HODGE THEORY & PERIOD DOMAINS

This set of notes was taken in Michael Kemeny's Hodge Theory course at Univesity of Wisconsin–Madision in Fall 2019 by Connor Simpson, who is responsible for all errors. If you find an error, please contact Connor at `csimpson6@wisc.edu`

This course assumes differentiable manifolds, algebraic topology, and complex analysis. Algebraic geometry is helpful for intuition, but not required.

1. Introduction

The goal of Hodge theory is to provide a dictionary

$$\{\text{geometric objects}\} \leftrightarrow \{\text{more linear data}\}.$$

Geometric objects are hard, but linear algebra is relatively easy, so this is helpful. For us, the main geometric objects will be Riemann surfaces (curves) and K3 surfaces, and our two main goals for the course will be to prove Torelli theorems for both of them.

Definition 1.1. A **Riemann surface** is a complex projective manifold of dimension 1.

Projective 1-manifolds are compact, so a Riemann surface is a compact real 2-manifold; in other words, a connected sum of tori. The **genus** g of a Riemann surface is defined by $2g = \dim H_1(X, \mathbb{C})$. Genus classifies Riemann surfaces up to homeomorphism; however, there are infinitely many Riemann surfaces of a given dimension g that are not isomorphic as varieties.

Remark 1.2. The curves of genus g are parameterized by M_g, the **moduli of curves** of genus g. For $g > 1$, M_g has dimension $3g - 3$. If $g = 1$, we're studying elliptic curves, and things are somewhat complicated...

In the case of curves, our theory will be particularly nice: isomorphism classes of Riemann surfaces will correspond bijectively via the **period map** with abelian varieties called Picard groups. This result is the **Torelli theorem** for curves. Our second main goal will be to prove the Torelli theorem for K3 surfaces, which is similar in flavor.

Part 1. Classical Hodge Theory

2. Complex manifolds

In this course, all spaces will be Hausdorff and second-countable (i.e. they have a countable base).

Definition 2.1. Let $f : U \subset \mathbb{C}^n \to \mathbb{C}^n$ be given by $\mathbf{z} \mapsto (f_1(\mathbf{z}), \dots, f_n(\mathbf{z}))$. We say that f is **holomorphic** if each of the component functions f_i is separately holomorphic in each variable; that is, if for all i, j, $f_j(z_1, \dots, z_n)$ is holomorphic in z_i with $z_1, \dots, \hat{z}_i, \dots z_n$ held constant.

Definition 2.2. Let X be a topological space. A **complex chart** on X is an open $U \subset X$ with a homeomorphism $\phi : U \to V \subset \mathbb{C}^n$ with V open. A **holomorphic atlas** on X is a collection of complex charts $\{(U_i, \phi_i)\}_i$ such that

(i) $\bigcup_i U_i = X$
(ii) The transition maps $\phi_{ij} := \phi_i \circ \phi_j^{-1}$ are holomorphic on $\phi_j(U_i \cap U_j)$.

A **complex n-manifold** is a topological space with a holomorphic atlas in which the codomain of all the charts is an open subset of $\mathbb{C}^n$.

Remark 2.3. Since $\mathbb{C}^n \cong \mathbb{R}^{2n}$, a complex manifold is also a real manifold.

Example 2.4 (Projective space). As a set, let $\mathbb{P}^1_{\mathbb{C}} := (\mathbb{C}^2 \setminus \{(0,0)\})/(x,y) \sim (\lambda x, \lambda y)$ with $\lambda \in \mathbb{C}^*$. The charts are the usual ones: the subset $U_1 = \{[x : y] : x \neq 0\}$ is homeomorphic to $\mathbb{C}$ via $\phi_1 : [x : y] \mapsto \frac{y}{x}$ and similarly, we defined $\phi_2 : U_2 = \{[x : y] : y \neq 0\} \to \mathbb{C}$ given by $[x : y] \mapsto \frac{x}{y}$. We define the topology of $\mathbb{P}^1_{\mathbb{C}}$ by setting it equal to the coarsest topology that makes ϕ_1, ϕ_2 homeomorphisms. The transition function is the inversion map $\phi_2 \circ \phi_1^{-1} : z \mapsto z^{-1}$, which is holomorphic $\phi_1(U_1 \cap U_2) = \mathbb{C}^*$.

Exercise 2.5. Let $\psi_i : S^2 \setminus \{(0, 0, (-1)^{i+1})\} \to \mathbb{R}^2$ with $i = 1, 2$ be the stereographic projections from the north and south poles of S^2. Show that they are homeomorphisms. Further, show that ψ_1 and $\bar{\psi}_2 := ((x, y) \mapsto (x, -y)) \circ \psi_2$ give charts that glue to the sphere. Finally, show that these charts can be composed with the standard charts on $\mathbb{P}^1$ to give a diffeomorphism $S^2 \to \mathbb{P}^1_{\mathbb{C}}$.

From exercise 2.5, it follows that $\mathbb{P}^1_{\mathbb{C}}$ is compact and has genus 0. Exercise 2.5 is enough of a pain for $\mathbb{P}^1$ to make it clear that defining diffeomorphisms ad-hoc is not something that we want to do in the future for more complicated examples..

Let X and Y be complex manifolds.

Definition 2.6. A **morphism of complex manifolds** $f : X \to Y$ is a continuous map such that for each chart $\phi : X \supset U \to \mathbb{C}^n$ and $\psi : Y \supset V \to \mathbb{C}^n$

$$\mathbb{C}^n \supset \phi(U \cap V) \xrightarrow{\phi^{-1}} U \xrightarrow{f} V \xrightarrow{\psi} \mathbb{C}^n$$

is holomorphic.

An **isomorphism** is a morphism with an inverse. The group of **automorphisms** of a complex manifold X is

$$\mathrm{Aut}(X) := \{\text{isomorphisms } X \to X\}.$$

Example 2.7 (Automorphisms of $\mathbb{P}^1_{\mathbb{C}}$). The automorphisms of $\mathbb{P}^1_{\mathbb{C}}$ are called **Möbius transformations**. Explicitly, they are $\mathrm{PGL}_2(\mathbb{C}) := \{A \in \mathrm{GL}_2(\mathbb{C})\}/\sim$ where $A \sim \lambda A$ for all $\lambda \in \mathbb{C}^*$, where $[A] \cdot [x : y] = [A \cdot (x, y)]$. Since $\mathrm{GL}_2(\mathbb{C}) \subset \mathbb{C}^4$ is an open subset, $\mathrm{GL}_2(\mathbb{C})$ is a complex 4-manifold. Hence, $\mathrm{PGL}_2(\mathbb{C})$ is a complex 3-manifold.

3. Plane curves

Definition 3.1. $\mathbb{P}^n_{\mathbb{C}} = (\mathbb{C}^{n+1} \setminus \{0\})/\sim$ where $\lambda(x_0, \ldots, x_n) \sim (x_0, \ldots, x_n)$ for all $\lambda \in \mathbb{C}^*$.

The usual charts show that $\mathbb{P}^n$ is a complex n-manifold.

Proposition 3.2. $\mathbb{P}^n_{\mathbb{C}}$ *is compact*

Proof. The sphere $S^{2n+1} \subset \mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$ is compact and we have a surjection $S^{2n+1} \to \mathbb{P}^n_{\mathbb{C}}$ given by $(x_0, \dots, x_n) \mapsto [x_0 : \cdots : x_n]$. □

We will use $\mathbb{P}^n$ as our basic ambient space. A quick way to define compact complex manifolds is to take the zero set of a homogeneous polynomial system in $\mathbb{P}^n$. In the particular case of the **complex projective plane** $\mathbb{P}^2_{\mathbb{C}}$, a single homogeneous polynomial defines a curve.

Let $S = \mathbb{C}[x, y, z]$ and write S_d for the vector space of homogeneous polynomials of degree d. Given a homogeneous polynomial $F \in S_d$, the **vanishing** or **zero set** of F will be denoted $Z(F) := \{[x : y : z] : F(x, y, z) = 0\}$.

Remark 3.3. In affine space, the collection of zero sets of holomorphic functions is bigger than the collection of zero sets of polynomials. However, in projective space, there is no loss of generality in dealing with just polynomials! We can't easily justify this right now.

Theorem 3.4. *If $F \in S_d$ is nonsingular (i.e. its Jacobian is surjective at all points), then $Z(F)$ is a compact complex manifold of dimension 1.*

To prove this, we require

Theorem 3.5 (Implicit function theorem in two variables)**.** *Let $f \in \mathbb{C}[u, v]$. If the Jacobian of f is surjective at $(u_0, v_0) \in Z(f)$, then there is a holomorphic function $g(w)$ of one variable with $g(u_0) = v_0$ and a neighborhood U so that $(u_0, v_0) \in U \cong \{(w, g(w)) : w \in \mathbb{C}\}$.*

Exercise 3.6. Let g be holomorphic. Show that the graph of g is a complex manifold that is isomorphic to $\mathbb{C}$ (the isomorphism is projection onto the first coordinate).

Proof of theorem 3.4. $Z(F)$ is compact because it is a closed subset of a compact space.

Intersecting $Z(F)$ with one of the standard charts, we get the vanishing of a dehomogenization of$F$ in $\mathbb{C}^2$. The image of $Z(F)$ under the chart is nonsingular by the chain rule. Hence, the image of $Z(F)$ is covered by opens that are diffeomorphic to $\mathbb{C}$. Hence, $Z(F)$ is a complex 1-manifold. □

Exercise 3.7. Show that the image of $Z(F)$ in the proof above is non-singular using the chain rule.

Exercise 3.8. Consider the function $UP : \mathbb{C}^3 \times S_d \to \mathbb{C}$ defined by $(p, F) \mapsto F(p)$. Show this function is holomorphic and that $(p, F) \mapsto 0$ if and only if $(\lambda_1 p, \lambda_2 F) \mapsto 0$ for all $\lambda_1, \lambda_2 \in \mathbb{C}^*$. Hence, $Z(UP)$ is a closed subspace of $\mathbb{P}^2 \times \mathbb{P}(S_d)$.

Exercise 3.9. Similarly, show there is a holomorphic function $U_x : \mathbb{C}^3 \times S_d \to \mathbb{C}$ given by $(p, F) \mapsto \frac{dF}{dx}(p)$.

Definition 3.10. The **moduli space of plane curves** of degree d is

$$U_d := \mathbb{P}(S_d) \setminus \mathrm{pr}_2(Z(UP)) \cap \mathrm{pr}_2(Z(U_x)) \cap \mathrm{pr}_2(Z(U_y)) \cap \mathrm{pr}_2(Z(U_z))$$

where pr_2 is the projection onto the second coordinate and UP and U_x, U_y, U_z are as in the preceding exercises.

3.1. **Genus and degree.** Suppose we're given a curve $Z(F) \subset \mathbb{P}^2$ with $F \in S_d$ non-singular. The only invariant that we've defined so far is genus, so let's see what the genus could be.

- $f \in S_1$**:** Then $Z(F)$ is a line in $\mathbb{P}^2$, so $Z(F) \cong \mathbb{P}^1$ and has genus 1.
- $f \in S_2$**:** All plane conic are isomorphic, so we need to compute the genus of one of them.If $F = xz - y^2$, then $Z(F)$ is the image of the Veronese embedding $\mathbb{P}^1 \to \mathbb{P}^2$ given by $[u : v] \mapsto [u^2 : uv : v^2]$, so $Z(F)$ has genus 1.
- $f \in S_3$**:** This leads to elliptic curves. Turns out that the genus is 1 always.

In general, we would like a formula for the genus of a plane curve in terms of its degree.

4. A review of de Rham cohomology

The material that we review here is all in Spivak's "A comprehensive introduction to differential geometry".

Let X be a differentiable n-manifold.

Definition 4.1. The **tangent bundle** $\pi : T_X \to X$ is a vector bundle on X of rank $\dim X$ defined as follows:

- On each chart $\phi_U : U \to \mathbb{R}^n$, $\pi^{-1}(U) = U \times \mathbb{R}^n$. On each patch $U \times \mathbb{R}^n$, T_X has $2n$ coordinate functions $(x_1, \dots, x_n \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$ where
$$x_i := U \xrightarrow{\phi_U} \mathbb{R}^n \xrightarrow{\mathrm{pr}_i} \mathbb{R}$$
are the coordinates of ϕ_U and $\frac{\partial}{\partial x_i}$ is the ith coordinate function on $\mathbb{R}^n$.
- Transition functions
$$\mathbb{R}^n \times \mathbb{R}^n \to (U \cap V) \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$$
given by $(\phi_{UV}, \mathrm{Jac}(\phi_{UV})$ with $\phi_{UV} := \phi_U \circ \phi_V^{-1}$.

Definition 4.2. A **section** of a vector bundle $\pi : E \to X$ over $U \subset X$ is a map $s : U \to E$ such that $\pi \circ s = \mathrm{Id}_U$. We denote the set of sections of E over U by $E(U)$; for each U, $E(U)$ is a $C^\infty(U)$-module. Sections of the tangent bundle are called **vector fields**.

Definition 4.3. The **cotangent bundle** is $\Omega_X := T_X^\vee$, the dual bundle of T_X. Sections of the cotangent bundle are **differential 1-forms**.

We can think of vector fields and differential forms as operators: a vector field v acts on C^∞ functions $f : X \to \mathbb{R}$. If locally on $U \times \mathbb{R}^n$, $v = \sum_i c_i \frac{\partial}{\partial x_i}$ is a section of T_X, then $v(f) := \sum_i c_i \frac{\partial f}{\partial x_i} \in C^\infty(X)$. The fact that the transition functions are given by $\mathrm{Jac}(\phi_{UV})$ ensures that this action is well-defined.

In turn, a differential form acts on vector fields. Given $f : X \to \mathbb{R}$, we can define a differential form $df : X \to \Omega_X$ by
$$x \mapsto \left(\sum_i c_i \frac{\partial}{\partial x_i} \mapsto \sum_i c_i \frac{\partial f}{\partial x_i} \right)$$
on the chart $\phi_i : U_i \to \mathbb{R}^n$. The function $d : C^\infty(X) \to \Omega_X(U)$ defined by $f \mapsto df$ is the **exterior derivative**.

Definition 4.4. If $E \to X$ is a vector bundle, we can define the **wedge product** $\bigwedge^p E$ by

$$\left(\bigwedge^p E\right)(U) := \bigwedge^p (E(U)).$$

In the case of the cotangent bundle, we write $\Omega_X^p := \bigwedge^p \Omega_X$, and sections of Ω_X^p are **differential** p**-forms**.

If $x_i : U \xrightarrow{\cong} \mathbb{R}^n \xrightarrow{\mathrm{pr}_i} \mathbb{R}$ is a coordinate function, we an consider $dx_i \in \Omega_X^1(U)$. Note that the forms $\{dx_i\}$ are the dual basis of the coordinates $\{\frac{\partial}{\partial x_i}\}$ for T_X over U.

Therefore, if $\omega \in \Omega_X^p(U)$ is a local p-form, then we can write it as

$$\omega = \sum_{i_1 < \cdots < i_p} c_{i_1,\ldots,i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

with the coefficients $c_{i_1}, \ldots, c_{i_p}$ smooth functions.

4.1. **Action of** p**-forms on** p**-tuples of vector fields.** If V and W are vector spaces over k, then there is a tautological isomorphism between $(\bigwedge^p V)^\vee$ and alternating maps $V \times V \times \cdots \times V \to k$. There is also a non-tautological natural isomorphism $\bigwedge^p V^\vee \to (\bigwedge^p V)^\vee$ defined by

$$\phi_1 \wedge \cdots \wedge \phi_p \mapsto \left(v_1 \wedge \cdots \wedge v_p \mapsto \sum_{\sigma \in S_p} \mathrm{sgn}(\sigma) \phi_1(v_{\sigma(1)}) \cdots \phi_p(v_{\sigma(p)}) \right)$$

Now, suppose that $\omega \in \Omega_X^p(X)$ is a p-form. On a trivializing open U for Ω_X, we now have isomorphisms

$$\omega|_U \in \left(\bigwedge^p \Omega_X\right)(U) = \bigwedge^p(\Omega_X(U)) = \bigwedge^p(T_X^\vee(U)) \cong \left(\bigwedge^p T_X(U)\right)^\vee \cong \mathrm{Alt}_p(T_X(U))$$

given by

$$\omega|_U = \sum_{i_1 < \cdots < i_p} c_{i_1,\ldots,i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p} \mapsto$$

$$\left((v_1, \ldots, v_p) \mapsto \sum_{\substack{i_1 < \cdots < i_p \\ \sigma \in S_p}} \mathrm{sgn}(\sigma) c_{i_1,\ldots,i_p} dx_{i_1}(v_{\sigma(1)}) \wedge \cdots \wedge dx_{i_p}(v_{\sigma(p)}) \right)$$

It follows from this discussion that given p vector fields $v_1, \ldots, v_p \in T_X(U)$, $\omega(v_1, \ldots, v_p) \in C^\infty(U)$ is a $C^\infty(U)$-linear combination of determinants of minors of $[v_1, \ldots, v_p]$.

4.2. **The de Rham complex and cohomology.** We now extend the exterior derivative for all p to a map $d : \Omega^p(X) \to \Omega^{p+1}(X)$ satisfying

(i) $d \circ d = 0$
(ii) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$, where α for any k-form α.

Locally, it is given by the formula

$$\omega = \sum_{i_1 < \cdots < i_p} c_{i_1,\ldots,i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p} \mapsto d\omega := \sum_{i_1 < \cdots < i_p} dc_{i_1,\ldots,i_p} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

and one can check that these maps glue.

Definition 4.5. The **de Rham complex** is

$$0 \longrightarrow \Omega^0_X(X) \xrightarrow{d} \Omega^1_X(X) \xrightarrow{d} \cdots \Omega^n_X(X) \longrightarrow 0$$

and its cohomology is **de Rham cohomology**. We denote that pth de Rham cohomology group by H^p_{dR}.

Theorem 4.6 (de Rham's theorem)**.** *For any differentiable manifold X, there exists a natural isomorphism $H^p_{dR}(X) \to H^p(X;\mathbb{R})$ for all p, where H^p is the pth singular cohomology group.*

The isomorphism in Theorem 4.6 is defined as follows. Let $[\omega] \in H^p_{dR}$ with $d\omega = 0$. In order to define the isomorphism, we need to associate to ω a function $C_p(X) \to \mathbb{R}$, where $C_p(X)$ is the free $\mathbb{R}$-vector space with basis maps $\gamma : \Delta^p \to X$. To do this, define $\omega(\gamma) := \int_\gamma \omega := \int_{\Delta^p} \gamma^*\omega$.

5. Holomorphic vector bundles

Suppose that X is a complex manifold.

Definition 5.1. A **topological complex vector bundle** $\pi : E \to X$ of rank m is consists of the following data:

- An open cover $\{U_i\}$ of X,
- Trivializations $\tau_i : \pi^{-1}(U_i) \xrightarrow{\cong} U_i \times \mathbb{C}^m$ so that

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\tau_i} & U_i \times \mathbb{C}^n \\ \big\downarrow \pi & \swarrow \mathrm{pr}_1 & \\ U_i & & \end{array}$$

  commutes,
- Transition functions

$$\tau_i := \tau_i \circ \tau_j^{-1} : U_i \cap U_j \times \mathbb{C}^m \to U_i \cap U_j \times \mathbb{C}^m$$

  that are linear on each fibre $E_u := u \times \mathbb{C}^m$ with $u \in U_i \cap U_j$.

The fact that the transition functions are linear on fibers implies that we can write the transition function τ_{ij} as a matrix $T_{ij} \in \mathrm{GL}_n(C^0(U_i \cap U_j))$, where $C^0(U_i \cap U_j)$ denotes continuous functions on $U_i \cap U_j$. The matrices T_{ij} are the **transition matrices**. Denote by $\mathcal{A}(U_i \cap U_j)$ the set of holomorphic function on $U_i \cap U_j$.

Definition 5.2. A topological complex vector bundles $E \to X$ is **holomorphic** if the entries of the transition matrices T_{ij} are holomorphic functions; that is, if $T_{ij} \in \mathrm{GL}_m(\mathcal{A}(U_i \cap U_j))$ for all i, j.

A holomorphic vector bundle is a complex manifold because the gluing maps are all holomorphic, and moreover, $E \to X$ is a map of complex manifolds.

Convention. All vector bundles are holomorphic unless otherwise stated.

Exercise 5.3. Suppose we are given a complex manifold X an open cover $\{U_i\}$ and transition matrices $\{T_{ij}\}$ with $T_{ij} \in \mathrm{GL}_m(\mathcal{A}(U_i \cap U_j))$ satisfying

(i) $T_{ii} = \mathrm{Id}_{U_i}$
(ii) $T_{kj}T_{ji} = T_{ki}$

Show that this data determines a vector bundle E with T_{ij} as its transition matrices.

Definition 5.4. Let $\pi_i : E_i \to X$ for $i = 1, 2$. A map $f : E_1 \to E_2$ is a **morphism of vector bundles** if $\pi_1 = \pi_2 \circ f$. A **isomorphism** of vector bundles is a morphism of vector bundles with an inverse.

Exercise 5.5. A vector bundles is determined up to isomorphism by the data of its transition matrices.

Example 5.6 (Holomoprhic tangent bundle). Suppose that X is a complex manifold with atlas $\{\phi_i : U_i \to \mathbb{C}^n\}$. The atlas gives transition functions $\phi_{ij} = \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$. Now, define $T_{ij} = \mathrm{Jac}_{\mathbb{C}}(\phi_{ij})$, where $\mathrm{Jac}_{\mathbb{C}}$ is the **complex Jacobian**, which is the matrix of partials with respect to complex arguments. These transition functions will satisfy the cocycle condition by the chain rule. Therefore, using these transition functions gives the **holomorphic tangent bundle** $\mathcal{T}_X$ of X.

6. Complexification of the tangent bundle

We now have two vector bundles on a complex manifold X: the holomorphic tangent bundle $\mathcal{T}_X$, of rank $\dim_{\mathbb{C}} X$, and the complexification of the real bundle, $T_X \otimes \mathbb{C}$, which has rank $2 \dim_{\mathbb{C}} X$ over $\mathbb{C}$. We would like to compare them.

6.1. **The Jacobian.** If V is a vector space over $\mathbb{R}$ with basis $\{e_1, \dots, e_v\}$, then the **complexification** $V \otimes_{\mathbb{R}} \mathbb{C}$ is a vector space of dimension v over $\mathbb{C}$ and dimension $2v$ over $\mathbb{R}$. If we have a linear map $\phi : V \to V$, then it naturally extends to a map $\tilde{\phi} : V \otimes_{\mathbb{R}} \mathbb{C} \to V \otimes_{\mathbb{R}} \mathbb{C}$ given by $e_i \otimes 1 \mapsto \phi(e_i) \otimes 1$.

Now, consider a smooth function $f : \mathbb{R}^2 \to \mathbb{R}^2$. Identifying $\mathbb{R}^2$ with $\mathbb{C}$, we can write $f(x, y) = u(x, y) + iv(x, y)$ with u, v smooth real-valued functions. The Jacobian of f is

$$\mathrm{Jac}(f) = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}.$$

Complexifying, we obtain $\mathrm{Jac}(f) \otimes \mathbb{C} : \mathbb{C}^2 \to \mathbb{C}^2$.

If f is holomorphic, then the Cauchy-Riemann equations say that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Based on this, define $\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ and $\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$, and make the change of coordinates $w_1 = \frac{1}{2}(e_1 - ie_2)$ and $w_2 = \frac{1}{2}(e_1 + ie_2)$, where $\{e_1, e_2\}$ is a basis for $\mathbb{C}^2$.

Exercise 6.1. Show that $\mathrm{Jac}(f)(w_1) = \frac{\partial f}{\partial z} w_1 + \frac{\partial \bar{f}}{\partial z} w_2$ and $\mathrm{Jac}(f)(w_2) = \frac{\partial f}{\partial \bar{z}} w_1 + \frac{\partial \bar{f}}{\partial \bar{z}} w_2$. In these new coordinates,

$$\mathrm{Jac}(f) = \begin{bmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial \bar{z}} \\ \frac{\partial \bar{f}}{\partial z} & \frac{\partial \bar{f}}{\partial \bar{z}} \end{bmatrix}.$$

If f is holomorphic, then Cauchy-Riemann says that $\frac{\partial f}{\partial \bar{z}} = 0 = \overline{\frac{\partial f}{\partial \bar{z}}} = \frac{\partial \bar{f}}{\partial z}$, so we have diagonalized the Jacobian.

6.2. **Decomposing the complexified tangent bundle.** Suppose that X is a complex curve with local coordinate z. Let $\mathcal{T}_X$ be its holomorphic tangent bundle with local coordinate $\frac{\partial}{\partial z}$ and transition matrices $\frac{\partial \phi_{ij}}{\partial z}$, and let $T_X \otimes \mathbb{C}$ be the complexified smooth tangent bundle, which has local coordinates $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$. Both $\mathcal{T}_X$ and $T_X \otimes \mathbb{C}$ are complex topological bundles with transition functions that are smooth.

Define an inclusion

$$\begin{aligned}\iota_1 : \mathcal{T}_X &\to T_X \otimes \mathbb{C} \\ \frac{\partial}{\partial z} &\mapsto \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).\end{aligned}$$

To check that this is a map of bundles, we need to check that it behaves properly with respect to gluing; in other words, we must check that the following diagram commutes

$$\begin{array}{ccc} \mathcal{T}_X(U_i) & \longrightarrow & (T_X \otimes \mathbb{C})(U_i) \\ \uparrow & & \uparrow \\ \mathcal{T}_X(U_j) & \longrightarrow & (T_X \otimes \mathbb{C})(U_j) \end{array} \qquad \begin{array}{ccc} \frac{\partial \phi_{ij}}{\partial z}\left(\frac{\partial}{\partial z}\right) & \longmapsto & \mathrm{Jac}(\phi_{ij})\left(\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\right) \\ \uparrow & & \uparrow \\ \frac{\partial}{\partial z} & \longmapsto & \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \end{array}$$

Using the new coordinates that we defined in Exercise 6.1, we see that $\mathrm{Jac}(\phi_{ij})(w_1) = \frac{\partial \phi_{ij}}{\partial z}\left(\frac{\partial}{\partial z}\right)$ because ϕ_{ij} is holomorphic; therefore, the diagram commutes.

Definition 6.2. Let $T_X^{1,0} := \mathrm{im}(i : \mathcal{T}_X \hookrightarrow T_X \otimes \mathbb{C})$

Definition 6.3. For any smooth complex topological vector bundle E, let $\bar{E}$ be the vector bundle with transition matrices $\bar{\tau}_{ij}$, where τ_{ij} is the transition matrix of E.

Remark 6.4. In the preceding definition, we are working with a bundle whose transition matrices τ_{ij} have complex valued functions as their entries. The entries are smooth as function to $\mathbb{R}^2$, but may not be holomorphic!

Connsider $\overline{\mathcal{T}_X}$, with local coordinate $\frac{\partial}{\partial \bar{z}}$ (this is just a formal symbol). The transition matrix for $\overline{\mathcal{T}_X}$ is $\overline{\frac{\partial \phi_{ij}}{\partial z}} = \frac{\partial \bar{\phi}_{ij}}{\partial \bar{z}}$. As before, we have an inclusion $\iota_2 : \overline{\mathcal{T}_X} \hookrightarrow T_X \otimes \mathbb{C}$, given by $\frac{\partial}{\partial \bar{z}} \mapsto \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$. Write $T_X^{0,1} := \mathrm{im}(\iota_2)$.

Exercise 6.5. Check that $\frac{\partial}{\partial z}(z) = 1$, that $\frac{\partial}{\partial z} = 0$, that $\frac{\partial}{\partial z}(\bar{z}) = 0$, and that $\frac{\partial}{\partial \bar{z}}(\bar{z}) = 1$.

Proposition 6.6. $T_X \otimes \mathbb{C} \cong T_X^{1,0} \oplus T_X^{0,1}$

Proof. This follows from the fact that w_1 and w_2 are an eigenbasis for the Jacobian of a holomorphic function. Fill in the details as an exercise. □

Exercise 6.7. Show that for X of any dimension, Proposition 6.6 still holds.

7. Decomposition of cohomology

When dualized, Proposition 6.6 gives the following decomposition of the space of 1-forms

$$\Omega_X \otimes \mathbb{C} = \Omega_X^{1,0} \oplus \Omega_X^{1,0}$$

with $\Omega_X^{i,j} = \left(T_X^{i,j}\right)^{\vee}$. Write dz_i for the dual basis in $\Omega_X^{1,0}$ and $d\bar{z}_i$ for the dual basis of $\Omega_X^{0,1}$, so that $\Omega_X \otimes \mathbb{C}$ has basis $dz_i, d\bar{z}_i$ for $1 \leq i \leq \dim X$.

In turn, this gives a decomposition of the space of p-forms for each p:

$$\bigwedge^p (\Omega_X \otimes \mathbb{C}) = \bigoplus_{a+b=p} \bigwedge^a \Omega_X^{1,0} \otimes \bigwedge^b \Omega_X^{0,1}$$

Locally, elements of $\bigwedge^p (\Omega_X \otimes \mathbb{C})$ are of the form

$$\alpha = \sum_{|I|+|J|=p} c_{I,J} dz_I \wedge d\bar{z}_J$$

where $c_{I,J}$ is a smooth complex-valued function, $I = \{i_1, \dots, i_{|I|}\}$, $J = \{j_1, \dots, j_{|J|}\}$, $dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_{|I|}}$, and $d\bar{z}_J$ is defined similarly. Local sections of $\bigwedge^a \Omega_X^{1,0} \otimes \bigwedge^b \Omega_X^{0,1}$ are (a,b)**-forms** and written $\sum_{|I|=a,|J|=b} c_{I,J} dz_I \wedge d\bar{z}_J$.

7.1. **Dolbeault cohomology.** Complexifying the exterior derivative gives $d : \Omega_X^p \otimes \mathbb{C} \to \Omega_X^{p+1} \otimes \mathbb{C}$. If α is an (a,b)-form, then

$$d\alpha = \sum_{|I|=a,|J|=b} dc_{I,J} \wedge dz_I \wedge d\bar{z}_J \text{ with } dc_{I,J} \in \Omega^{1,0} \oplus \Omega^{0,1}.$$

Hence, $d\alpha$ is the sum of a $(a+1,b)$-form and a $(a,b+1)$-form.

Definition 7.1. If α is an (a,b)-form, then let $\partial\alpha \in \Omega^{a+1,b}$ and $\bar{\partial}\alpha \in \Omega^{a,b+1}$ be the forms such that $d\alpha = \partial\alpha + \bar{\partial}\alpha$.

Exercise 7.2. Let $f : X \to \mathbb{C}$ be a smooth complex-valued function. Show that f is holomorphic if and only if $\bar{\partial} f = 0$. This is analogue to how a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is holomorphic if and only if $\frac{\partial}{\partial \bar{z}} f = 0$.

Lemma 7.3. *The operators ∂ and $\bar{\partial}$ satisfy:*
(i) $\partial^2 = \bar{\partial}^2 = 0$
(ii) $\partial\bar{\partial} + \bar{\partial}\partial = 0$

Proof. The differential is $d = \partial + \bar{\partial}$, so

$$0 = d^2 = (\partial + \bar{\partial})^2 = \partial^2 + (\partial\bar{\partial} + \bar{\partial}\partial) + \bar{\partial}^2.$$

Collecting terms of the same degree (as maps of graded vector spaces) gives the desired result. □

The lemma above allows us to make the following definition.

Definition 7.4. The **Dolbeault complex** is

$$\cdots \to \Omega_X^{p,q-1}(X) \xrightarrow{\bar{\partial}} \Omega_X^{p,q}(X) \xrightarrow{\bar{\partial}} \Omega_X^{p,q+1}(X) \to \cdots$$

The (p,q)th **Dolbeault cohomology** group is

$$H^{p,q}(X) := \ker(\Omega^{p,q}(X) \to \Omega^{p,q+1}(X)) / \operatorname{im}(\Omega^{p,q-1}(X) \to \Omega^{p,q}(X)).$$

Exercise 7.5. Let $\alpha = c_{I,J} dz_I \wedge d\bar{z}_J$. Show that $\bar{\partial}\alpha = 0$ if and only if α is holomorphic with respect to all z_i with $i \notin J$ (i.e. the coefficient functions are holomorphic).

8. Sheaves and sheaf cohomology

While vector bundles are nice, they sadly don't form an abelian category.

Example 8.1. For a divisor D, consider

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0.$$

The first two terms of this sequence are vector bundles, but $\mathcal{O}_D$ is usually not a vector bundle. Hence, vector bundles are not closed under taking cokernels.

Hence, to give ourselves tools like the long exact sequence for cohomology, we need to enlarge our category and work with sheaves instead.

8.1. **Sheaves.**

Definition 8.2. A **presheaf** $\mathcal{F}$ on X of abelian groups (or modules, rings, etc) is the following data:

- For each $U \subset X$ open, a set $\mathcal{F}(U)$.
- For each inclusion of opens $V \subset U$, **restriction maps** $\mathcal{F}(V) \xleftarrow{\gamma_{UV}} \mathcal{F}(U)$

satisfying $\gamma_{U,U} = \mathrm{Id}_U$ and $\gamma_{WV} \circ \gamma_{VU} = \gamma_{UW}$ for all $W \subset V \subset U$.

Definition 8.3. A **sheaf** $\mathcal{F}$ on X of abelian groups (or modules, rings, etc) is a presheaf satisfying the additional condition that if $U = \bigcup_i U_i$ is an open cover and $u_i \in \mathcal{F}(U_i)$ are sections such that

$$\gamma_{U_i \cap U_j, U_i}(u_i) = \gamma_{U_i \cap U_j, U_j}(u_j) \text{ for all } i, j,$$

then there is a unique section $u \in \mathcal{F}(U)$ such that $\gamma_{U_i,U}(u) = u_i$ for all i.

Exercise 8.4. Let $E \to X$ be a vector bundle. Show that the sets $E(U)$ with restriction maps defined by restriction of functions form a sheaf.

Example 8.5 (Sheaves).

(i) The **constant sheaf** $\overline{\mathbb{Z}}$ on X is defined by

$$\overline{\mathbb{Z}} := \{\text{locally constant functions } U \to \mathbb{Z}\}$$

with restriction maps given by restriction of functions.

(ii) The **structure sheaf** $\mathcal{O}_X$ is given by

$$\mathcal{O}_X(U) := \{\text{holomorphic functions } f : U \to \mathbb{C}\}$$

with restriction maps given by restriction of functions. Observe that this is a sheaf of rings, since the product of two holomorphic functions is holomorphic.

(iii) The sheaf of **nowhere-zero functions** is $\mathcal{O}_X^*$ defined by

$$\mathcal{O}_X^*(U) := \{f \in \mathcal{O}_X(U) : \forall u \in U,\ f(u) \neq 0\}$$

and restriction maps given by restriction of functions. This is a sheaf of abelian groups under multiplication (this makes $\mathcal{O}_X^*$ quite different from $\mathcal{O}_X$!).

8.1.1. *Sections of vector bundles.* Recall that we can define a holomorphic vector bundle E of rank m by defining it to be $\mathbb{C}^m$ on a trivializing cover $\{U_i\}$ and specifying transition matrices $T_{ij} \in \mathrm{GL}_m(\mathcal{O}_X(U_i \cap U_j))$ satisfying the cocycle condition.

We equivalently use this data to determine the sheaf of sections of E as follows. On each trivial open U_i, set $E(U_i) = \mathcal{O}_X^{\oplus m}$. On overlaps $U_i \cap U_j$, we identify sections by $s_i = T_{ij} \circ s_j$.

8.2. **Morphisms of sheaves.**

Definition 8.6. A **morphism** of presheaves $\phi : \mathcal{F} \to \mathcal{G}$ is specified by a morphism $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for each open U such that the following diagram commutes for all $V \subset U$ opens

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi(U)} & \mathcal{G}(U) \\ \downarrow{\scriptstyle \gamma_{V,U}} & & \downarrow{\scriptstyle \gamma_{V,U}} \\ \mathcal{F}(V) & \xrightarrow{\phi(U)} & \mathcal{G}(V) \end{array}$$

Given a morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$, we can define the **image, kernel, and cokernel presheaves** of ϕ by $\mathrm{im}(\phi)^p(U) = \mathrm{im}(\phi(U))$, $\mathrm{coker}(\phi)^p(U) = \mathrm{coker}(\phi(U))$, and $\ker(\phi)^p(U) = \ker(\phi(U))$. These are presheaves, but not necessarily sheaves, even if both $\mathcal{F}$ and $\mathcal{G}$ are sheaves. However, the following proposition lets us fix this problem.

Proposition 8.7. *Given any presheaf $\mathcal{F}$, there exists a unique sheaf $\mathcal{F}^+$ and a map $\theta : \mathcal{F} \to \mathcal{F}^+$ such that any $\phi : \mathcal{F} \to \mathcal{G}$ with $\mathcal{G}$ a sheaf factors uniquely through θ.*

The sheaf $\mathcal{F}^+$ is the **sheafification** of $\mathcal{F}$.

Definition 8.8. Given $\phi : \mathcal{F} \to \mathcal{G}$ a map of sheaves, define the **kernel, image, and cokernel sheaves** to be the sheafification of the presheaf kernel, image, and cokernel presheaves, respectively. We call ϕ **injective** if $\ker(\phi) = 0$ and we say that ϕ is **surjective** if $\mathrm{coker}(\phi) = 0$.

8.3. **Stalks.** Let X be a complex manifold and $\mathcal{F}$ a presheaf on X.

Definition 8.9. For $x \in X$, the **stalk** of $\mathcal{F}$ at x, denoted $\mathcal{F}_x$ is

$$\mathcal{F}_x := \{(U, s) \,:\, x \in U \subset X \text{ open}, s \in \mathcal{F}(U)\} \,/\, \sim$$

where $(U, s) \sim (V, t)$ if there is $W \subset U \cap V$ such that $\gamma_{W,U}(s) = \gamma_{W,V}(t)$.

Proposition 8.10. *A morphism of sheaves on X $\phi : \mathcal{F} \to \mathcal{G}$ induces a map $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ for all $x \in X$.*

Proposition 8.11. *A map $\phi : \mathcal{F} \to \mathcal{G}$ of sheaves on X is injective, resp. surjective if and only if $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective, resp. surjective, for all $x \in X$.*

Example 8.12 (Exponential exact sequence). Let X be a complex manifold. We will write $\mathcal{O}_X$ for the sheaf of holomorphic functions, $\mathcal{O}_X^*$ for the sheaf of non-vanishing holomorphic functions under multiplication, $\underline{\mathbb{Z}}$ for the locally constant sheaf $\mathbb{Z}$, and $\mathcal{A}$ for the sheaf of smooth functions.

The **exponential map** is $\phi : \mathcal{O}_X \to \mathcal{O}_X^*$ is given by $\mathcal{O}_X(U) \ni f \mapsto \exp(2\pi i f)$. Note that this is a surjection because one can locally take logarithms in $\mathbb{C}$. The kernel of the exponential map is the locally constant sheaf $\underline{\mathbb{Z}}$. Hence, we have a short exact sequence

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0.$$

8.4. **Čech cohomology.** Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a locally finite covering indexed by an ordered set I. Write $U_{i_0,\ldots,i_p} := \bigcap_{j=i_0,i_1,\ldots,i_p} U_j$. Let $\mathcal{F}$ be a sheaf and let $C^p(\mathcal{U}, \mathcal{F}) := \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0,\ldots,i_p})$.

Define a map $\delta : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$ by

$$(\delta(\alpha))_{i_0,\ldots,i_{p+1}} = \sum_k (-1)^k \alpha_{i_0,\ldots,\hat{i}_k,\ldots,i_{p+1}}|_{U_{i_0,\ldots,i_{p+1}}}.$$

Proposition 8.13. $\delta^2 = 0$

Let $\check{H}^p(\mathcal{U}, \mathcal{F}) := \ker(\delta : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F}))/\,\mathrm{im}(\delta)$ and note that if $\mathcal{U}'$ refines $\mathcal{U}$, then there is a map $\check{H}^p(\mathcal{U}, \mathcal{F}) \to \check{H}^p(\mathcal{U}', \mathcal{F})$.

Definition 8.14. The pth **Čech cohomology group** is

$$\check{H}^p(X, \mathcal{F}) := \varinjlim_{\mathcal{U}} \check{H}^p(\mathcal{U}, \mathcal{F}).$$

Remark 8.15. A theorem of Leray says that to compute Čech cohomology, it suffices to take a sufficiently fine open cover. In particular, if using the Zariski topology, then a cover by affine opens is fine enough.

8.4.1. *Line bundles and cohomology of* $\mathcal{O}_X^*$. Let X be a complex manifold and fix a cover $\mathcal{U} = \{U_i\}_{i\in I}$. Recall that a line bundle with trivializing cover $\mathcal{U}$ is given by the data of $g_{ij} \in \mathcal{O}_X^*(U_i \cap U_j)$ satisfying $g_{ki}g_{ij} = g_{kj}$. This data gives an element $g = (g_{ij}) \in C^1(\mathcal{U}, \mathcal{O}_X^*)$.

Since the group operation in $\mathcal{O}_X^*(U)$ is multiplication, we can write $(\delta g)_{ijk} = g_{j,k} \cdot g_{i,k}^{-1} \cdot g_{i,j}$. Note that

$$g \in \ker \delta \iff g_{jk}g_{ik}^{-1}g_{ij} = 1 \iff g_{ij}g_{jk} = g_{ik}.$$

But this is the cocycle condition! Hence, the kernel of δ contains the data of all line bundles on X that are trivialized by $\mathcal{U}$.

Definition 8.16. The **Picard group**, denoted $\mathrm{Pic}(X)$, is the group of line bundles on X with operation $\otimes$.

Exercise 8.17. Show that

$$\begin{aligned} \mathrm{Pic}(X) &\xrightarrow{\cong} \check{H}^1(\mathcal{O}_X^*) \\ L &\mapsto \overline{[g_{ij}]} \end{aligned}$$

is an isomorphism.

Recall that given any cohomology theory, we have a long exact sequence. Hence, from the exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

we obtain a long exact sequence

$$\cdots \to \check{H}^1(\mathcal{O}_X) \to \check{H}^1(\mathcal{O}_X^*) \xrightarrow{c_1} \check{H}^2(X, \mathbb{Z}) \to \cdots$$

Philosophically, this beaks $\mathrm{Pic}(X)$ up into a continuous part coming from $\check{H}^1(\mathcal{O}_X)$ and a discrete part coming from $\check{H}(\underline{\mathbb{Z}})$.

Definition 8.18. If $L \in \mathrm{Pic}(X) = \check{H}^1(\mathcal{O}_X^*)$, then $c_1(L)$ is the **first Chern class** of L.

8.4.2. *Smooth functions (or "Why Čech cohomology is useless for differential geometers").* Let $\mathcal{A}$ be the sheaf of smooth functions on a complex manifold X. Recall that $\Omega_X^{a,b}$ is the (a,b)th part of $\bigwedge^{a+b} \Omega_{X,\mathbb{R}} \otimes \mathbb{C}$, and note that for each open $U \subset X$, $\Omega_X^{a,b}(U)$ is a module over $\mathcal{A}(U)$; in order words, $\Omega_X^{a,b}$ is an $\mathcal{A}$-module.

Proposition 8.19. $\check{H}^p(\Omega_X^{a,b}) = 0$ *for* $p > 0$.

Proof. Given $\mathcal{U} = \{U_i\}$ countable and locally finite, there is a partition of unity subordinate to $\mathcal{U}$; that is, there are smooth functions $p_i : X \to \mathbb{R}$ such that $p_i^{-1}(\mathbb{R} \setminus \{0\}) \subset U_i$ and such that for all $x \in X$, $\sum_i p_i(x) = 1$ (the fact that the cover is locally finite implies that this sum is always finite).

If $\sigma = (\sigma_{i_0,\ldots,i_p}) \in Z^p(\mathcal{U}, \Omega_X^{a,b})$, define $\tau \in C^{p-1}(\mathcal{U}, \Omega_X^{a,b})$ by

$$\tau_{j_0,\ldots,j_{p-1}} = \sum_k p_k \sigma_{k,j_0,\ldots,j_{p-1}}$$

where we consider $\sigma_{k,j_0,\ldots,j_{p-1}}$ as a smooth function on $U_{j_0} \cap \cdots \cap U_{j_{p-1}}$. One can check that $\delta\tau = \sigma$. □

Exercise 8.20. Do the check in the previous proof.

Remark 8.21. This shows why Čech cohomology is completely useless in differential geometry.

9. Resolutions and cohomology

The following result can be found in most introductory differential geometry texts (such as Spivak, page 225). Call a p-form **closed** if $d\alpha = 0$ and **exact** if there is a β such that $\alpha = d\beta$.

Lemma 9.1 (Poincaré lemma)**.** *If M is a contractible smooth manifold and α is a closed p-form with $p > 0$, then α is exact.*

Since sufficiently small opens in manifolds are isomorphic to $\mathbb{R}^n$, the Poincaré lemma implies that the following is an exact complex of sheaves

$$0 \to \underline{\mathbb{R}} \to \Omega^0_{M,\mathbb{R}} \to \Omega^1_{M,\mathbb{R}} \to \Omega^2_{M,\mathbb{R}} \to \cdots$$

We can use this to get a new (integration-free!) proof of de Rham's theorem.

Theorem 9.2. *For $i \geq 0$,*

$$\check{H}^i(X, \underline{\mathbb{R}}) \cong H^i_{dR}(X) = \ker(d^i : \Omega^i_{X,\mathbb{R}}(X) \to \Omega^{i+1}_{X,\mathbb{R}}(X)) / \operatorname{im}(d^{i-1}).$$

Proof. For $i = 0$, consider the short exact sequence

$$0 \to \underline{\mathbb{R}} \to \Omega^0_{X,\mathbb{R}} \to \operatorname{im}(d^0) \to 0. \tag{1}$$

Since $\check{H}^0(X, \cdot) = \Gamma(X, \cdot)$, the beginning of the long exact sequence arising from the short exact sequence (1) is

$$0 \to \check{H}^0(\underline{\mathbb{R}}) \to \underbrace{\check{H}^0(\Omega^0_{X,\mathbb{R}})}_{=\Omega^0_{X,\mathbb{R}}(X)} \to \check{H}^0(\operatorname{im} d^0) \to \cdots,$$

so $\check{H}^0(X, \underline{\mathbb{R}}) \cong \ker(d^0) = H^0_{dR}(X)$.

For $i > 0$, we again use long exact sequences. By (1) and Proposition 8.19, $\check{H}^i(\underline{\mathbb{R}}) \cong \check{H}^{i-1}(\operatorname{im} d^0)$. Similarly, by using the long exact sequences associated to the short exact sequences

$$0 \to \operatorname{im}(d^{j-1}) = \ker(d^j) \to \Omega^j_{X,\mathbb{R}} \xrightarrow{d^j} \operatorname{im}(d^j) \to 0, \tag{2}$$

for each $1 \leq j \leq i-2$, we find that

$$\check{H}^i(\underline{\mathbb{R}}) \cong \check{H}^{i-1}(\operatorname{im} d^0) \cong \check{H}^{i-2}(\operatorname{im} d^1) \cong \cdots \check{H}^1(\operatorname{im} d^{i-2}).$$

The beginning of the long exact sequence associated to (2) when $j = i-1$ reads

$$0 \to \check{H}^0(\operatorname{im} d^{i-2}) \to \check{H}^0(\Omega^{i-1}_{X,\mathbb{R}}) \to \check{H}^0(\operatorname{im} d^{i-1}) \to \check{H}^1(\operatorname{im} d^{i-2}) \to 0.$$

Hence,

$$\begin{aligned}
\check{H}^i(\underline{\mathbb{R}}) &\cong \check{H}^1(\operatorname{im} d^{i-2}) \cong \check{H}^0(\operatorname{im} d^{i-1}) / \operatorname{im}(\Omega^{i-1}_{X,\mathbb{R}}(X) \to \check{H}^0(\operatorname{im} d^{i-1})) \\
&= \check{H}^0(\ker d^i) / \operatorname{im}(\Omega^{i-1}_{X,\mathbb{R}}(X) \to \check{H}^0(\operatorname{im} d^{i-1})) \\
&= \ker(d^i : \Omega^i_{X,\mathbb{R}}(X) \to \Omega^{i+1}_{X,\mathbb{R}}(X)) / \operatorname{im}(d^{i-1} : \Omega^{i-1}_{X,\mathbb{R}}(X) \to \Omega^i_{X,\mathbb{R}}(X)) \\
&= H^i_{dR}(X)
\end{aligned}$$

□

Let's abstract our proof technique for this.

Definition 9.3. If $\mathcal{F}$ is a sheaf, then a **resolution** of $\mathcal{F}$ is a complex

$$\mathcal{G}_\bullet : \mathcal{G}_0 \xrightarrow{\phi_0} \mathcal{G}_1 \xrightarrow{\phi_1} \mathcal{G}_2 \to \cdots$$

with a morphism $\mathcal{F} \to \mathcal{G}_0$ such that $0 \to \mathcal{F} \to \mathcal{G}_0 \to \mathcal{G}_1 \to \mathcal{G}_2 \to \cdots$ is exact.

Example 9.4.

$$\Omega^\bullet : \Omega^0_{M,\mathbb{R}} \to \Omega^1_{M,\mathbb{R}} \to \Omega^2_{M,\mathbb{R}} \to \cdots$$

is a resolution of $\underline{\mathbb{R}}$.

Definition 9.5. A resolution $\mathcal{G}_\bullet$ of $\mathcal{F}$ is **acyclic** if $\check{H}^p(\mathcal{G}_i) = 0$ for all $p > 0$.

Exercise 9.6. Let $\mathcal{R}_\bullet$ be an acyclic resolution of $\mathcal{F}$. Then

$$\check{H}^p(\mathcal{F}) \cong \ker(\mathcal{R}_p(M) \to \mathcal{R}_{p+1}(M))/\operatorname{im}(\mathcal{R}_{p-1}(M) \to \mathcal{R}_p(M)).$$

(Hint: this will mirror our proof of de Rham's theorem.)

9.1. **A resolution for Dolbeault cohomology.** The following result can be found on page 47 of Huybrechts' *Complex Geometry: An Introduction.*

Theorem 9.7 ($\bar\partial$ Poincaré lemma). *Let β be an open disc $D \subset \mathbb{C}^n$ and let $\alpha \in \Omega^{p,1}(D)$ satisfy $\bar\partial\alpha = 0$. Then there exists a $\beta \in \Omega^{p,q-1}(D)$ such that $\bar\partial\beta = \alpha$.*

Corollary 9.8. *If X is a complex manifold, then we have an acyclic resolution*

$$0 \to \Omega^p_X \to \Omega^{p,0}_X \xrightarrow{\bar\partial} \Omega^{p,1}_X \xrightarrow{\bar\partial} \Omega^{p,2}_X \to \cdots$$

where Ω^p_X is the sheaf of holomorphic p-forms and $\Omega^{p,i}_X$ is the sheaf of smooth (p,i)-forms.

Corollary 9.9 (Dolbeault's theorem)**.** $\check{H}^q(X, \Omega^p_X) = H^{p,q}(X)$

10. Almost-complex structures

Let V be a finite-dimensional vector space over $\mathbb{R}$.

Definition 10.1. An **almost complex structure** on V is a linear map $I : V \to V$ such that $I^2 = -\operatorname{Id}_V$.

An almost complex structure gives V the structure of a complex vector space, with $(a + bi) \cdot v := av + bI(v)$. Additionally, if we complexify V, then the possible eigenvalues of $I : V_\mathbb{C} \to V_\mathbb{C}$ are $\pm i$.

Definition 10.2. If V has almost complex structure, let

$$V^{1,0} \subset V_\mathbb{C} \text{ be the } i\text{-eigenspace of } I \text{ and}$$
$$V^{0,1} \subset V_\mathbb{C} \text{ be the } (-i)\text{-eigenspace of } I.$$

Lemma 10.3. $V_\mathbb{C} = V^{1,0} \oplus V^{0,1}$

Proof. Let $v \in V_\mathbb{C}$. Then

$$v = \frac{1}{2}(\underbrace{v - iI(v)}_{\in V^{1,0}}) + \frac{1}{2}(\underbrace{v + iI(v)}_{\in V^{0,1}})$$

□

Lemma 10.4. *Complex conjugation is an isomorphism (over $\mathbb{R}$) $V^{1,0} \to V^{0,1}$.*

Proof. Let $v = x + iy$ with $x, y \in V$. The $V^{1,0}$ part is

$$\frac{1}{2}(v - iI(v)) = \frac{1}{2}(x + iy - i(I(x) + iI(y)))$$

and its conjugate is $\frac{1}{2}(\bar{v} + iI(\bar{v}))$, which is the $V^{0,1}$ part of $\bar{v}$. □

Exercise 10.5. Show that if V has an almost-complex stucuture, then $\dim_{\mathbb{R}} V$ is even.

Definition 10.6. Let $\langle \cdot, \cdot \rangle$ be a symmetric bilinear form on V with almost complex structure I. We say I is **compatible** with $\langle \cdot, \cdot \rangle$ if $\langle v, w \rangle = \langle I(v), I(w) \rangle$ for all $v, w \in V$.

Convention. For the remainder of this section, we will work with $(V, I, \langle \cdot, \cdot \rangle)$ with V a real vector spcae, I an almost complex structure, and $\langle \cdot, \cdot \rangle$ a positive definite symmetric bilinear form compatible with I (which we extend to be sesquilinear on $V_{\mathbb{C}}$ when needed).

Exercise 10.7. Check that if $(V, I, \langle , \rangle)$ is of real dimension ≥ 4, then we can write $(V, I, \langle , \rangle) = (W_1, I_1, \langle , \rangle_1) \oplus (W_2, I_2, \langle , \rangle_2)$ for some (non-unique) nonzero subspaces with almost-complex structures $(W_1, I_1, \langle , \rangle_1)$.

Definition 10.8. The **fundamental form** ω on V is the bilinear pairing

$$\omega(v, w) := \langle I(v), w \rangle$$

for $v, w \in V$.

We next establish a pair of basic properties of the ω.

Lemma 10.9. *ω is alernating; that is, $\omega \in \bigwedge^2 V^\vee$.*

Proof. Let $v, w \in V$.

$$\omega(v, w) = \langle I(v), w \rangle = \left\langle I^2(v), I(w) \right\rangle = -\langle v, I(w) \rangle = -\langle I(w), v \rangle = -\omega(w, v).$$

□

Lemma 10.10. *$\omega(I(v), I(w)) = \omega(v, w)$ for all $v, w \in V$.*

Proof.

$$\omega(I(v), I(w)) = \left\langle I^2(v), I(w) \right\rangle = \langle I(v), w \rangle = \omega(v, w)$$

□

From an almost-complex structure on V, we also obtain an almost-complex structure on $V^\vee$ given by $I \cdot f := f \circ I \in V^\vee$. This gives a compatible decomposition $V_{\mathbb{C}}^\vee = (V^\vee)^{1,0} \oplus (V^\vee)^{0,1} \cong (V^{1,0})^\vee \oplus (V^{0,1})^\vee$. Hence, we may define $\bigwedge^{p,q} V^\vee := \bigwedge^p (V^\vee)^{1,0} \otimes \bigwedge^q (V^\vee)^{0,1} \subset \bigwedge^{p+q} V_{\mathbb{C}}^\vee$.

Exercise 10.11. Show that $\omega \in \bigwedge^{1,1} V^\vee$. (Hint: $\omega \in \bigwedge^2 V_{\mathbb{C}}^\vee = \bigwedge^{2,0} V^\vee \oplus \bigwedge^{1,1} V^\vee \oplus \bigwedge^{0,2} V^\vee$, and for any $\alpha \in \bigwedge^{2,0} V^\vee$, we have $\alpha(I(v), I(w)) = -\alpha$.)

10.1. **Operators on the exterior algebra.** Let $(V, I, \langle \cdot, \cdot \rangle)$ be as above, with fundamental form ω.

Definition 10.12. Given a finite-dimensional vector space W, the **exterior algebra** is $\bigwedge^* W := \bigoplus_k \bigwedge^k W$. This is a graded algebra with non-commutative product $\wedge$, and it is finite-dimensional over the ground field of W.

There are three operators on $\bigwedge^* V^\vee$ (extensible by $\mathbb{C}$-linearity to $\bigwedge^* V_{\mathbb{C}}^\vee$) that will be of great interest.

Definition 10.13. The **Lefschetz operator** is $L : \bigwedge^* V_{\mathbb{C}}^\vee \to \bigwedge^* V_{\mathbb{C}}^\vee$ given by $\alpha \mapsto \omega \wedge \alpha$. Note that if we extend L to $\bigwedge^* V_{\mathbb{C}}^\vee$, then by Exercise 10.11, $L(\bigwedge^{p,q} V^\vee) \subset \bigwedge^{p+1,q+1} V^\vee$.

If $(W, \langle , \rangle)$ is a **Euclidean space** (a finite dimensional vector space with a positive definite inner product), then there is a natural inner product on $\bigwedge^* W^\vee$. To define it, pick an orthonormal basis $e_1, \ldots, e_d$ for W and declare

$$\{e_I^\vee := e_{i_1}^\vee \wedge \cdots \wedge e_{i_{|I|}}^\vee : i_1 < i_2 < \cdots < i_{|I|},\ |I| \leq d\}$$

to be an orthonormal basis of $\bigwedge^* W^\vee$.

Definition 10.14. The **dual Lefschetz operator** is $\Lambda : \bigwedge^* V^\vee \to \bigwedge^* V^\vee$, is defined to be the **adjoint** of L; that is, defined by the property

$$\langle L\alpha, \beta \rangle = \langle \alpha, \Lambda\beta \rangle \text{ for all } \alpha, \beta \in \textstyle\bigwedge^* V^\vee.$$

where $\langle , \rangle$ is the form induced on $\bigwedge^* V^\vee$ by the inner product on V.

Exercise 10.15. Prove that Λ has type $(-1, -1)$; that is, $\Lambda(\bigwedge^{p,q} V^\vee) \subset \bigwedge^{p-1,q-1} V^\vee$.

Definition 10.16. The **counting operator** is $H : \bigwedge^* V^\vee \to \bigwedge^* V^\vee$, which is defined by $H|_{\bigwedge^k V^\vee} := (k - n) \cdot \mathrm{Id}_{\bigwedge^k V^\vee}$, where $2n = \dim_{\mathbb{R}} V$.

Given ring elements A and B, write $[A, B] := AB - BA$ for their commutator.

Theorem 10.17. *The operators L, Λ, H satisfy*

(i) $[H, L] = 2L$
(ii) $[H, \Lambda] = -2\Lambda$
(iii) $[L, \Lambda] = H$

Remark 10.18. The complex vector space spanned by L, Λ, H with product given by commutator forms a 3-dimensional Lie algebra. One might naturally wonder which one. Recall that

$$\mathfrak{sl}_2(\mathbb{C}) := \{2 \times 2 \text{ matrices with trace } 0\}$$

has basis $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. There is a map of Lie algebras sending B to H, X to L, and Y to Λ; hence, the operators L, Λ, H make $\bigwedge^* V^\vee$ into an $\mathfrak{sl}_2(\mathbb{C})$ representation.

Exercise 10.19. Prove that B, X, and Y satisfy the relations described in Theorem 10.17. Hence, there is a natural $\mathfrak{sl}_2(\mathbb{C})$ action on $\bigwedge^* V^\vee$.

Proof of Theorem 10.17(i) and (ii).

(i) Let $\alpha \in \bigwedge^k V^\vee$.

$$\begin{aligned}
[H,L](\alpha) &= HL(\alpha) - LH(\alpha) \\
&= H(\omega \wedge \alpha) - (k-n)L(\alpha) \\
&= (k+2-n)(\omega \wedge \alpha) - (k-n)(\omega \wedge \alpha) \\
&= 2(\omega \wedge \alpha) = 2L(\alpha)
\end{aligned}$$

(ii) Let $\alpha \in \bigwedge^k V^\vee$.

$$\begin{aligned}
[H,\Lambda](\alpha) &= H\Lambda(\alpha) - \Lambda H(\alpha) \\
&= H\Lambda(\alpha) - (k-n)\Lambda(\alpha) \\
&= (k-n-2)(\Lambda(\alpha)) - (k-n)(\Lambda(\alpha)) \\
&= -2\Lambda(\alpha)
\end{aligned}$$

where the penultimate equality follows from Exercise 10.15.

□

Proof of Theorem 10.17(iii). We proceed by induction on $\dim_{\mathbb{R}} V = 2n$.

If $\dim_{\mathbb{R}} V > 2$, then by Exercise 10.7, $(V, I, \langle,\rangle) = (W_1, I_1, \langle,\rangle_1) \oplus (W_2, I_2, \langle,\rangle_2)$ for some nonzero orthogonal subspaces W_1 and W_2 with almost-complex structures. By standard linear algebra, $\bigwedge^* V^\vee \cong \bigwedge^* W_1^\vee \otimes \bigwedge^* W_2^\vee$, so it suffices to check the identity for $\alpha_1 \otimes \alpha_2$ with $\alpha_1 \in \bigwedge^{k_1} W_1^\vee$ and $\alpha_2 \in \bigwedge^{k_2} W_2^\vee$.

Observe that the Lefschetz operator of V decomposes as $L = L_1 \otimes 1 + 1 \otimes L_2$, where$L_1$ and L_2 are Lefschetz operators of W_1 and W_2, respectively. The dual Lefschetz operator decomposes similarly. This allows us to compute

$$\begin{aligned}
[L,\Lambda](\alpha_1 \otimes \alpha_2) &= L(\Lambda(\alpha_1 \otimes \alpha_2)) - \Lambda(L(\alpha_1 \otimes \alpha_2)) \\
&= L(\Lambda_1(\alpha_1) \otimes \alpha_2 + \alpha_1 \otimes \Lambda_2(\alpha_2)) \\
&\quad - \Lambda(L_1(\alpha_1) \otimes \alpha_2 + \alpha_1 \otimes L_2(\alpha_2)) \\
&= \big(L_1\Lambda_1(\alpha_1) \otimes \alpha_2 + \Lambda_1(\alpha_1) \otimes L_2(\alpha_2)\big) \\
&\quad + \big(L_1(\alpha_1) \otimes \Lambda_2(\alpha_2) + \alpha_1 \otimes L_2\Lambda_2(\alpha_2)\big) \\
&\quad - \big(\Lambda_1 L_1(\alpha_1) \otimes \alpha_2 + L_1(\alpha_1) \otimes \Lambda_2(\alpha_2)\big) \\
&\quad - \big(\Lambda_1(\alpha_1) \otimes L_2(\alpha_2) + \alpha_1 \otimes \Lambda_2 L_2(\alpha_2)\big) \\
&= [L_1,\Lambda_1](\alpha_1) \otimes \alpha_2 + \alpha_2 \otimes [L_2,\Lambda_2](\alpha_2) \\
&= (k_1 - n_1)\alpha_1 \otimes \alpha_2 + \alpha_2 \otimes (k_2 - n_2)(\alpha_2) \qquad \text{by induction} \\
&= (k_1 + k_2 - (n_1 + n_2))\alpha_1 \otimes \alpha_2 = H(\alpha_1 \otimes \alpha_2)
\end{aligned}$$

Now, all that remains is to check the base case, when $\dim_{\mathbb{R}} V = 2$. In this case,

$$\bigwedge^* V^\vee = \bigwedge^0 V^\vee \oplus V^\vee \oplus \bigwedge^2 V^\vee.$$

Since L has total degree 2, it acts by 0 on the top two homogeneous components. On $\mathbb{R} \cong \bigwedge^0 V^\vee$, it acts by $L : 1 \mapsto \omega$. Similarly, since Λ has degree -2 and is the adjoint of L, it acts on $\bigwedge^2 V^\vee$ by $\omega \mapsto 1$ and by 0 on $\bigwedge^0 V^\vee \oplus V^\vee$. One can now easily check case-by case that $[L,\Lambda](\alpha) = H(\alpha)$ for elements α of each of the possible degrees $0,1,2$, completing the proof. □

Lemma 10.20. *For $i \geq 1$ and $\alpha \in \Lambda^k V^\vee$,*

$$[L^i, \Lambda](\alpha) = i(k-n+i-1)L^{i-1}(\alpha).$$

Proof. We proceed by induction on i. When $i=1$, $[L,\Lambda](\alpha)=H(\alpha)=(k-n)\alpha$ by Theorem 10.17. For $i>1$,

$$\begin{aligned}[L^i,\Lambda](\alpha) &= L^i\Lambda(\alpha)-\Lambda L^i(\alpha)\\ &= L(L^{i-1}\Lambda(\alpha)-\Lambda L^{i-1}(\alpha))+L\Lambda L^{i-1}(\alpha)-\Lambda L^i(\alpha)\\ &= L\big([L^{i-1},\Lambda](\alpha)\big)+[L,\Lambda](L^{i-1}(\alpha))\\ &= L\big((k-n+i-2)(i-1)L^{i-2}(\alpha)\big)+(k+2(i-1)-n)L^{i-1}(\alpha)\\ &= i(k-n+i-1)L^{i-1}(\alpha)\end{aligned}$$

□

10.2. **The Lefschetz decomposition.**

Definition 10.21. $\alpha\in\Lambda^k V^\vee$ is **primitive** if $\Lambda\alpha=0$. We denote the subspace of degree k primitive elements by $P^k\subset\Lambda^k V^\vee$.

Theorem 10.22 (Lefschetz decomposition)**.** $\Lambda^k V^\vee=\bigoplus_i L^i P^{k-2i}$

Proof. Recall that by Theorem 10.17, we have an $\mathfrak{sl}_2(\mathbb{C})$-representation on $\bigwedge^* V^\vee$. Hence, we can decompose $\bigwedge^* V^\vee$ as a direct sum of irreps. We show that if $W\subset\bigwedge^* V^\vee$ is a nonzero irrep, then $W=\operatorname{span}\{v,Lv,L^1v,\ldots\}$ for some primitive element v.

Pick $w\neq 0\in\Lambda^m W$, where $m=\max\{m:\Lambda^m W\neq 0\}$. If w is not homogeneous, then we may cancel off all but one of the nonzero homogeneous components of of w using the operators $(k-n)\operatorname{Id}-H$ for $0\le k\le 2n$ (the resulting vector will still be in $\Lambda^m W$ because $\Lambda H(w)=H\Lambda(w)-2\Lambda(w)=0$). Set v to be the primitive element that results from this process.

Since the elements of $\{v,Lv,L^2v,\ldots\}$ are homogeneous, $W':=\operatorname{span}_{\mathbb{R}}\{v,Lv,L^2v,\ldots\}$ is closed under H. It is also closed under L, and by Lemma 10.20

$$\Lambda L^i(v)=cL^{i-1}(v)+L^i\Lambda(v)=cL^{i-1}(v)+0$$

for some constant c, so W' is also closed under Λ, and $W'=W$, as desired. □

11. Hermitian and projective manifolds

In this section, we discuss how to globalize the structures discussed in the previous section. Let X be a complex manifold with real tangent bundle $T_{X,\mathbb{R}}$ with fiber coordinates $\frac{\partial}{\partial x_i},\frac{\partial}{\partial y_i}$. Denote the complexification by $T_{X,\mathbb{C}}:=T_{X,\mathbb{R}}\otimes\mathbb{C}$, and let $\mathcal{T}_X$ be the holomorphic tangent bundle.

We have an isomorphism of smooth bundles (not holomorphic!) given by the composition

$$T_{X,\mathbb{R}}\hookrightarrow T_{X,\mathbb{C}}=T^{1,0}\oplus T^{0,1}\to T^{1,0}\cong\mathcal{T}_X.$$

Hence, the multiplication $i:\mathcal{T}_X\xrightarrow{\cong}\mathcal{T}_X$ induces an almost-complex structure $I:T_{X,\mathbb{R}}\to T_{X,\mathbb{R}}$ given by $\frac{\partial}{\partial x_i}\mapsto\frac{\partial}{\partial y_i}$ and $\frac{\partial}{\partial y_i}\mapsto-\frac{\partial}{\partial x_i}$.

To finish the analogy with the vector space picture, we need a compatible inner product.

Definition 11.1. A **Riemannian metric** is $g\in\operatorname{Sym}^2\Omega_{X,\mathbb{R}}(X)$ such that $g_x: T_{x,\mathbb{R}}\times T_{x,\mathbb{R}}\to\mathbb{R}$ is positive definite for all $x\in X$ and such that the function $x\mapsto g_x(v(x),u(x))$, with $v,u\in T_{X,\mathbb{R}}(X)$ is smooth.

Definition 11.2. A **Hermitian manifold** is a complex manifold together with a Riemannian metric $g \in \mathrm{Sym}^2 \Omega_{X,\mathbb{R}}(X)$ such that g is compatible with I; that is, $g(I(u), I(v)) = g(u, v)$ for any $u, v \in T_{x,\mathbb{R}}$.

Definition 11.3. If X is a Hermitian manifold, the **fundamental form** is $\omega := g(I(\cdot), \cdot)$.

Note that ω is a real $(1,1)$-form; that is, $\omega \in \Omega^2_{X,\mathbb{R}}(X) \cap \Omega^{1,1}_X(X) \subset \Omega^2_{X,\mathbb{C}}(X)$. As in the vector space case, we can define the exterior algebra $\bigwedge^* \Omega_{X,\mathbb{R}}$ a smooth vector bundle of finite rank, and the operators $L, H, \Lambda : \bigwedge^* \Omega_{X,\mathbb{R}} \to \bigwedge^* \Omega_{X,\mathbb{R}}$ all globalize appropriately (for example, L is locally given by wedging with ω).

Definition 11.4. A Hermitian manifold is **Kähler** if the fundamental form is closed; that is, $d\omega = 0$.

11.1. **Projective manifolds.** We would eventually like to have the operators L, H, Λ act on the cohomology of X. One problem with out current set-up is that there is no reason for $\bigoplus_{p,q} H^{p,q}(X)$ to be finite-dimensional. To fix this problem, we could assume that X is compact or, as we will do in this course, impose the stronger condition that X is projective.

Definition 11.5. A morphism $i : X \to Y$ of complex manifolds is an **immersion** if i is injective and the induced holomorphic map on holomorphic tangent bundles is $\mathcal{T}_{X,x} \to \mathcal{T}_{Y,i(x)}$ is injective.

Definition 11.6. A **projective manifold** is a complex manifold such that there exists an immersion $i : X \to \mathbb{P}^n$ for some n such that the image is closed.

Note that if X is projective, then the compactness of the image of the immersion $i : X \to \mathbb{P}^n$ implies that i is a homeomorphism. Therefore, X must be compact.

Theorem 11.7 (Serre)**.** *If X is a projective complex manifold and E is a holomorphic vector bundle of finite rank, then $\check{H}^i(X, E)$ is finite dimensional over $\mathbb{C}$ for all i.*

Recall from Corollary 9.9 that $H^{p,q}(X) \cong \check{H}^q(X, \Omega^p_X)$, where Ω^p_X denotes the pth wedge power of the holomorphic cotangent bundle.

Remark 11.8. We will see later that we have a decomposition $H^p(X, \mathbb{C}) = \bigoplus_{a+b=p} H^{a,b}(X)$. Hence, if X is projective, then Theorem 11.7 and Corollary 9.9 imply that $\dim H^p(X, \mathbb{C}) < \infty$.

Remark 11.9. Theorem 11.7 is very false if X is not compact!

Remark 11.10. If X is a projective variety, it is not bad to prove Theorem 11.7. The hard part is showing that a holomorphic bundle on a projective complex manifold is the same as a coherent sheaf on a projective variety, which is among the main results of Serre's GAGA.

We have operators $L, H, \Lambda : \bigwedge^* \Omega_{X\mathbb{R}} \to \bigwedge^* \Omega_{X,\mathbb{R}}$. Fibrewise, these operators are the same as the ones that we studied on vector spaces in Section 10. Taking global sections, we obtain operators $L, H, \Lambda : \bigwedge^* \Omega_{X\mathbb{R}}(X) \to \bigwedge^* \Omega_{X,\mathbb{R}}(X)$. We will later see that these operators descend to operators on cohomology.

11.2. **Projective manifolds are Kähler.** The main result of this section is

Theorem 11.11. *Projective manifolds are Kähler.*

To show that a Hermetian manifold is Kähler, we need to show that $\omega = g(I(\cdot), \cdot)$ is closed. It turns out that it is enough to work with the fundamental form ω. In particular, if we have such an ω, then we can define $g := \omega(\cdot, I(\cdot))$. If we define g this way, then it is symmetric because

$$g(a, b) = \omega(a, Ib) = \omega(Ia, -b) = \omega(b, Ia) = g(b, a)$$

and compatible with I because

$$g(Ia, Ib) = \omega(Ia, I^2 b) = \omega(a, Ib) = g(a, b).$$

11.2.1. *The case of* $\mathbb{P}^n$*.* Before proving Theorem 11.11, we prove the following special case. Our strategy for the proof will be to construct an I-compatible real $(1, 1)$-form ω'. We will then show that $\omega'(\cdot, I(\cdot))$ is positive definite and hence a Riemannian metric.

Theorem 11.12. $\mathbb{P}^n$ *is Kähler*

Let $\{U_i\}$ be the standard charts on $\mathbb{P}^n$, where U_i is the open of $\mathbb{P}^n$ where the ith coordinate is nonzero and the homeomorphism to

$$U_i \to \mathbb{C}^n \text{ is given by } [x_0 : \cdots : x_n] \mapsto \left(\frac{x_0}{x_i}, \ldots, \widehat{\frac{x_i}{x_i}}, \ldots, \frac{x_n}{x_i} \right).$$

Define $(1, 1)$-forms $\omega_j \in \Omega^{1,1}(U_j)$ by

$$\omega_j := \frac{i}{2\pi} \partial\bar{\partial} \log \left(\sum_{\ell=0}^{n} \left| \frac{x_\ell}{x_j} \right|^2 \right).$$

Recall that if $f \in \Omega^0(X) = C^\infty(X)$, then $\partial\bar{\partial} f \in \Omega^{1,1}_X(X)$. The function $\frac{x_\ell}{x_j}$ is the ℓth coordinate function $z_\ell : U_j \to \mathbb{C}$. When we take absolute value and square it, we are getting $z_\ell \bar{z}_\ell$. This is a smooth function, even if it is not a holomorphic one, so ω_j is in fact a $(1, 1)$-form.

Lemma 11.13. $\partial\bar{\partial} f$ *is a closed form for any* $f \in \Omega^0(X)$*.*

Proof.

$$d(\partial\bar{\partial} f) = (\partial + \bar{\partial})(\partial\bar{\partial} f) = \partial^2 \bar{\partial} f + \bar{\partial}\partial\bar{\partial} f = 0 - \partial\bar{\partial}^2 f = 0$$

where the final equality is because $\partial\bar{\partial} + \bar{\partial}\partial = 0$. □

Lemma 11.14. *The forms* $\omega_j \in \Omega^{1,1}_X(U_j)$ *glue to give a global* $(1, 1)$*-form* $\omega_{FS} \in \Omega^{1,1}_X(X)$*, called the* ***Fubini-Study form****.*

Proof. Let $\phi_j : U_j \xrightarrow{\cong} \mathbb{C}^n$ be the usual charts with $U_j = \{x_j \neq 0\}$ on $\mathbb{P}^n$. We must show that $\omega_j|_{U_j \cap U_k} = \omega_k|_{U_j \cap U_k}$. Since $\{x_j, x_k \neq 0\} = U_j \cap U_k$, we have

$$\begin{aligned}\log\left(\sum_{\ell=0}^{n}\left|\frac{x_\ell}{x_j}\right|^2\right) &= \log\left(\sum_{\ell=0}^{n}\left|\frac{x_k}{x_j}\right|^2\left|\frac{x_\ell}{x_k}\right|^2\right)\\ &= \log\left(\left|\frac{x_k}{x_j}\right|^2\sum_{\ell=0}^{n}\left|\frac{x_\ell}{x_k}\right|^2\right)\\ &= \log\left(\left|\frac{x_k}{x_j}\right|^2\right) + \log\left(\sum_{\ell=0}^{n}\left|\frac{x_\ell}{x_k}\right|^2\right)\end{aligned}$$

It now suffices to show that $\partial\bar{\partial}\log\left(\left|\frac{x_k}{x_j}\right|^2\right) = 0$. For convenience, write $z_k := \frac{x_k}{x_j} \in \mathbb{C}$.

In this notation,

$$\begin{aligned}\partial\bar{\partial}\log\left(\left|\frac{x_k}{x_j}\right|^2\right) &= \partial\bar{\partial}\log\left(|z_k|^2\right)\\ &= \partial\bar{\partial}\log\left(z_k\bar{z}_k\right)\\ &= \partial\left(\frac{1}{z_k\bar{z}_k}z_k\,d\bar{z}_k\right)\\ &= \partial\left(\frac{1}{\bar{z}_k}d\bar{z}_k\right) = 0\end{aligned}$$

because the $\bar{\partial}$ and ∂ acts by differentiation with respect to the $\bar{z}_k$'s and the z_k's, respectively, in local coordinates. Therefore, the ω_j's glue to the Fubini-Study form ω_{FS}.

Next, we must show that $\omega_{FS} \in \Omega^{1,1}_{\mathbb{P}^n,\mathbb{R}}(\mathbb{P}^n) \subset \Omega^2_{\mathbb{P}^n,\mathbb{C}}(\mathbb{P}^n)$, or equivalently, that $\omega_{FS} = \overline{\omega_{FS}}$ (Note the subscript $\mathbb{R}$. The Fubini-Study form is an element of the smooth, real cotangent bundle!).

For any real-valued differentiable f, we have $\overline{\partial\bar{\partial}f} = \bar{\partial}\partial\bar{f} = \bar{\partial}\partial f = -\partial\bar{\partial}f$. Since $\omega_j = \frac{i}{2\pi}\partial\bar{\partial}f$ for f differentiable and real-valued, it follows that

$$\overline{\omega_j} = -\frac{i}{2\pi}\overline{\partial\bar{\partial}f} = \frac{i}{2\pi}\partial\bar{\partial}f = \omega_j.$$

Therefore, $\omega_{FS} = \overline{\omega_{FS}}$, so it is real, as desired.

Finally, ω_{FS} is closed by Lemma 11.13. □

Proof of Theorem 11.12. We use ω_{FS} to define a Riemannian metric $g(\cdot,\cdot) := \omega_{FS}(\cdot, I(\cdot))$. We need to check that g is positive definite; that is, $g(v,v) := w_x(v, Iv) > 0$ for all $v \neq 0 \in T_{x,\mathbb{R}}$. To do this locally, write $\omega_{FS} = i\sum h_{ij}dz_i \wedge d\bar{z}_j$ and consider the matrix $H = [h_{ij}]_{ij}$. The fact that w_{FS} is a Kähler metric implies that H is Hermitian at every point of X, so we need only check that it is positive definite. This check is left as an exercise. □

Exercise 11.15. Check that g is positive definite. (Hint: Huybrechts page 118)

Hence, $\mathbb{P}^n_{\mathbb{C}}$ has a Kähler metric ω_{FS}. In particular, $d\omega_{FS} = 0$ and $\bar{\partial}w_{FS} = 0$, so we can think of ω_{FS} as a class $[\omega_{FS}] \in H^{1,1}(\mathbb{P}^n) = H^1(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(\mathbb{P}^n))$.

11.2.2. *The general case.* We now proceed to the general case of Theorem 11.11.

Proposition 11.16. *If $f : X \to Y$ is a closed immersion and $\omega \in \Omega^2_{Y,\mathbb{R}}(Y)$ is Kähler, then the pullback $f^*\omega \in \Omega^2_{X,\mathbb{R}}(X)$ is also Kähler.*

Proof. For any $x \in X$, we have $df_x : T_{x,\mathbb{R}} \to T_{f(x),\mathbb{R}}$. The pullback is defined by $(v, w) \mapsto f^*\omega(v, w) := \omega(df(v), df(w))$.

Since f is a closed immersion, $\dim Y \geq \dim X$; therefore, there are local coordinates $z_1, \ldots, z_{\dim Y}$ in a neighborhood of $f(x) \in Y$ such that $z'_i := z_i \circ f$ for $i = 1, \ldots, \dim X$ are local coordinates in a neighborhood of $x \in X$ (by the implicit function theorem). If we write ω in the basis $dz_i \wedge d\bar{z}_j$, then $f^*\omega = \hat{i} \sum_{0 \leq i,j \leq n} h_{ij} \circ f dz'_i \wedge d\bar{z}'_j$. The coefficients $[h_{ij} \circ f]_{i,j \leq n}$ form a positive definite Hermitian matrix because $[h_{ij}]_{0 \leq i,j \leq m}$ is positive definite and Hermitian; therefore, $f^*\omega$ is Kähler. □

Proof of Theorem 11.11. This follows immediately from Proposition 11.16, applied to the Fubini-Study form. □

12. Harmonic forms

Throughout this section, we will assume for simplicity that X is a compact, connected Hermitian complex manifold with $\dim_{\mathbb{R}} X = 2n$. These assumptions will not all be needed all the time, but they will make the exposition simpler. Recall from smooth manifold theory that a **volume form** is a nonzero global section of $\Omega^{2n}_{X,\mathbb{R}}$.

Definition 12.1. $\mathrm{Vol} := \frac{\omega^n}{n!}$, where $\omega \in \Omega^2_{X,\mathbb{R}}(X)$ is the fundamental form of X.

Exercise 12.2. Explain why we multiplied by $\frac{1}{n!}$.

Recall that locally, $\omega = \frac{i}{2} \sum h_{ij} dz_i \wedge d\bar{z}_j$, where $[h_{ij}]$ form a Hermetian matrix, which defines a Hermitian inner product $H(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}) = h_{ij}$ on each holomorphic tangent space $\mathcal{T}_{X,x}$. Now, let $e_1, \ldots, e_n$ be an orthonormal basis for $\mathcal{T}_{X,x}$. Then $e_1, ie_1, e_2, ie_2, \ldots, e_n, ie_n$ gives a basis over $\mathbb{R}$ for $\mathcal{T}_{X,x}$.

Proposition 12.3. $\mathrm{Vol}(e_1 \wedge ie_1 \wedge e_2 \wedge ie_2 \wedge \cdots \wedge e_n \wedge ie_n) = 1$

Proof. Exercise. □

Volume forms are useful because we can integrate against them. Recall from our section on linear algebra that we can extend the Riemannian metric g to be a metric on $\Omega^k_{X,\mathbb{R}}$, defined on $\{e^\vee_{i_1} \wedge \cdots \wedge e^\vee_{i_k}\}$, where $e_1, \ldots, e_{2n}$ is an orthornormal basis for $T_{X,\mathbb{R}}$. We denote the metric on $\Omega^k_{X,\mathbb{R}}$ by g as well.

Let $\alpha, \beta \in \Omega^k_{X,\mathbb{R}}(X)$ be smooth k-forms.

Definition 12.4.

$$(\alpha, \beta)_{L^2} := \int_X g(\alpha, \beta) \,\mathrm{Vol}$$

regarding $g(\alpha, \beta)$ as a smooth function on X given by $x \mapsto g_x(\alpha(x), \beta(x))$.

This definition makes sense because we assumed that X is compact. Also observe that $(\alpha, \alpha)_{L^2} \geq 0$ and that $(\alpha, \alpha) = 0 \iff \alpha = 0$.

12.1. **More operators.** The objects we have defined give us three isomorphisms.

- The metric g gives an isomorphism $m : \Omega^k_{X,\mathbb{R}} \xrightarrow{\cong} \mathrm{Hom}(\Omega^k_{X,\mathbb{R}}, \mathbb{R})$ by $\alpha \mapsto g(\alpha, \cdot)$.
- The volume form defines an isomorphism $\mathbb{R} \to \Omega^{2n}_{X,\mathbb{R}}$ given by $1 \mapsto \mathrm{Vol}$.
- Recall that form a vector space, $\bigwedge^k V \cong \left(\bigwedge^{\dim V - k} V\right)^\vee$ and that for a vector bundle, $\bigwedge^k \mathcal{E} \cong \left(\bigwedge^{\mathrm{rank}(\mathcal{E})-k} \mathcal{E}\right)^\vee \otimes \det \mathcal{E}$. Hence, there is an isomorphism $p : \Omega^{2n-k} \to \mathrm{Hom}(\Omega^k_{X,\mathbb{R}}, \Omega^{2n}_{X,\mathbb{R}})$ given by $\alpha \mapsto \cdot \wedge \alpha$.

Putting all these together, we get the Hodge $*$ operator

Definition 12.5. Let

$$* := p^{-1} \circ m : \Omega^k_{X,\mathbb{R}} \xrightarrow{\cong} \Omega^{2n-k}_{X,\mathbb{R}}.$$

This map yields the **Hodge $*$ operator**

$$* : \Omega^k_{X,\mathbb{R}}(X) \xrightarrow{\cong} \Omega^{2n-k}_{X,\mathbb{R}}(X).$$

Exercise 12.6. For all $\alpha, \beta \in \Omega^k_{X,\mathbb{R}}(X)$,

$$\alpha \wedge *\beta = g(\alpha, \beta)\, \mathrm{Vol}\,.$$

In particular, $(\alpha, \beta)_{L^2} = \int g(\alpha, \beta)\, \mathrm{Vol} = \int \alpha \wedge *\beta$.

We now define some adjoints with respect to the L^2 metric. Recall the differential $d : \Omega^k_{X,\mathbb{R}}(X) \to \Omega^{k+1}_{X,\mathbb{R}}(X)$. Define its adjoint $d^* : \Omega^k_{X,\mathbb{R}}(X) \to \Omega^{k-1}_{X,\mathbb{R}}(X)$ by $d^* := (-1)^k *^{-1} d*$.

Proposition 12.7. *If $\alpha \in \Omega^k_{X,\mathbb{R}}$, then $(\alpha, d^*\beta)_{L^2} = (d\alpha, \beta)_{L^2}$.*

Proof. By the Leibniz rule, $d(\alpha \wedge *\beta) = d\alpha \wedge *\beta + (-1)^k \alpha \wedge d(*\beta)$. Since $d(\alpha \wedge *\beta)$ is closed, this implies that the right-hand side is

$$(d\alpha, \beta)_{L^2} = \int_X d\alpha \wedge *\beta = (-1)^{k+1} \int_X \alpha \wedge d(*\beta) = \int_X \alpha \wedge *d^*\beta = (\alpha, d^*\beta)_{L^2}.$$

□

Definition 12.8. The **Laplacian** $\Delta_d : \Omega^k_{X,\mathbb{R}}(X) \to \Omega^k_{X,\mathbb{R}}(X)$ is given by $\Delta_d := dd^* + d^*d$.

The name of the Laplacian is justified by the fact that when $k = 0$, $\Delta_d(f) = -\sum \frac{\partial^2 f}{\partial x_i^2}$.

Proposition 12.9. *Δ_d is self adjoint with respect to the L^2-metric*

Proof.

$$\begin{aligned}(\alpha, dd^*\beta + d^*d\beta) &= (\alpha, dd^*\beta) + (\alpha, d^*d\beta) \\ &= (d^*\alpha, d^*\beta) + (d\alpha, d\beta) \\ &= (dd^*\alpha, \beta) + (d^*d\alpha, \beta) \\ &= (\Delta_d \alpha, \beta)\end{aligned}$$

□

For later convenience, we also record the following

Lemma 12.10. $(\alpha, \Delta_d(\beta))_{L^2} = (d^*\alpha, d^*\beta) + (d\alpha, d\beta)$

Definition 12.11. Call α a **harmonic** k-form if $\Delta_d(\alpha) = 0$. For the set of all harmonic k-forms, write

$$\mathcal{H}^k_{\Delta_d} := \{\alpha \in \Omega^k_{X,\mathbb{R}}(X) : \Delta_d(\alpha) = 0\}$$

Proposition 12.12. *α is harmonic if and only if $d\alpha = d^*\alpha = 0$.*

Proof. The implication $\Leftarrow$ is immediate. For the other direction, suppose $\Delta_d(\alpha) = 0$. Then

$$0 = (\alpha, \Delta_d(\alpha)) = (d\alpha, d\alpha) + (d^*\alpha, d^*\alpha)$$

so the positive definiteness of $(\cdot,\cdot)_{L^2}$ implies that $d\alpha = 0$ and $d^*\alpha = 0$. □

12.2. **Harmonic forms and de Rham cohomology.** Proposition 12.12 implies that harmonic forms are all closed, which gives a map $\mathcal{H}^k_{\Delta_d}(X) \to H^k_{dR}(X)$ with $\alpha \mapsto [\alpha]$.

Proposition 12.13. *The natural map $\mathcal{H}^k_{\Delta_d}(X) \to H^k_{dR}(X)$ is injective*

Proof. Suppose that α is harmonic and $\alpha = d\beta$. Then

$$(\alpha, \alpha)_{L^2} = (\alpha, d\beta)_{L^2} = (d^*\alpha, \beta)_{L^2} = (0, \beta)_{L^2} 0$$

by Proposition 12.12. □

Showing that the natural map is surjective requires the following hard analysis result (elliptic differential operators).

Theorem 12.14 (Hodge decomposition)**.** $\Omega^k_{X,\mathbb{R}}(X) = \mathcal{H}^k_{\Delta_d}(X) \oplus \Delta_d(\Omega^k_{X,\mathbb{R}}(X))$

Corollary 12.15. $\mathcal{H}^k_{\Delta_d}(X) \xrightarrow{\cong} H^k_{dR}(X)$

Proof. It remains to show that $H^k_{\Delta_d}(X) \hookrightarrow H^k_{dR}(X)$ is onto. Let $\alpha \in \Omega^k_{X,\mathbb{R}}(X)$ be closed. By Theorem 12.14, $\alpha = \beta + \Delta_d(\gamma)$ with β harmonic. Expanding this expression, $\alpha = \beta + dd^*(\gamma) + d^*d(\gamma)$. Since α, β, and $dd^*(\gamma)$ are both closed, it follows that $d^*d\gamma$ is closed as well.

Therefore,

$$(d^*d\gamma, d^*d\gamma)_{L^2} = (d\gamma, dd^*d\gamma)_{L^2} = 0,$$

so $\alpha = \beta + dd^*\gamma$ and $[\alpha] = [\beta]$. □

12.3. **Duality theorems.** We will deduce several duality results using the Hodge $*$ operator $* : \Omega^k_{X,\mathbb{R}}(X) \to \Omega^{2n-k}_{X,\mathbb{R}}(X)$. On k-forms, the Hodge $*$ is given by $\cdot \wedge *\alpha = g(\cdot, \alpha)\,\mathrm{Vol}$.

Exercise 12.16. Let's look at $*_k$ on the fibre $\Omega^k_{x,\mathbb{R}}$. Choose a basis $e_1, \ldots, e_{2n}$ of $\Omega_{x,\mathbb{R}}$ such that $\mathrm{Vol} = e_1 \wedge \cdots \wedge e_{2n}$. Show that if $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_{2n-k}$ so that $\{i_1, \ldots, i_k, j_1, \ldots, j_{2n-k}\} = \{1, \ldots, 2n\}$ then $*(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \mathrm{sgn}(\sigma)(e_{j_1} \wedge \cdots \wedge e_{j_k})$ where σ is the permutation whose 1-line notation is $i_1, \ldots, i_k j_1, \ldots, j_{2n-k}$.

Proposition 12.17. $*^2 = (-1)^{k(2n-k)} : \Omega^k_{X,\mathbb{R}}(X) \to \Omega^k_{X,\mathbb{R}}(X)$

Proof. By Exercise 12.16, $*^2(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \mathrm{sgn}(\sigma') e_{i_1} \wedge \cdots e_{i_k}$, where σ' is the permutation that takes $i_1, i_2, \ldots, i_k, j_1, \ldots, j_{2n-k}$ to $j_1, \ldots, j_{2n-k}, i_1, \ldots, i_k$. The sign of this permutation is $(-1)^{k(2n-k)}$. □

Proposition 12.18. *For all* $\alpha \in \Omega^k_{X,\mathbb{R}}(X)$, $\Delta_d * (\alpha) = *\Delta_d \alpha$.

Proof. Exercise (Hint: use $\Delta_d = dd^* + d^*d$; the hard part is the signs). □

Proposition 12.19. $d^* = (-1)^k *^{-1} d*$ *and* $d^* = - * d*$

Proof. This follows from the fact that $k - (k+1)^2$ is odd. □

Corollary 12.20. *If* α *is Harmonic, then* $*\alpha$ *is Harmonic.*

Corollary 12.21. $*$ *induces an isomorphism* $\mathcal{H}^k_{\Delta_d}(X) \xrightarrow{\cong} \mathcal{H}^{2n-k}_{\Delta_d}(X)$.

Corollary 12.22 (Poincaré duality)**.** *The pairing* $H^k_{\Delta_d}(X) \times H^{2n-k}_{\Delta_d}(X) \to \mathbb{R}$ *given by* $(\alpha, \beta) \mapsto \int_X \alpha \wedge \beta$ *is non-degenerate.*

Proof. For each $\alpha \in \mathcal{H}^k_{\Delta_d}(X)$, we need a Harmonic $2n-k$ form β such that $\int \alpha \wedge \beta \neq 0$. Set $\beta = *\alpha$. Now, $\int \alpha \wedge *\alpha = \int g(\alpha, \alpha) \operatorname{Vol} = (\alpha, \alpha)_{L^2} \neq 0$. □

13. The complex case

We would like to refine the duality theorems above to work on Dolbeault cohomology.

Recall the complexified tangent bundle $T_{X,\mathbb{C}} = T^{1,0} \oplus T^{0,1}$, and the decomposition $\Omega_{X,\mathbb{C}} = \Omega^{1,0} \oplus \Omega^{0,1}$. If g is an inner product on $T_{X,\mathbb{R}}$, then we extend this to a Hermetian inner product h on $T_{X,\mathbb{C}}$ by the formula $h(v, u) := g(v, \bar{u})$, where g is extended to be $\mathbb{C}$-linear.

Lemma 13.1. *The decomposition* $T_{X,\mathbb{C}} = T^{1,0} \oplus T^{0,1}$ *is orthogonal with respect to* $h(u, v)$.

Proof. Consider the fibre $T_{x,\mathbb{C}} = T^{1,0}_x \oplus T^{0,1}_x$. If $v \in T_{x,\mathbb{C}}$, then its components are

$$\frac{1}{2}(v - iI(v)) \in T^{1,0}_x$$
$$\frac{1}{2}(v + iI(v)) \in T^{0,1}_x$$

Now, for $v, w \in T_{x,\mathbb{C}}$,

$$\begin{aligned} h(v - iIv, w + iIw) &= g(v - iIv, \bar{w} - iI\bar{w}) \\ &= g(v, \bar{w}) - ig(v, I\bar{w}) - ig(Iv, \bar{w}) - g(Iv, I\bar{w}) = 0 \end{aligned}$$

because g is I-invariant, so the terms above cancel. □

Extending h to $\Omega^k_{X,\mathbb{C}}(X)$, we find that the decomposition $\Omega^k_{X,\mathbb{C}}(X) = \bigoplus_{a+b=k} \Omega^{a,b}(X)$ is also an orthogonal decomposition with respect to h. This naturally gives us a norm

Definition 13.2. For $\alpha, \beta \in \Omega^{p,q}(X)$, the L^2**-norm** is

$$(\alpha, \beta)_{L^2} := \int_X h(\alpha, \beta) \operatorname{Vol} \in \mathbb{C}$$

where $\operatorname{Vol} = \frac{\omega^n}{n!}$.

13.1. **Kähler identities.** Let X be a complex Hermitian manifold of dimension $2n$. Let g be a Riemannian metric on $\Omega_{X,\mathbb{R}}$ and h a Hermitian metric on $\Omega^k_{X,\mathbb{C}} \cong \Omega_{X,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Expanding all our definitions, recall that

$$h(\alpha, \beta)\,\mathrm{Vol} = g(\alpha, \bar{\beta})\,\mathrm{Vol} = \alpha \wedge *\bar{\beta}$$

and that $\alpha \in \Omega^{p,q}_X(X)$ and $\beta \in \Omega^{p',q'}_X(X)$, then $h(\alpha, \beta) = 0$ unless $p = p'$ and $q = q'$ by Lemma 13.1.

Definition 13.3. Let $\bar{\partial} := - * \partial *$ and $\partial^* := - * \bar{\partial} *$.

Proposition 13.4. *∂^* is adjoint to ∂ and $\bar{\partial}^*$ is adjoint to $\bar{\partial}$ with respect to the complexified L^2 norm.*

The following exercises will be useful for the proof.

Exercise 13.5. If $\beta \in \Omega^{p,q}_X(X)$, then $*\beta \in \Omega^{n-q,n-p}_X(X)$.

Exercise 13.6. $*\bar{\beta} = \overline{*\beta}$.

Proof of Proposition 13.4. This proof will follow a plan roughly the same as the proof of Proposition 12.7. Let $\alpha \in \Omega^{p-1,q}_X(X)$ and $\beta \in \Omega^{p,q}(X)$. We need to show that $(\partial\alpha, \beta) = (\alpha, \partial^*\beta)$.

Expanding the left-hand side, we get

$$(\partial\alpha, \beta)_{L^2} = \int_X h(\partial\alpha, \beta)\,\mathrm{Vol} = \int_X g(\partial\alpha, \bar{\beta})\,\mathrm{Vol} = \int_X \partial\alpha \wedge \overline{*\beta}$$

By Exercises 13.5 and 13.6,

$$\alpha \wedge \overline{*\beta} \in \Omega^{n-1,n}(X) \implies \bar{\partial}(\alpha \wedge \overline{*\beta}) \in \Omega^{n-1,n+1}(X) = 0$$

because $\bigwedge^{n+1} \Omega^{0,1}(X) = 0$. Hence, $d(\alpha \wedge \overline{*\beta}) = \partial(\alpha \wedge \overline{*\beta})$, which implies that $\int_X \partial(\alpha \wedge \overline{*\beta}) = 0$ by Stokes' theorem. One can now complete this proof by using the Leibniz rule as in the proof of Proposition 12.7. □

As before, we now define our Laplacian operators $\Delta_\partial = \partial\partial^* + \partial^*\partial : \Omega^{p,q}(X) \to \Omega^{p,q}(X)$ and $\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^* + \bar{\partial}$. We also define our other operators $L := \omega \wedge - : \Omega^{p,q}(X)|to\Omega^{p+1,q+1}(X)$, Λ the adjoint of L.

Theorem 13.7. *If X is Kähler (i.e. $d\omega = 0$), then we have the following identities.*

(i) $[\bar{\partial}, L] = [\partial, L] = 0$ *and* $[\bar{\partial}^*, \Lambda] = [\partial^*, \Lambda] = 0$.
(ii) $[\bar{\partial}^*, L] = i\partial$, $[\partial^*, L] = -i\bar{\partial}$, $[\Lambda, \bar{\partial}] = -i\partial^*$, *and* $[\Lambda, \partial] = i\bar{\partial}^*$

Lemma 13.8. $\Lambda = *^{-1} L *$

Proof. Let $\alpha \in \Omega^k_{X,\mathbb{C}}(X)$, $\beta \in \Omega^{k-2}_{X,\mathbb{C}}(X)$. Then

$$\begin{aligned}
g(\alpha, L\beta)\,\mathrm{Vol} &= g(L\beta, \alpha)\,\mathrm{Vol} \\
&= L\beta \wedge *\alpha \\
&= \omega \wedge \beta \wedge *\alpha \\
&= \beta \wedge \omega \wedge *\alpha \quad \text{because } \omega \text{ is a 2-form} \\
&= \beta \wedge L(*\alpha) \\
&= g(\beta, *^{-1} L * \alpha)\,\mathrm{Vol} \\
&= \beta \wedge L(*\alpha) = g(\beta, *^{-1} L * \alpha)\,\mathrm{Vol}
\end{aligned}$$

□

Proof of Theorem 13.7.

(i) We show that $[\bar\partial^*, \Lambda] = [\partial^*, \Lambda] = 0$. If $\alpha \in \Omega^{p,q}(X)$, then

$$\begin{aligned}[\bar\partial^*, \Lambda](\alpha) &= \bar\partial^*(*^{-1}L * \alpha) - *^{-1}L * (\bar\partial^* \alpha)\\ &= - * \partial * *^{-1} L * \alpha + *^{-1} L * * \partial * \alpha \\ &= - * \partial L(*\alpha) + (-1)^k *^{-1} L\partial(*\alpha) \quad = - * [\partial, L](*\alpha) = 0 \quad \text{because } [\partial, L] = 0\end{aligned}$$

(ii) We won't prove this. It uses the Lefschetz decomposition $\Omega^k_X = \bigoplus L^i(P^{k-2i})$ where $P^j = \ker(\Lambda : \Omega^j_{X,\mathbb{C}} \to \Omega^{j-2}_{X,\mathbb{C}})$. For the remainder, see page 120 of Huybrechts.

□

These relations can be used to prove the following, which allows us to refine our knowledge of Harmonic forms to work on (p, q)-forms.

Theorem 13.9. $\Delta_\partial = \Delta_{\bar\partial} = \frac{1}{2}\Delta_d$

Lemma 13.10. $\partial\bar\partial^* + \bar\partial^*\partial = 0$

Proof.

$$i(\partial\bar\partial^* + \bar\partial^*\partial) = \partial[\Lambda, \partial] + [\Lambda, \partial]\partial = \partial\Lambda\partial - \partial^2\Lambda + \Lambda\partial^2 - \partial\Lambda\partial = 0$$

□

Proof of Theorem 13.9.

$$\begin{aligned}\Delta_\partial &= \partial\partial^* + \partial^*\partial = i\partial[\Lambda, \bar\partial] + i[\Lambda, \bar\partial]\\ &= i\partial(\Lambda\bar\partial - \bar\partial\Lambda) + i(\Lambda\bar\partial - \bar\partial\Lambda)\partial\\ &= i(\partial\Lambda\bar\partial - \partial\bar\partial\Lambda + \Lambda\bar\partial\partial - \bar\partial\Lambda\partial)\\ &= i(([\partial, \Lambda] + \Lambda\partial)\bar\partial - \partial\bar\partial\Lambda + \Lambda\bar\partial\partial - \bar\partial([\Lambda, \partial] + \partial\Lambda)\\ &= i(-i\bar\partial^* + \Lambda\partial)\bar\partial - \partial\bar\partial\Lambda + \Lambda\bar\partial\partial - \bar\partial(i\bar\partial^* + \partial\Lambda)\end{aligned}$$

Now, the identities $\partial^2 = \bar\partial^2 = \partial\bar\partial + \bar\partial\partial = 0$ imply that the above is equal to $\bar\partial^*\bar\partial + \bar{\bar\partial}\partial^* = \Delta_{\bar\partial}$.

Next, we show that $\frac{1}{2}\Delta_d = \Delta_\partial$.

$$\begin{aligned}\Delta_d &= dd^* + d^*d\\ &= (\partial + \bar\partial)(\partial^* + \bar\partial^*) + (\partial^* + \bar\partial^*)(\partial + \bar\partial)\\ &= \Delta_\partial + \Delta_{\bar\partial} + \text{cross-terms}\\ &= \Delta_\partial + \Delta_{\bar\partial} + (\partial\bar\partial^* + \bar\partial\partial^* + \partial^*\bar\partial + \bar\partial^*\partial)\\ &= \Delta_\partial + \Delta_{\bar\partial} + \overline{\partial\bar\partial^* + \bar\partial^*\partial} = \Delta_\partial + \Delta_{\bar\partial}\end{aligned}$$

where the final equality is by Lemma 13.10. □

Theorem 13.9 has a number of important consequences. Among them is the following lemma, which allows us to decompose Harmonic forms into harmonic p, q-forms.

Lemma 13.11. *Suppose X is Kähler and $\alpha \in \Omega^k_{X,\mathbb{C}}(X)$ such that $\alpha = \sum_{p+q=k} \alpha^{p,q}$ is a decomposition as (p, q)-forms. If α is harmonic, then so is $\alpha^{p,q}$ for all p, q.*

Proof. $\Delta_d(\alpha) = 0$ by assumption. This implies that $\Delta_\partial(\alpha) = 0$ by Theorem 13.9, so

$$\sum_{p+q=k} \Delta_\partial(\alpha^{p,q}) = 0 \implies \Delta_\partial(\alpha^{p,q}) = 0$$

for all p, q because the decomposition of $\Omega^k_{X,\mathbb{C}}(X)$ into p, q-forms is a direct sum. □

Definition 13.12.

$$\mathcal{H}^{p,q}_{\Delta_{\bar\partial}}(X) := \{\alpha \in \Omega^{p,q}(X) : \Delta_{\bar\partial}(\alpha) = 0\} \tag{3}$$

$$\mathcal{H}^{p,q}_{\Delta_\partial}(X) := \{\alpha \in \Omega^{p,q}(X) : \Delta_\partial(\alpha) = 0\} \tag{4}$$

If X is Kähler, then $\mathcal{H}^{p,q}_{\Delta_{\bar\partial}}(X) = \mathcal{H}^{p,q}_{\Delta_\partial}(X)$.

Corollary 13.13. *If X is compact and Kähler, then*

$$\mathcal{H}^k_{\Delta_d}(X, \mathbb{C}) = \bigoplus_{p+q=kp,q\geq 0} \mathcal{H}^{p,q}_\Delta(X)$$

where $\mathcal{H}^{p,q}_\Delta(X) = \{\alpha \in \Omega^{p,q}_{X,\mathbb{C}}(X) : \Delta_d(\alpha) = 0\} = \{\alpha \in \Omega^{p,q}_{X,\mathbb{C}}(X) : \Delta_\partial(\alpha) = \Delta_{\bar\partial}(\alpha) = 0\}$.

By Corollary 12.15, $\mathcal{H}^k_{\Delta_d}(X, \mathbb{C}) = H^k_{dR}(X, \mathbb{C}) \cong \check{H}^k(X, \underline{\mathbb{C}})$. We can understand the right-hand side of the statement above in similar terms. There is a linear map $\mathcal{H}^{p,q}_\Delta(X) \to H^{p,q}(X) := \ker(\bar\partial : \Omega^{p,q} \to \Omega^{p,q+1}) = \mathrm{im}(\bar\partial)$. Showing that this is injective is not too bad. By applying hard theorems from elliptic operator theory, one obtains the following result

Theorem 13.14. *$\mathcal{H}^{p,q}_\Delta(X) \to H^{p,q}(X)$ is an isomorphism.*

Hence, $H^{p,q}(X) \cong \check{H}^q(X, \Omega^p_X)$, where Ω_X is the holomorphic cotangent bundle (which by GAGA corresponds to the algebraic cotangent bundle). Putting all these isomorphisms together, we obtain

Corollary 13.15. *If X is compact and Kähler, then*

$$\check{H}^k(X, \mathbb{C}) \cong H^k(X, \mathbb{C}) \cong \bigoplus_{\substack{p+q=k \\ p,q\geq 0}} H^{p,q}(X) \cong \bigoplus_{\substack{p+q=k \\ p,q\geq 0}} H^p(\Omega^q_X).$$

Recall that complex conjugation gives and isomorphism $\overline{\Omega^{p,q}} \cong \Omega^{q,p}$. If $\alpha \in \Omega^{p,q}(X)$ is harmonic, then

$$(\Delta_d(\alpha) = 0 \implies \Delta_d(\bar\alpha) = 0) \implies \overline{H^{p,q}(X)} \cong \overline{\mathcal{H}^{p,q}_\Delta(X)} \cong \mathcal{H}^{q,p}_\Delta(X) \cong H^{q,p}(X)$$

This is important enough, that we dignify it with the title of Theorem

Theorem 13.16. $\overline{H^{p,q}(X)} \cong H^{q,p}(X)$

We have now established all the Hodge theory that we will need and we can move on to applications!

Part 2. The Torelli Theorems

Our first goal in this new age will be to establish the Torelli theorem for curves. Let C be a projecive complex variety of dimension 1. Recall from Exercise 8.17 that $\mathrm{Pic}(C) \cong H^1(\mathcal{O}_X^*)$ and that the boundary map

$$H^1(\mathcal{O}_X^*) \stackrel{\deg}{\to} H^2(C, \underline{\mathbb{Z}}) \cong \mathbb{Z}$$

associates to each line bundles L an integer, which we will denote by $\deg L$.

Since $\dim C = 1$, $\check{H}^2(C, \mathcal{O}_C) = 0$, so deg is surjective. We now study its kernel.

14. Jacobians

Definition 14.1. The **Jacobian** of C is

$$\mathrm{Jac}^0(C) := \ker \deg = \{L : \deg L = 0\}.$$

We will soon see that the Jacobian is not only an abelian group, but also a compact complex manifold. The Torelli theorem says roughly that we can completely recover C from $\mathrm{Jac}^0(C)$ and a little bit of extra data.

Consider the long exact sequence arising from the exponential sequence

$$\cdots \to \check{H}^1(C, \underline{Z}) \to \check{H}^1(C, \mathcal{O}_C) \to \check{H}^1(C, \mathcal{O}_C^*) \to \underbrace{\check{H}^2(C, \underline{\mathbb{Z}})}_{\cong \mathbb{Z}} \to 0.$$

There is a short exact subsequence

$$0 \to \check{H}^1(C, \underline{Z}) \to \check{H}^1(C, \mathcal{O}_C) \to \mathrm{Jac}^0(C) \to 0$$

For exactness on the left, note that the exponential map $\mathbb{C} \cong H^0(\mathcal{O}_C) \to H^0(\mathcal{O}_C^*)\mathbb{C}^*$ is surjective. Hence,

$$\mathrm{Jac}^0(C) \cong \check{H}^1(C, \mathcal{O}_C)/i(\check{H}^1(C, \underline{\mathbb{Z}})) \cong H^{0,1}(C)/\check{H}^1(C, \underline{\mathbb{Z}}).$$

From Hodge theory, we have $H^{0,1}(C) \cong H^{1,0}(C) \cong \check{H}^0(C, \omega_C) \cong \mathbb{C}^g$, and we know that $H^1(C, \underline{\mathbb{Z}}) \cong H_1(C, \mathbb{Z}) \cong \mathbb{Z}^{2g}$. Hence, $\mathrm{Jac}^0(C) \cong \mathbb{C}^g/\mathbb{Z}^{2g}$, a complex torus.

Definition 14.2. A **torus** is a quotient $\mathbb{C}^n/\Lambda$, where Λ is a free abelian group spanning $\mathbb{C}^n$ over $\mathbb{R}$.

A complex torus is always compact and has a natural complex structure that makes the surjection $\mathbb{C}^n \to \mathbb{C}^n/\Lambda$ holomorphic. A complex torus is also a group, so $\mathrm{Jac}^0(C)$ is an abelian variety.

Remark 14.3. We haven't shown that $\check{H}^1(C, \underline{\mathbb{Z}})$ is full rank in $H^{0,1}(C)$. We'll talk about it more in the future.

14.1. Getting a better handle on the Jacobian.

To get a better handle on the Jacobian $H^{0,1}(C)/\check{H}^1(C, \underline{\mathbb{Z}})$, we'll use **Serre duality**. Let X be a compact Kähler manifold of dimension $2n$. Define a pairing $H^{p,q}(X) \times H^{n-p,n-q}(X) \to \mathbb{C}$ by $(\alpha, \beta) \mapsto \int_X \alpha \wedge \beta$. This pairing is positive definite: given $\alpha \in H^{p,q}(X)$, $*\alpha \in H^{n-q,n-p}(X)$, so $\overline{*\alpha} H^{n-p,n-q}(X)$ and

$$(\alpha, \overline{*\alpha}) = \int \alpha \wedge \overline{*\alpha} = \int h(\alpha, \alpha) \,\mathrm{Vol} > 0.$$

Hence, we have the Serre duality isomorphism $H^{p,q}(X) \xrightarrow{\cong} (H^{n-p,n-q}(X))^\vee$ by $\alpha \mapsto (\alpha, \cdot)$.

Applying this to $H^{0,1}(C)$, we obtain an isomorphism $H^{0,1}(C) \cong H^{1,0}(C)^\vee = (H^0(\omega_C))^\vee$, and from this, it follows that

$$\mathrm{Jac}^0(C) \cong H^{0,1}(C)/\check{H}^1(C,\underline{\mathbb{Z}}) \cong H^{1,0}(C)^\vee/H_1(C,\mathbb{Z}).$$

We can understand the right-hand side explicitly in terms of forms and singular homology. Consider an element $[\alpha] \in H^{1,0}(X)$ with α a $(1,0)$-form. Locally, we can write $\alpha = f\,dz = f\,dx + if\,dy$ where f is a smooth complex-valued function. Since α is closed, it follows from Stokes' theorem that for any loop γ representing a class $[\gamma] \in H_1(C,\mathbb{Z})$, we obtain a well-defined complex number $\int_\gamma \alpha \in \mathbb{C}$. This yields a nice description

$$\mathrm{Jac}^0(C) \cong \mathrm{coker}\left(H_1(C,\mathbb{Z}) \longrightarrow H^{1,0}(C)^\vee\right)$$
$$[\gamma] \longmapsto \int_\gamma \cdot$$

14.2. **A bilinear pairing.** Besides the Jacobian, we will need a little bit of extra data to determine C. Recall $\check{H}^1(C,\underline{\mathbb{C}}) \cong H^{1,0}(C) \oplus H^{0,1}(C)$. We have a bilinear pairing $Q : \check{H}^1(C,\underline{\mathbb{C}}) \times \check{H}^1(C,\underline{\mathbb{C}}) \to \mathbb{C}$ given by $Q(\alpha,\beta) = \int \alpha \wedge \beta$. This pairing has a number of properties:

Proposition 14.4.

(i) *If* $\alpha,\beta \in H^{1,0}(C)$ *or* $\alpha,\beta \in H^{0,1}(C)$*, then* $\int \alpha\wedge\beta = 0$*.*
(ii) *If* $\alpha \neq 0 \in H^{1,0}(C)$*, then* $iQ(\alpha,\bar{\alpha}) > 0$ *(and is real!)*

Proof.

(i) This follows immediatley because $H^{2,0}(C) = H^{0,2}(C) = 0$.
(ii) To understand Q, let's work out what the $*$ operator does. In this context, you can work out from Propositions that $* : dx \mapsto dy$ and $* : dy \mapsto -dx$. Hence, in the basis $dz = dx + idy$, $* : dz \mapsto -i\,dz$. Now,

$$iQ(\alpha,\bar{\alpha}) = \int \alpha \wedge (i\overline{\alpha}) = \int \alpha \wedge \overline{(-i\alpha)} = \int \alpha \wedge \overline{*\alpha} > 0$$

□

15. Hodge theory on abelian varieties

Let C be a curve of genus g.

Definition 15.1. The **Hodge metric** of C is the positive definite Hermitian metric on $H^{1,0}(C)$ defined by

$$h(x,y) := iQ(x,\bar{y}) = i\int x \wedge \bar{y}$$

We can now precisely state our goal.

Theorem 15.2. *C is completely determined by the data of:*

(i) *The g-dimensional complex vector space $H^{1,0}(C)$*
(ii) *The rank $2g$ lattice $\Lambda = H_1(C,\mathbb{Z})$ embedded with full rank in $H^{1,0}(C)^\vee$*
(iii) *The Hodge metric h*

To proceed, we need to describe the Hodge theory of the g-dimensional torus.

15.1. **Hodge theory of complex tori.** Let $X = \mathbb{C}^g/\Lambda$ with Λ a rank $2g$ lattice spanning $\mathbb{C}^g$ over $\mathbb{R}$. We will work out the Hodge theory for the torus X. For each $x \in X$, we have an isomorphism $\tau_x : X \to X$ given by $v \mapsto x + v$, which induces an isomorphism $d\tau_x : T_{X,0} \to T_{X,x}$. It follows that the tangent bundle of X is trivial, since the group action gives us a nonvanishing global section.

15.1.1. *Tori are Kähler.* To see that X is Kähler, take local coordinates $x_1, \dots, x_g$ on X, which yield 1-forms $dx_1, \dots, dx_g$ (more geometrically, $d(x+a) = dx$ because $da = 0$). Now take $\omega = \sum dx_i \wedge d\bar{x}_i$ to be our Kähler form. This is obviously Hermitian and positive definite, since its matrix is the identity.

15.1.2. *Cohomology of tori.* In sheaf notation, the statement that the tangent bundle of X is trivial is that $\mathcal{O}_X \otimes \underline{T_{x,0}} \xrightarrow{\cong} \mathcal{T}_X$. Dualizing, we obtain $\Omega_X \xrightarrow{\cong} \mathcal{O}_X \otimes \underline{\Omega_{X,0}}$. Hence,

$$H^{p,q}(X) = H^q(\Omega_X^p) \cong H^q(\textstyle\bigwedge^p \underline{\Omega_{X,0}} \otimes_{\mathbb{C}} \mathcal{O}_X) \cong \textstyle\bigwedge^p \Omega_{X,0} \otimes H^q(\mathcal{O}_X). \tag{5}$$

Since X is Kähler, we now obtain

$$H^q(\mathcal{O}_X) \cong H^{0,q}(X) \cong \overline{H^{q,0}(X)} \cong \bigwedge^q \overline{\Omega_{X,0}},$$

and substituting this into Equation 5, we obtain

Proposition 15.3. $H^{p,q}(X) \cong \bigwedge^p \Omega_{X,0} \otimes \bigwedge^q \overline{\Omega_{X,0}}$

and

Corollary 15.4. $H^p(X, \mathbb{C}) = \bigoplus_{a+b=p} \bigwedge^a \Omega_{X,0} \otimes \bigwedge^b \bar{\Omega}_{X,0} \cong \bigwedge^p(\Omega_{X,0} \otimes \overline{\Omega_{X,0}})$

Since $H^1(X, \mathbb{C}) \cong \Omega_{X,0} \otimes \overline{\Omega_{X,0}}$, we have found a new proof that the cohomology ring of a torus is the exterior algebra.

15.1.3. *Jacobians.* If $X = \mathrm{Jac}^0(C)$, then $T_{\mathrm{Jac}^0(C),0} = H^{1,0}(C)^\vee$, so $\Omega_{\mathrm{Jac}^0(C),0} = H^{1,0}(C)$ and $\bar{\Omega}_{\mathrm{Jac}^0(C),0} \cong H^{0,1}(C)$. The following results are direct consequences of this observation and the preceding discussion.

Proposition 15.5. $H^{p,q}(\mathrm{Jac}^0(C)) \cong \bigwedge^p H^{1,0}(C) \otimes \bigwedge^q H^{0,1}(C)$

Corollary 15.6. $H^p(\mathrm{Jac}^0(C), \mathbb{C}) \cong \bigwedge^p H^1(C, \mathbb{C})$

We have a bilinear form Q on $H^1(C, \mathbb{C})$, so we can identify $H^1(C, \mathbb{C}) \cong H^1(\mathrm{Jac}^0(C), \mathbb{C})$ with its dual. Hence, we can identify $H^2(\mathrm{Jac}^0(C), \mathbb{C})$ with alternating 2-forms $H^1(C, \mathbb{C}) \times H^1(C, \mathbb{C}) \to \mathbb{C}$.

Now, regard Q as an element of $H^2(\mathrm{Jac}^0(C), \mathbb{C})$. By Proposition 14.4, $Q \in H^{1,1}(\mathrm{Jac}^0(C))$.

Let q be the restriction of Q to $H^1(C, \mathbb{Z}) \subset H^1(C, \mathbb{C})$.

Proposition 15.7. *q is integer-valued; that is,*

$$q \in \mathrm{Alt}_2(H^1(C, \mathbb{Z})^\vee, \mathbb{Z}) \cong \mathrm{Alt}_2(H^1(C, \mathbb{Z}), \mathbb{Z}) \cong H^2(C, \mathbb{Z})$$

Proof. Exercise. The second isomorphism comes from identifying $H^1(C, \mathbb{Z})$ with its dual via q. □

It follows that $q \in H^2(\mathrm{Jac}^0(C), \mathbb{Z}) \cap H^{1,1}(\mathrm{Jac}^0(C))$.

Theorem 15.8 (Lefschetz theorem for (1,1)-forms)**.** *If X is a Kähler manifold,*

$$\mathrm{Pic}(X) = H^1(\mathcal{O}_X^*) \to H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$$

is surjective.

Proof. Next time □

Corollary 15.9. *$q = c_1(\theta)$ for some line bundle θ on $\mathrm{Jac}^0(C)$.*

Remark 15.10. If you are an algebraic geometer, you immediately wonder what line bundle θ is. To specify a line bundle on a variety, you can give a divisor. Any point $p \in C$ gives an isomorphism $\mathrm{Jac}^0(C) \to \mathrm{Jac}^{g-1}(C)$ given by $L \mapsto L((g-1)p)$.

By Riemann-Roch, $\chi(L) = \deg(L) + 1 - g$, so if $L \in \mathrm{Pic}^{g-1}(C)$, then $h^0(L) - h^1(L) = 0$.

In general, $L \in \mathrm{Pic}^{g-1}(C)$ has zero sections; therefore, θ corresponds to the set of effective line bundles, regarded as a subscheme of $\mathrm{Jac}^{g-1}(C)$.