq n.$$

Then, we apply $\mathrm{Hom}(A/I, \bullet)$. We similarly get that it suffices to show

$$\ell_A(\mathrm{Ext}^1(A/I, A)) \leq n$$

using that $\mathrm{Ext}^3(\bullet, \bullet) = 0$. Since A is Gorenstein, we have

$$\mathrm{Ext}^i_{A/m}(A/\mathfrak{m}_A, A) = 0$$

and

$$\mathrm{Ext}^2_A(A/\mathfrak{m}_A, A) \simeq A/\mathfrak{m}_A.$$

We then have, for $I \subset \mathfrak{m}_A$,

$$0 \longrightarrow N \longrightarrow A/I \longrightarrow A/\mathfrak{m}_A \longrightarrow 0 \tag{19.5}$$

with $I \subset \mathfrak{m}_A$. Then, $\ell_A(N) = n - 1$. Then, by induction, we get $\mathrm{Ext}^2(A/I, A) = n$. □

Exercise 19.2. Show that the depth of $(x, y) \subset k[x, y]$ is one.

Corollary 19.3. *If X is a smooth surface, $X^{[n]} = \mathcal{H}^n_X$ is smooth of dimension $2n$.*

Proof. It suffices to show that X has dimension n. Then you use connectedness of the Hilbert scheme. □

20. 6/2/17

Today, we'll discuss Voisin's description of syzygies. Let X be a smooth projective surface. Last time, we proved Fogarty's theorem, that $X^{[n]} := \mathcal{H}^n_X$ is smooth of dimension $2n$.

Recall that the trick to proving this was to show $\dim T_p X^{[n]} \leq 2n$ and then appealing to Hartshorne's theorem on connectedness of the Hilbert schemes.

We have the universal subscheme

$$\begin{tikzcd} Z \arrow[rr, "{(p,q)}"] \arrow[dr, "q"] & & X \times X^{[n]} \arrow[dl] \\ & X^{[n]}. & \end{tikzcd} \tag{20.1}$$

Definition 20.1. Let $L \in \operatorname{Pic} X$. Define the **tautological bundle** $L^{[n]}$ by

$$L^{[n]} := q_* p^* \mathcal{L}.$$

Remark 20.2. The fiber over $[z]$, for $z \subset X$ a 0 dimensional subscheme is isomorphic to $H^0(\mathcal{L} \otimes \mathcal{O}_Z)$, which is n dimensional. Therefore, $L^{[n]}$ has rank n.

Remark 20.3. The tautological bundle is useful in Gromov Witten theorem, various invariants. If you're interested in learning more, look at Ellingsrudd-Gottsche-Lehn.

Proposition 20.4 (Ellingsrudd-Gottsche-Lehn)**.** *We have*

$$H^0\left(X^{[n]}, \det L^{[n]}\right) = \wedge^n H^0(X, L).$$

Proof. Here is a sketch of the proof. Consider the evaluation map

$$H^0(X, L) \otimes \mathcal{O}_{X^{[n]}} \to \mathcal{L}^{[n]}$$

which over $[z] \in X^{[n]}$ sends $H^0(X, L) \to H^0(X, L \otimes \mathcal{O}_z)$.

Take the nth wedge of the above evaluation map and taking global sections, we obtain a map

$$\wedge^n H^0(X, L) \to H^0(\det L_n).$$

We will prove this is an isomorphism by constructing an inverse. Recall that if X is any factorial variety and $U \subset X$ is any set whose complement has codimension at least 2 with $H^0(X, L) \simeq H^0(U, L)$ is an isomorphism. There is a Hilbert chow morphism

$$\rho : X^{[n]} \to X^{(n)} = X^n / S^n$$

where the latter denotes zero cycles of degree n and is called the **symmetric product**. We also have a projection

$$\pi : X^n \to X^{(n)},$$

Let $U \subset X^{[n]}$ be the subscheme parameterizing reduced 0-dimensional whose support has length at least $n-1$ (so we allow one degree 2 point). Let $V = \pi^{-1}(U) \subset X^n$. Let

$$\widetilde{V} := \mathrm{Bl}_\Delta V,$$

where Δ is the diagonals $\Delta = \cup_{i,j} \Delta_{i,j}$ with $\Delta_{i,j} = \{x_i = x_j\}$.

Remark 20.5. One can think of $X^{[n]}$ as a resolution of singularities for $X^{(n)}$.

Fact 20.6 (Fogarty)**.** The quotient of $\widetilde{V}$ under the S_n action is an open set whose complement has codimension at least 2 $W \subset X^{[n]}$.

The relevant diagram is

$$\begin{tikzcd} \widetilde{V} \arrow[r, "q"] \arrow[d, "p"] & V \subset X^n \arrow[d, "\pi"] \\ X^{[n]} \supset W \arrow[r, "\rho"] & X^{(n)} \end{tikzcd} \tag{20.2}$$

On $X^{[n]}$ we have a rank n bundle $L^{[n]}$ and we also have a bundle $L^n = p_1^* L \oplus \cdots \oplus p_n^* L$, where $p_i : X^n \to X$ is the ith projection. There is a natural map

$$p^* L^{[n]} \to q^* L^n$$

given as follows: If $z \in \widetilde{V}$, the fiber of f is

$$H^0(L \otimes \mathcal{O}_{p(z)}) \to \oplus_i H^0(L \otimes X_i)$$

where $q(z) = (x_1, \ldots, x_n)$.

The map f is an isomorphism on the open set of tuples of distinct points. Therefore, the morphism of vector bundles is injective (as it is such on the generic fiber, and any subsheaf of a torsion free sheaf is torsion free). Let E be the exceptional divisor. Then, $\operatorname{coker} f$ is supported on the exceptional divisor. That is, we have

$$0 \longrightarrow p^* \mathcal{L}^{[n]} \xrightarrow{f} q^* L^n \longrightarrow \operatorname{coker} f \longrightarrow 0 \tag{20.3}$$

It turns out that $\operatorname{coker} f$ has rank 0 and is supported on E, and it turns out that $\det \operatorname{coker} f \simeq \mathcal{O}_X(E)$. Therefore,

$$\det p_* \mathcal{L}^{[n]} \simeq L^{\boxtimes n}(-E).$$

We have

$$\begin{aligned} H^0(X^{[n]}, \det L^{[n]}) &\simeq H^0(W, \det L^{[n]}) \\ &\simeq H^0(\widetilde{V}, p^* \det L^{[n]})^{S_n} \\ &\simeq H^0(q^* L^{\boxtimes n}(-E))^{S_n} \\ &\subset H^0(q^* L^{\boxtimes n})^{S_n} \\ &\simeq H^0(V, L^{[n]})^{S_n} \\ &\simeq H^0(X^n, L^{[n]})^{S_n} \end{aligned}$$

Summarizing, we have a map

$$H^0(X^{[n]}, \det L^{[n]}) \to H^0(X^n, L^{\boxtimes n})^{S_n}.$$

For $\sigma \in S_n$ and $t_i \in H^0(X, L)$, we define

$$\sigma\left(p_1^* t_1 \otimes p_2^* t_2 \otimes \cdots \otimes p_n^* t_n\right) := \operatorname{sgn}(\sigma)\left(p_1^* t_{\sigma(1)} \otimes \cdots p_n^* t_{\sigma(n)}\right).$$

That is, S_n acts by permuting factors and taking determinants. Therefore,

$$H^0(X^n, L^{\boxtimes n})^{S_n} \simeq \wedge^n H^0(X, L).$$

Exercise 20.7. Verify that this construction is inverse to the map in the other direction, yielding an isomorphism.

□

21. 6/5/17

Today, we'll start by discussing Curvilinear schemes.

Definition 21.1. We say $[Z] \subset X^{[n]}$ is called **curvilinear** if

$$\mathcal{O}_{Z,p} \simeq \mathbb{C}[t]/(t^\ell).$$

The value of this concept of curvilinear schemes is that if we fix $x \in \operatorname{Supp}(Z)$, we can then define the residual of x for any curvilinear scheme.

Definition 21.2. For Z a curvilinear scheme and $x \in \operatorname{Supp}(Z)$, the **residual** τ_x is

$$\tau_x(z) := \begin{cases} Z_y & \text{if } y \neq x \\ \operatorname{Spec} \mathbb{C}[t]/(t^\ell - 1) & \text{if } y = x \end{cases}$$

Exercise 21.3 (Easy exercise)**.** Let X be a surface. The set of curvilinear schemes $X^{[n]}_{\text{curv}} \subset X^{[n]}$ is open and its complement has codimension 2.

Let

$$\mathcal{Z}_n \subset X^{[n]}_{\text{curv}} \times X$$

be the universal subscheme. We have the **residual morphism**

$$\begin{aligned} \tau \colon \mathcal{Z}_n &\to X^{[n-1]}_{\text{curv}} \times X \\ ([Z], x) &\mapsto (\tau_x[Z], x) \end{aligned}$$

The point of this is that it gives you inductive arguments.

Theorem 21.4 (Voisin)**.** *Let*

$$\begin{tikzcd} \mathcal{Z}_n \arrow[r, "q"] \arrow[d, "p"] & X^{[n]}_{\text{curv}} \\ X. & \end{tikzcd} \tag{21.1}$$

Let L *be a line bundle on* X*. There is an isomorphism*

$$K_{n-1,1}(X, L) \simeq H^0\left(\mathcal{Z}_n, q^* \det L^{[n]}\right) / q^* H^0\left(X^{[n]}_{\text{curv}}, \det L^{[n]}\right).$$

Proof. Recall last class we showed $H^0(\det L^{[n]}) = \wedge^n H^0(X, L)$. We have

$$\begin{tikzcd} X^{[n]}_{\text{curv}} \times X \arrow[r, "p_2"] \arrow[d, "p_1"] & X \\ X^{[n]}_{\text{curv}} & \end{tikzcd} \tag{21.2}$$

There is a morphism

$$\phi : q^* L^{[n]} \to \tau^* \left(p_1^* L^{[n-1]} \oplus p_2^* L\right).$$

This map sends $[Z] \in \mathcal{Z}_n$ for

$$\tau(Z) = \left(Z', x\right)$$

to

$$H^0(L \otimes \mathcal{O}_Z) \to H^0(L \otimes \mathcal{O}_{Z'}) \oplus H^0(x).$$

Let $D \subset Z_n$ be the complement of

$$\{(Z, x) \in \mathcal{Z}_n : Z \text{ is reduced at } x\}.$$

We have $\tau(D) = \mathcal{Z}_{n-1}$. We have that ϕ is injective outside of D. The cokernel of ϕ is D (and not 2D), even scheme theoretically. We then have

$$q^* \det L^{[n]} \simeq \tau^* \left(\det L^{[n]} \boxtimes L\right)(-D),$$

which implies

$$H^0\left(\mathcal{Z}_n, q^* \det L^{[n]}\right) = \ker\left(H^0\left(\tau^* \det L^{[n]} \boxtimes L\right) \to H^0\left((\tau^* \det L^{[n]} \boxtimes L)|_D\right)\right).$$

Then, τ is an isomorphism away from D and it sends D to $\mathcal{Z}_{n-1}$. One then shows

$$H^0\left(\mathcal{Z}_n, q^* \det L^{[n]}\right) = \ker\left(H^0\left(X^{[n-1]} \times X, \det L^{[n]} \boxtimes L\right) \to H^0\left(\mathcal{Z}_{n-1}, L^{[n]} \boxtimes L\right)\right).$$

So we have an exact sequence
(21.3)

$$\begin{tikzcd} 0 \arrow[r] & H^0\left(\mathcal{Z}_n, q^* \det L^{[n]}\right) \arrow[r] & H^0\left(X^{[n-1]} \times X, \det L^{[n]} \boxtimes L\right) \arrow[dl] \\ & H^0\left(\mathcal{Z}_{n-1}, L^{[n]} \boxtimes L\right) \arrow[r] & 0 \end{tikzcd}$$

By Proposition 20.4, we have an equality

$$H^0\left(X^{[n-1]}_{\mathrm{curv}} \times X, \det L^{[n-1]} \boxtimes L\right) \simeq \wedge^{n-1} H^0(L) \otimes H^0(L).$$

We obtain a corresponding commutative diagram
(21.4)

$$\begin{tikzcd} \wedge^{n-1} H^0(L) \otimes H^0(L) \arrow[rr] \arrow[dr] & & H^0(\mathcal{Z}_{n-1}, \det L^{[n-1]} \boxtimes L) \arrow[dl] \\ & \wedge^{n-1} H^0(L) \otimes H^0(L^2). & \end{tikzcd}$$

Repeating the above argument, we obtain a map

$$H^0(\mathcal{Z}_{n-1}, \det L^{[n-1]} \boxtimes L) \to H^0(X^{[n-2]}_{\mathrm{curv}} \times X, \det L^{[n-2]} \boxtimes L^2.$$

We similarly obtain

$$\begin{tikzcd}
\wedge^{n-2} H^0(L) \arrow[d] \arrow[dr, "\delta"] & \\
H^0(\mathcal{Z}_n, q^* \det L^{[n]}) \arrow[r] & \wedge^{n-1} H^0(L) \otimes H^0(L) \arrow[d] \arrow[dd, bend right=60, "\delta"'] \\
& H^0(\mathcal{Z}_{n-1}, \det L^{[n-1]} \boxtimes L) \arrow[d] \\
& \wedge^{n-2} H^0(L) \otimes H^0(L^2)
\end{tikzcd} \tag{21.5}$$

We can identify

$$\ker \left(\wedge^{n-1} H^0(X, L) \otimes H^0(L) \xrightarrow{\delta} \wedge^{n-2} H^0(L) \otimes H^0(L^2) \right)$$

with

$$H^0(\mathcal{Z}_n, q^* \det L^{[n]}).$$

and $\operatorname{im} \delta^{n-2} \simeq q^* H^0(X^{[n]}_{\text{curv}} \times X, \det L^{[n]})$. □

Theorem 21.5 (Voisin). *Let* X *be a K3 surface and* $\mathcal{L}$ *and ample line bundle. Suppose* $\operatorname{Pic} X \simeq \mathbb{Z}L$. *We have* $L^2 = 2g - 2$, *and assume* $g = 2 - k$. *Then,* $K_{k-1,1}(C, \omega_C) = K_{k-1,1}(X, L) = 0$ *for* $C \in |L|$.

By Lazarsfeld's result, if $C \in |L|$ is general, then it is Brill Noether general for $\operatorname{Gon}(C) = k + 1$.

22. 6/7/17

Theorem 22.1 (Voisin). *Let* X *be a K3 surface,* L *a line bundle and* $\operatorname{Pic} X \simeq \mathbb{Z}L$. *We have* $L^2 = 2g - 2$ *and* $g = 2k$. *Then,* $K_{k,1}(X, L) \simeq K_{k,1}(C, \omega_C) = 0$.

Remark 22.2. If $C \in |L|$ is general then Lazarsfeld's result says C is Brill-Noether-Petri general. We have

$$\rho(g, 1, k + 1) = g - 2h^1 = 2k - 2k = 0,$$

since $2 - h^1 = (k + 1) + 1 - 2k$, so $h^1 = k$. So, C has gonality $k + 1$, by the Brill-Noether theorem.

So, C carries a base point free $D \in W^1_{k+1}$. We have the Lazarsfeld-Mukai bundle

$$0 \longrightarrow F \longrightarrow H^0(C, D_{\otimes} \mathcal{O}_X \longrightarrow i_* D \longrightarrow 0. \tag{22.1}$$

We have that F has rank 2.

Lemma 22.3. *If G is a vector bundle with the same Chern character of F, we have $F \simeq G$.*

Proof. We showed previously

$$\chi(F \otimes F^\vee) = 2 - 2\rho(D) = 2.$$

Suppose G is any vector bundle with $\mathrm{Ch}(G) = \mathrm{Ch}(F)$. Then,

$$\chi(F \otimes G^\vee) = 2,$$

since the Euler characteristic only depends on the Chern character, by Hirzebruch Riemann Roch. Then, either $H^0(F \otimes G^\vee) > 0$ or $H^0(F^\vee \otimes G) \simeq H^2(F \otimes G^\vee) > 0$. Therefore, either there is a nonzero $G \to F$ or $F \to G$. But since F and $F^\vee$ are stable, and $\mu_1(F) = \mu_1(G)$ and F is stable, we have seen $F \simeq G$. □

22.1. **An alternate construction of F.** Using Lemma 22.3, we can now give an alternate construction. The following construction is either due to Serre or Griffiths-Harris.

Let $D \in W^1_{k+1}(C)$. We have

$$\chi(\mathcal{O}_C(D)) = k + 1 + 1 - 2k = 2 - k.$$

We know

$$h^0(\mathcal{O}_C(D)) = 2,$$

so

$$h^0(\omega_C - D) = h^1(\mathcal{O}(D)) = k.$$

We have an exact sequence

$$0 \longrightarrow \mathcal{O}(-D) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_D \longrightarrow 0 \tag{22.2}$$

which we twist to

$$0 \longrightarrow \omega_C - D \longrightarrow \omega_C \longrightarrow \omega_C|_D \longrightarrow 0 \tag{22.3}$$

Note that $\omega_C|_D \simeq \mathcal{O}_D$. We obtain

$$h^0(\omega_C) = 2k, h^0(\omega_C|_C) = k + 1.$$

Chasing the exact sequences and using Riemann Roch, we deduce

$$H^0(C, \omega_C) \to H^0(D, \omega_C|_D)$$

has 1-dimensional cokernel.

On a K3 surface, we have the sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow L \longrightarrow L|_C \longrightarrow 0 \tag{22.4}$$

where $L|_C \simeq \omega_C$. We have $h^1(\mathcal{O}_X) = 0$. Therefore

$$\begin{tikzcd} H^0(X, L) \arrow[rr, two heads] \arrow[dr, "f"'] & & H^0(C, \omega_C) \arrow[dl] \\ & H^0(D, \omega_C|_D) & \end{tikzcd} \tag{22.5}$$

Note that f has corank 1 if and only if $D \in W_{k+1}$ with $D \in \mathrm{Pic}^{k+1}(C)$. Look at the ideal sheaf of D as a codimension 2 subscheme of X. We have the sequence

$$0 \longrightarrow \mathcal{I}_D \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0 \tag{22.6}$$

We have the following facts:

Fact 22.4.
(1) $\mathrm{Ext}^1(\mathcal{O}_D, L^\vee) = 0$
(2) $\mathrm{Ext}^2(\mathcal{O}_D, L^\vee) \simeq H^0(L^\vee \otimes \mathcal{O}_D)$, thinking of the latter as a vector space.

To prove the above facts, use the local to global Ext sequence.

We now apply the functor $\mathrm{Hom}\left(\bullet, L^\vee\right)$ to the prior exact sequences. We obtain
(22.7)

$$0 \longrightarrow \mathrm{Ext}^1(\mathcal{O}_X, L^\vee) \longrightarrow \mathrm{Ext}^1(I_D, L^\vee) \longrightarrow H^0(L^\vee \otimes \mathcal{O}_D) \longrightarrow \mathrm{Ext}^2(\mathcal{O}, L^\vee).$$

Using the facts, we obtain
(22.8)

$$0 \longrightarrow H^1(L^\vee) \longrightarrow \mathrm{Ext}^1(I_D, L^\vee) \longrightarrow H^0(L^\vee \otimes \mathcal{O}_D) \longrightarrow H^0(L)^\vee,$$

where we used

$$\mathrm{Ext}^2(\mathcal{O}, L^\vee) = H^2(L^\vee) = H^0(L)^\vee.$$

Dualizing the above sequence, we obtain
(22.9)

$$H^0(L) \xrightarrow{\mathrm{res}_D} H^0(L \otimes \mathcal{O}_D) \longrightarrow \mathrm{Ext}^1(I_D, L^\vee)^\vee \longrightarrow H^1(L) \longrightarrow 0.$$

We have shown res_D is not surjective with corank 1 if and only if $D \in W^1_{k+1}$. If $D \in W^1_{k+1}$, we get

$$\begin{aligned} 0 \neq e &\in \mathrm{Ext}^1(I_D, L^\vee)^\vee \\ &\simeq \mathrm{Ext}^1(I_D \otimes L, \omega_X)^\vee \end{aligned}$$

Therefore,

$$e^\vee \in \mathrm{Ext}^1(I_D \otimes L, \mathcal{O}_X)$$

This element corresponds to an extension

$$0 \longrightarrow \mathcal{O}_X \longrightarrow V \longrightarrow I_D \otimes L \longrightarrow 0. \tag{22.10}$$

Let $E := F^\vee$. Via a local computation, one obtains that V is a vector bundle with $\mathrm{Ch}\, V = \mathrm{Ch}(F^\vee) = \mathrm{Ch}(E)$. So, $V \simeq E \simeq F^\vee$, using Lemma 22.3.

Note that our construction in this case yields

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{s} E \longrightarrow I_D \otimes L \longrightarrow 0. \tag{22.11}$$

We then obtain a global section s of E vanishing on $D \in W^1_{k+1}$. It follows that the map $E \to I_D \otimes L$ can be written as $\wedge s$ (sending $t \mapsto t \wedge s$). So the exact sequence above is

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{s} E \xrightarrow{\wedge s} I_D \otimes L \longrightarrow 0. \tag{22.12}$$

For any $t \in H^0(E)$, we can see that $V(t)$ is some $W^1_{k+1}(C')$ for some C'.

Motivated by the above description, Voisin studies

$$\begin{aligned} \phi : \mathbb{P}\left(H^0(E)\right) &\to X^{[k+1]} \\ s &\mapsto V(s). \end{aligned}$$

One can verify ϕ is a closed immersion. Let's use this an Voisin's description of syzygies via the Hilbert scheme.

Define

$$\mathbb{P}\left(H^0(E)\right)_{\mathrm{curv}} := \mathbb{P}\left(H^0(E)\right) \cap X^{[k+1]}_{\mathrm{curv}}.$$

Let

$$\begin{tikzcd}
\mathcal{Z}_{k+1} \arrow[rr] \arrow[dr, "q"'] & & X^{[k+1]}_{\mathrm{curv}} \arrow[dl] \\
& X^{[k+1]} &
\end{tikzcd} \tag{22.13}$$

Let $W = q^* \left(\mathbb{P}\left(H^0(E)\right)_{\mathrm{curv}}\right)$.

Voisin wants to prove the following. She doesn't actually prove this, she has to do some alterations, but for simplicity, we'll just pretend she proves this:

(1) $H^0(W, q^* \det L^{[k+1]}) \simeq q^* H^0 \left(\mathbb{P}(H^0(E))_{\mathrm{curv}}, \det L^{[k+1]}\right)$.

(2) The restriction

$$H^0(\mathcal{Z}_{k+1}, q^* \det L^{[k+1]}) \to H^0(W, q^* \det L^{[k+1]}$$

is injective.

Let's see why these facts above imply Theorem 22.1. By our description of syzygies from previous lectures, we know

$$K_{k+1}(X, L) \simeq H^0(\mathcal{Z}_{k+1}, q^* \det L^{[k+1]}) / q^* H^0(X^{[k+1]}, \det L^{[k+1]}).$$

We want to show

$$H^0(\mathcal{Z}_{k+1}, q^* \det L^{[k+1]}) / q^* H^0(X^{[k+1]}, \det L^{[k+1]}) = 0.$$

That is, we want to show

$$H^0(\mathcal{Z}_{k+1}, q^* \det L^{[k+1]}) \simeq q^* H^0(X^{[k+1]}, \det L^{[k+1]}).$$

We know this is injective and it is an isomorphism when restricted to the curvilinear part. To do this, we use the trace map. This is essentially summing over the fibers. Explicitly, it is

$$H^0(\mathcal{Z}_{k+1}, q^* \det L^{[k+1]}) \to H^0(X^{[k+1]}, \det L^{[k+1]}).$$

We have $\deg q = k+1$ so $\mathrm{tr} \circ q^* = (k+1)\,\mathrm{id}$. Let

$$\alpha \in H^0(\mathcal{Z}_{k+1}, q^* \det L^{[k+1]}).$$

We want to show it comes from something in $q^* H^0(X^{[k+1]}, \det L^{[k+1]})$. We know that $\alpha|_W = q^* \beta$ for some

$$\beta \in H^0\left(\mathbb{P}\left(H^0(E)\right)_{\mathrm{curv}}, \det L^{[k+1]}\right).$$

We have

$$\beta = \frac{1}{k+1} \left(\mathrm{tr}\left(\alpha\right)\right)|_W. \tag{22.14}$$

If $\alpha' = \alpha - q^*\left(\frac{1}{k+1}\operatorname{tr}(\alpha)\right)$. We have by (22.14), that $\alpha'|_W = 0$. The second property (saying that the restriction to W is injective). We obtain $\alpha' = 0$ so $\alpha \in \operatorname{im} q^*$ as we wanted, completing the proof.