The rank of syzygies of canonical curves

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The Twisted Cubic

The *equations* defining algebraic curves embedded into projective space has been studied for centuries. Let's start with a warm-up example. Consider the extended complex line $\mathbf{P}^1 = \mathbb{C} \cup \{\infty\}$, which can be also be thought of as the Riemann sphere. We can embed $\mathbf{P}^1$ in projective 3 space by

$$\begin{aligned} \nu:\ \mathbf{P}^1(\mathbb{C}) &\to \mathbf{P}^3(\mathbb{C}) \\ [u:v] &\mapsto [u^3 : u^2v : uv^2 : v^3] \end{aligned}$$

We call the image $X = \nu(\mathbf{P}^1)$ the twisted cubic.

X can also be described as the intersection of three quadric surfaces:

$$\begin{aligned} f(x,y,z,w) &= yw - z^2, \\ g(x,y,z,w) &= yz - xw, \\ h(x,y,z,w) &= xz - y^2. \end{aligned}$$

Canonical curves

Let C be a smooth, projective curve of genus $g \geq 2$, over the complex numbers. Consider the canonical bundle ω_C. We may embed C into projective space $\mathbf{P}^{g-1}$ via sections of the line bundle ω_C.

$$\begin{aligned} \phi_{\omega_C} \ : \ C &\to \mathbf{P}^{g-1} \\ p &\mapsto [s_1(p), \ldots, s_g(p)] \end{aligned}$$

where $s_1, \ldots, s_g$ is a basis of $H^0(\omega_C)$.

The map $\phi_{\omega_C} \ : \ C \to \mathbf{P}^{g-1}$ is a closed embedding provided that C is not *hyperelliptic*, i.e. C is not a double cover of $\mathbf{P}^1$.

Question: What can we say about the equations of the canonical curve $C \subseteq \mathbf{P}^{g-1}$, when C is not hyperelliptic?

Petri's Theorem

The following theorem of Petri is one of the cornerstones of the theory of algebraic curves.

Theorem (Petri, 1924)

Let C be a nonhyperelliptic projective curve of genus $g \geq 4$. Suppose that C is not trigonal, and not isomorphic to a plane quintic. Then C may be defined by quadratic equations on $\mathbf{P}^{g-1}$.

Here *trigonal* means that C admits a $3:1$ cover of $\mathbf{P}^1$. Geometrically, Petri's Theorem tells us that canonical curves are (scheme theoretically) an intersection of quadric surfaces in $\mathbf{P}^{g-1}$, with two well-understood exceptions.

Question: How many quadrics does it take to define a canonical curve $C \subseteq \mathbf{P}^{g-1}$?

Noether's Theorem

A fundamental theorem of Max Noether tells us how many hypersurfaces of degree d pass through a canonical curve.

Theorem (M. Noether, 1880)

The vector space $H^0(I_{C/\mathbf{P}^{g-1}}(2))$ of quadrics in $\mathbf{P}^{g-1}$ containing a canonical curve C has dimension $\frac{(g-2)(g-3)}{2}$.

Question: What can we say about the quadrics defining the canonical curve? What is their rank?

The Theorem of Mark Green

Recall that a quadric $Q \subseteq \mathbf{P}^{g-1}$ has rank r, if, after a linear change of coordinates, Q can be written in terms of r variables. Equivalently, Q is a cone over a quadric in $\mathbf{P}^{r-1}$.

Example We saw in the first slide that the twisted cubic $\mathbf{P}^1 \subseteq \mathbf{P}^3$ was defined by quadrics $yw - z^2$, $xz - y^2$, $yz - xw$. The first two have rank 3 and the last has rank 4.

Theorem (M. Green, 1984)

Let $C \subseteq \mathbf{P}^{g-1}$ be a canonical curve. The vector space $H^0(I_{C/\mathbf{P}^{g-1}}(2))$ of quadrics containing C has a basis consisting of quadrics of rank at most four.

Explicit Torelli Theorem

Aside for Experts: Green's Theorem provides an explicit version of the Torelli Theorem.
Consider the Jacobian $\mathrm{Pic}^{g-1}(C)$ with Theta divisor Θ. Then the double points of Θ correspond to line bundles $L \in \mathrm{Pic}^{g-1}(C)$ with $h^0(L) = 2$ (Riemann Singularity Theorem).

Considering the Petri map $H^0(L) \otimes H^0(\omega_C \otimes L^{-1}) \to H^0(\omega_C)$, one can define rank four quadrics in $\mathbf{P}^{g-1}$ out of such line bundles.

Indeed, by Riemann–Roch, $h^0(\omega_C \otimes L^{-1}) = 2$. Let u, v be a basis for $H^0(L)$ and s, t a basis for $H^0(\omega_C \otimes L^{-1})$. Then

$$(us)(vt) - (vs)(ut) \in \mathrm{Sym}^2 H^0(\omega_C)$$

defines a quadric in $\mathbf{P}^{g-1}$ which vanishes on C.

Green's Theorem states that these quadrics define C. Thus C can be reconstructed from Θ and $\mathrm{Pic}^{g-1}(C)$.

Syzygies of the Twisted Cubic

We would now like to put the classical theory of the equations defining algebraic curves into a much broader context. To motivate the definitions, first consider once more the twisted cubic

$$\begin{aligned} \nu :\ & \mathbf{P}^1(\mathbb{C}) \to \mathbf{P}^3(\mathbb{C}) \\ & [u : v] \mapsto [u^3 : u^2 v : u v^2 : v^3] \end{aligned}$$

Which was defined by quadratic equations

$$\begin{aligned} f(x, y, z, w) &= yw - z^2, \\ g(x, y, z, w) &= yz - xw, \\ h(x, y, z, w) &= xz - y^2. \end{aligned}$$

in the homogeneous coordinates of $\mathbf{P}^3$.

The three equations defining X are not independent. Two relations amongst them:

$$x \cdot f(x,y,z,w) + y \cdot g(x,y,z,w) + z \cdot h(x,y,z,w) = 0$$
$$y \cdot f(x,y,z,w) + z \cdot g(x,y,z,w) + w \cdot h(x,y,z,w) = 0.$$

I.e. we have two linear relations amongst the defining equations for the twisted cubic, with coefficients given by homogeneous polynomials in $S := \mathbb{C}[x,y,z,w]$. We call such relations amongst the equations *syzygies*.

The term syzygies was coined by J. J. Sylvester in 1853 to refer to relations amongst equations defining projective varieties.

Graded Modules

Let's put the results above into an algebraic context. Let C be a curve, with very ample line bundle L on C. Consider the graded polynomial ring

$$S := \mathrm{Sym}(H^0(C, L)) \simeq \mathbb{C}[x_1, \cdots, x_n],$$

where $x_1, \ldots, x_n$ is a basis for $H^0(C, L)$.

We may define a graded S module

$$\Gamma_C(L) := \bigoplus_{n \geq 0} H^0(C, L^{\otimes n}),$$

where multiplication by the graded pieces $S_d := \mathrm{Sym}^d H^0(C, L)$ is defined in the natural way.

When $L = \omega_C$ and C is not-hyperelliptic, the theorem of Max Noether tells us that $\Gamma_C(\omega_C)$ is precisely the *homogeneous coordinate ring* S/I_C of the canonical curve C.

Minimal Free Resolutions

We would like to study the graded module $\Gamma_C(\omega_C)$ over the polynomial ring $S = \mathrm{Sym} H^0(\omega_C) \simeq \mathbb{C}[x_1, \cdots, x_g]$. How to study the structure of graded modules in detail?

Let M be a finitely generated, *graded* S-module.

Theorem (Hilbert, 1890)

M has a minimal free resolution

$$0 \to F_{n+1} \xrightarrow{\delta_{n+1}} \mathcal{F}_n \to \ldots \to F_1 \to F_0 \to M \to 0,$$

of length at most $n+1$.

Free means we can write

$$F_i = \bigoplus_j S(-i-j)^{b_{i,j}}.$$

$$(S_d(m) := \{\text{homog. polynomials of degree } d+m\})$$

Betti numbers

The *Betti numbers* of the module M are the $b_{i,j}$. View these as fundamental algebraic invariants of M.

The *Betti table* of M is the table with $(i,j)^{th}$ entry $b_{j,i}$.
The coordinate ring S/I_X of the twisted cubic $X \subseteq \mathbf{P}^3$ has min resolution

$$0 \leftarrow S/I_X \leftarrow S \xleftarrow{[yw-z^2, yz-xy, xz-y^2]} S(-2)^{\oplus 3} \xleftarrow{\begin{bmatrix} x & y \\ y & z \\ z & w \end{bmatrix}} S(-3)^{\oplus 2} \leftarrow 0$$

The *Betti table* is

1	2	3	4	...
$1 = b_{0,0}$	$0 = b_{1,0}$	$0 = b_{2,0}$	0	...
$0 = b_{0,1}$	$3 = b_{1,1}$	$2 = b_{2,1}$	0	...

Green's Conjecture

For a general curve C, Voisin's Theorem provides the Betti table of the coordinate ring $\Gamma_C(\omega_C)$

Theorem (Voisin 02,05)

The Betti table of a sufficiently general, canonical curve of odd genus $g = 2i + 3$ *has shape*

1	$\dots$	$i-1$	i	$i+1$	$i+2$	$\dots$	$2i$
$b_{1,1}$	$\dots$	$b_{i-1,1}$	$b_{i,1}$	0	0	$\dots$	0
0	$\dots$	0	0	$b_{i+1,2}$	$b_{i+2,2}$	$\dots$	$b_{2i,2}$

The Betti table of a sufficiently general, canonical curve of even genus $g = 2i + 2$ *has shape*

1	$\dots$	$i-1$	i	$i+1$	$i+2$	$\dots$	$2i-1$
$b_{1,1}$	$\dots$	$b_{i-1,1}$	$b_{i,1}$	0	0	$\dots$	0
0	$\dots$	0	$b_{i,2}$	$b_{i+1,2}$	$b_{i+2,2}$	$\dots$	$b_{2i-1,2}$

The Betti table of C has the following properties:

- Diagonal symmetry.
- The alternating sum of entries in the off-diagonal can always be computed by the Hilbert function of $C \subseteq \mathbf{P}^{g-1}$.

Betti numbers of canonical curves

In the case of a general canonical curve, we can compute this. We obtain all the Betti numbers! One finds, if $g = 2i + 3$ is odd:

$$b_{p,1} = \frac{(2i+2-p)(2i-2p+2)}{p+1}\binom{2i+2}{p-1}, \quad p \leq i$$

By the diagonal symmetry one can also determine $b_{p,2}$ from this formula. If $g = 2i + 2$, we have

$$b_{p,1} = \frac{(2i+2-p)(2i-2p+2)}{p+1}\binom{2i+2}{p-1}, \quad p \leq i$$

Koszul cohomology

Voisin's Theorem gives a vast generalization of Noether and Petri's Theorem. One way to see this is to introduce the Koszul Cohomology groups $K_{p,q}(C,\omega_C)$, which are defined to be the middle cohomology of

$$\bigwedge^{p+1} H^0(\omega_C)\otimes H^0(\omega_C^{\otimes q-1}) \to \bigwedge^{p} H^0(\omega_C)\otimes H^0(\omega_C^{\otimes q}) \to \bigwedge^{p-1} H^0(\omega_C)\otimes H^0(\omega_C^{\otimes q+1})$$

Then $K_{p,q}(C,\omega_C)$ is a vector space of dimension $b_{p,q}$.
For instance, $K_{1,1}(C,\omega_C)$ is the middle cohomology of

$$\bigwedge^{2} H^0(\omega_C) \to H^0(\omega_C)\otimes H^0(\omega_C) \to H^0(\omega_C^{\otimes 2})$$

This is precisely $\mathrm{Ker}\left(\mathrm{Sym}^2 H^0(\omega_C) \to H^0(\omega_C^{\otimes 2})\right) = H^0(I_{C/\mathbf{P}^{g-1}}(2))$, the space of quadrics containing C.

Rank of a syzygy

Thus Voisin's Theorem can be thought of as extending Noether's theorem on the dimension of $H^0(I_{C/\mathbf{P}^{g-1}}(2))$ to giving the dimension of *all* graded pieces in the resolution of a general canonical curve!

Question: Can we do the same for Green's Theorem stating that the ideal of a canonical curve is generated by quadrics of rank four?

Need a way to define the rank for *any* element $\alpha \neq 0 \in K_{p,1}(C, \omega_C)$ which, in the case $p = 1$, generalizes the notion of rank of a quadric.

Rank of a syzygy

Definition (Schreyer): We say that $\alpha \neq 0 \in K_{p,1}(C, \omega_C)$ has rank $\leq r$ if there exists a vector space $V \subseteq H^0(C, \omega_C)$ of dimension r such that α is represented by an element of

$$\bigwedge^p V \otimes H^0(C, \omega_C).$$

Quadrics of rank four correspond to syzygies $\alpha \in K_{1,1}(C, \omega_C)$ of rank *two*.

More precisely, let $L \in \mathrm{Pic}^{g-1}(C)$ be a double point of Θ, i.e. $h^0(L) = 2$. Then the two dimensional subspace $V \subseteq H^0(C, \omega_C)$ as above is $H^0(C, \omega_C \otimes L^{-1})$.

The Geometric Syzygy Conjecture

Our main result is the following, which was originally conjectured by Schreyer in the early 90s:

Theorem (K., 2021)

Let C be a general curve of genus g. Then $K_{p,1}(C, \omega_C)$ is spanned by syzygies of rank $p+1$, for all $p \geq 1$.

More precisely, $K_{p,1}(C, \omega_C)$ is spanned by syzygies α which are represented by elements of

$$\bigwedge^p V \otimes H^0(C, \omega_C),$$

where V is of the form $H^0(\omega_C \otimes L^{-1})$ for some line bundle L with $h^0(L) = 2$ and $\deg(L) = g - p$.

For any nondegenerate variety $X \subseteq \mathbf{P}^n$, any syzygy $\alpha \neq 0 \in K_{p,1}(X, \mathcal{O}_X(1))$ has rank at least $p+1$. So the spaces $K_{p,1}(C, \omega_C)$ are spanned by syzygies of the smallest possible rank.

Rational Normal Scrolls

Three main actors in the proof, 1) scrolls, 2) projection of syzygies and 3) K3 surfaces.

Let $\alpha \in K_{p,1}(C, \omega_C)$ be a syzygy of minimal possible rank $p+1$, which is represented by an element of $\bigwedge^p V \otimes H^0(C, \omega_C)$, with $V = H^0(\omega_C \otimes L^{-1})$ for some line bundle L with $h^0(L) = 2$ and $\deg(L) = g - p$.

Define a *rational normal scroll* $X_L \subseteq \mathbf{P}^{g-1}$ as

$$X_L := \bigcup_{D \in |L|} \mathrm{Span}(D) \subseteq \mathbf{P}^{g-1},$$

where for any divisor $D \subseteq C \subseteq \mathbf{P}^{g-1}$ in the linear system $|L| = \mathbf{P}(H^0(L))$, $\mathrm{Span}(D)$ denotes the span.

Then $X_L \subseteq \mathbf{P}^{g-1}$ is a variety of codimension p and degree $p+1$ containing C. Such varieties are called *varieties of minimal degree* and have been studied since the late 19^{th} century (by Bertini and others).

Rational Normal Scrolls

The syzygy $\alpha \in K_{p,1}(C, \omega_C)$ of minimal rank may be lifted to a syzygy $\widetilde{\alpha} \in K_{p,1}(X_L, \mathcal{O}_{X_L}(1))$ of the rational normal scroll X_L defined by it. Furthermore, *all* minimal rank syzygies arise in this manner, by a result on von Bothmer.

One can explicitly resolve the rational normal scroll $X_L \subseteq \mathbf{P}^{g-1}$ via an Eagon–Northcott complex (known since the 60s). We get that

$$b_{i,1}(X_L, \mathcal{O}_{X_L}(1)) = i\binom{p+1}{i+1},$$

and $b_{i,q}(X_L, \mathcal{O}_{X_L}(1)) = 0$ for $q \geq 2$.

Projection of Syzygies

A technique of Aprodu provides an inductive approach to study syzygies. Let C be a curve of genus g with a node p, and let $\mu : \widetilde{C} \to C$ be the resolution of singularities. Then, one may obtain the canonical embedding $\widetilde{C} \subseteq \mathbf{P}^{g-2}$ geometrically by projecting $C \subseteq \mathbf{P}^{g-1}$ from the node $p \in C$.

Aprodu showed how to extend the geometric construction of projection to an algebraic construction on syzygies. Precisely, he constructs a map

$$\mathrm{pr} \ : \ K_{p,1}(C, \omega_C) \to K_{p-1,1}(\widetilde{C}, \omega_{\widetilde{C}}).$$

By studying the way projection maps work on scrolls one shows

Lemma (K., 2021)

Suppose $\alpha \in K_{p,1}(C, \omega_C)$ is a syzygy of minimal rank $p+1$. Then $\mathrm{pr}(\alpha) \in K_{p-1,1}(\widetilde{C}, \omega_{\widetilde{C}})$ is a linear combination of syzygies of minimal rank p.

Projection of Syzygies

Moreoever, we find new exact sequences involving Aprodu's projection map. This gives us the following:

Proposition (K., 2021)

Let C be a general curve of genus g, and let $x, y \in C$ be general. Let D be the one-nodal, genus $g+1$ curve obtained by identifying x, y. Suppose $K_{p+1,1}(D, \omega_D)$ is spanned by syzygies of minimal rank. Then the same is true for $K_{p,1}(C, \omega_C)$.

This gives an inductive approach. Let C be a general curve of genus g. We wish to show $K_{p,1}(C, \omega_C)$ is spanned by syzygies of minimal rank. Set

$$m = g - 2p - 2$$

and let D be a general m-nodal curve obtained by identifying m pairs of points on C. It suffices to show $K_{p+m,1}(D, \omega_D)$ is spanned by syzygies of minimal rank. We have $g(D) = 2(p + m + 1)$ by these choices.

K3 surfaces and syzygies

Thus it suffices to resolve the special case where C is a curve of even genus $g = 2k$ where $p = k - 1$. Note that $K_{p,1}(C, \omega_C) = 0$ for $p \geq k$ by Voisin's Theorem, so this is the *extremal case*.

In this case, the conjecture has been solved using K3 surfaces. Let X be a K3 surface with Picard group generated by an ample line bundle L of even genus $g = 2k$, i.e. $(L)^2 = 2g - 2$.

Let E be the rank two *Lazarsfeld–Mukai bundle* associated to a bundle A on a smooth curve $C \in |L|$ of degree $k + 1$ with $h^0(A) = 2$. The dual $E^\vee$ fits into the exact sequence

$$0 \to E^\vee \to H^0(C, A) \otimes \mathcal{O}_X \to i_* A \to 0,$$

where $i : C \hookrightarrow X$ is the inclusion. E has invariants $\det(E) = L$, $h^0(E) = k + 2$, $h^1(E) = h^2(E) = 0$.

K3 surfaces

E does not depend on the choice of C or A! It is a very special bundle.

Theorem

We have natural isomorphism $\mathrm{Sym}^{k-2} H^0(X, E) \simeq K_{k-1,1}(X, L)$. *Further,* $\mathrm{K}_{k-1,1}(X, L)$ *is spanned by syzygies* α *of rank* $k+1$, *with associated spaces* $V_\alpha := H^0(X, L \otimes I_{Z(s)})$, *for* $s \in H^0(E)$.

It is easy to see that restriction to a hyperplane $C \in L$ induces an isomorphism $\mathrm{K}_{k-1,1}(X, L) \simeq \mathrm{K}_{k-1,1}(C, \omega_C)$.

Under this isomorphism syzygies $\alpha \in \mathrm{K}_{k-1,1}(X, L)$ of rank $k+1$ drop rank, and give syzygies of minimal rank k, with associated vector space $H^0(C, \omega_C \otimes A^{-1})$, for A a bundle of degree $k+1$ with $h^0(A) = 2$. This completes the proof.

Thank you!