

# Parking a Car and Lie Brackets

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The *controllability problem* deals with a system

$$\frac{dx}{dt} = f(x, u)$$

of ODEs where the vector field on right hand side depends on a parameter  $u$ . Here the *state*  $x$  takes values in a subset of Euclidean space  $\mathbb{R}^n$  and the *control*  $u$  takes values in another space. The problem is to choose, for given endpoints  $x_0$  and  $x_1$  in the state space, a curve  $u = u(t)$  in the control space so that the solution to the ODE satisfies  $x(0) = x_0$  and  $x(1) = x_1$ .

A good example of this is the problem of parking a car in a tight parking space. (This comes from [Ed Nelson's book \[3\]](#) pages 33-36.) The main control is the angle  $\theta$  between the direction of the car and the direction of the front wheels. The driver varies the control by turning the steering wheel. A second control is whether the car is in forward gear or reverse gear, i.e. whether the car is going forward or backward. The second control reverses the sign of the vector field. Call the forward motion of the car while the steering wheel is stationary the **drive motion** and the process of turning the steering wheel to the left while the car is at rest the **steer motion**. In the course of solving the control problem we will prove that by repeating the following **wriggle motion**

*turn the wheel a little bit to the right (reverse steer), drive backward a short distance (reverse drive), turn the wheel little bit to the left (steer), drive forward a short distance (drive)*

it is possible to rotate the car in a small confined area.

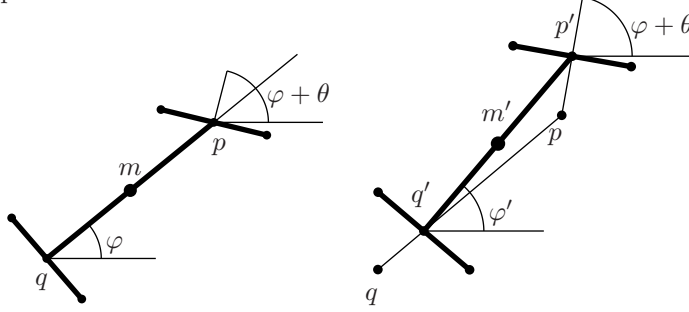
Let  $p$  denote the midpoint of the front axle,  $q$  denote the midpoint of the rear axle,  $\varphi$  denote the angle between the horizontal (i.e. the curb) and the line  $qp$ , and  $\theta$  denote the angle between the front wheels and the car, i.e.  $\theta$  is the angle between the line  $qp$  and the perpendicular to the front axle through  $p$ . The unit vectors perpendicular to the front and rear axles are

$$\begin{aligned} \text{normal}_{\text{front}} &= (\cos(\varphi + \theta), \sin(\varphi + \theta)), \\ \text{normal}_{\text{rear}} &:= (\cos(\varphi), \sin(\varphi)). \end{aligned} \tag{1}$$

If the angle  $\theta$  is held constant (i.e. the steering wheel is not turned) and the car moves forward, the points  $p$  and  $q$  move to points  $p'$  and  $q'$  related by

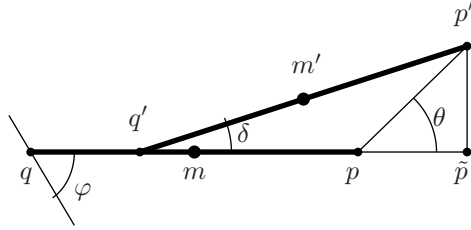
$$p' = p + h \text{normal}_{\text{front}} + o(h), \quad q' = q + k \text{normal}_{\text{rear}} + o(h) \tag{2}$$

where  $h$  is the distance that  $p$  moves and  $k$  is the distance that  $q$  moves. Equation (2) says that (to first order) the vector  $q' - q$  is perpendicular to the rear axle and the vector  $p' - p$  is perpendicular to the front axle. Ignoring the  $o(h)$  terms the picture is<sup>1</sup>



Here  $m$  denotes the center of the car, i.e. the midpoint of the line segment  $qp$ .

To write the equations (2) for the motion in coordinates, we must express  $k$  as a function of  $\varphi, \theta$  and  $h$ . Choose rectangular coordinates in the plane so that the  $x$ -axis is the horizontal. Fix an angle  $\theta$  for the steering wheel and let the car move from state  $(q, p)$  to state  $(q', p')$  as described above.



In the diagram the picture has been rotated through an angle of  $-\varphi$ ,  $\varphi$  and  $\varphi'$  denote the angles the lines  $qp$  and  $q'p'$  make with the horizontal, the point  $\tilde{p}$  on the line  $qp$  satisfies  $\|q' - q\| = \|\tilde{p} - p\|$ , and  $\delta := \angle pq'p'$ . Then  $\delta = \varphi' - \varphi$ ,  $\theta = \angle \tilde{p}pp'$ ,  $h = \|p' - p\|$ ,  $k = \|q - q'\| = \|\tilde{p} - p\|$ , and the length of the car (or more precisely the distance between the midpoints of the axles) is

$$\ell := \|p - q\| = \|p' - q'\| = \|q' - \tilde{p}\|.$$

Since  $\|p' - \tilde{p}\| = h \sin \theta = \ell \sin \delta$ , Equations (1) and (2) become

$$\frac{x(p') - x(p)}{h} = \cos(\varphi + \theta), \quad \frac{y(p') - y(p)}{h} = \sin(\varphi + \theta), \quad \frac{\sin(\varphi' - \varphi)}{h} = \frac{\sin \theta}{\ell},$$

<sup>1</sup> This picture is essentially the same as Figure 2 on page 34 in Nelson's book [3].

where we have discarded the terms of order  $h$ . Thus the motion is governed by the ODE

$$\dot{x}(p) = \cos(\varphi + \theta), \quad \dot{y}(p) = \sin(\varphi + \theta), \quad \dot{\varphi} = \frac{\sin \theta}{\ell}. \quad (3)$$

(The dot denotes differentiation with respect to the arclength parameter  $s$ .) The solution to (3) is  $\theta = \theta_0$ ,  $\varphi = \varphi_0 + cs$ , where  $c = \sin(\theta_0)/\ell$ , and

$$\begin{aligned} x(p) &= x(p_0) + \ell(\sin(\varphi + \theta) - \sin(\varphi_0 + \theta_0)) \\ y(p) &= y(p_0) - \ell(\cos(\varphi + \theta) - \cos(\varphi_0 + \theta_0)) \end{aligned}$$

Since  $\varphi$  is the angle that the vector  $p - q$  makes with the horizontal we have

$$x(q) = x(p) - \ell \cos \varphi, \quad y(q) = y(p) - \ell \sin \varphi.$$

Hence

$$\begin{aligned} \dot{x}(q) &= \dot{x}(p) + \sin \theta \sin \varphi = \cos(\varphi + \theta) + \sin \theta \sin \varphi \\ \dot{y}(q) &= \dot{y}(p) - \sin \theta \cos \varphi = \sin(\varphi + \theta) - \sin \theta \cos \varphi. \end{aligned} \quad (4)$$

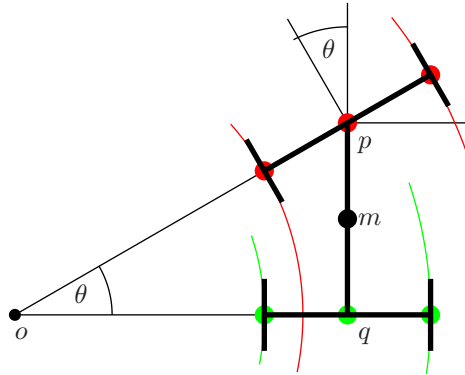
The center of the car has coordinates

$$x(m) = \frac{1}{2}(x(p) + x(q)), \quad y(m) = \frac{1}{2}(y(p) + y(q))$$

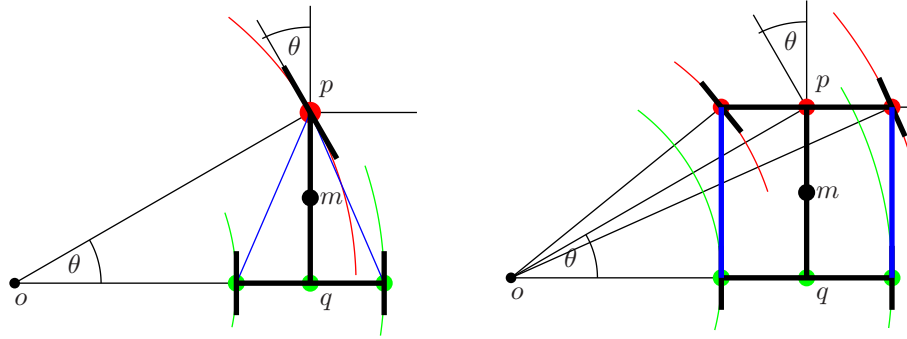
so

$$\begin{aligned} \dot{x}(m) &= \frac{1}{2}(\dot{x}(p) + \dot{x}(q)) = \cos(\varphi + \theta) + \frac{1}{2} \sin \theta \sin \varphi, \\ \dot{y}(m) &= \frac{1}{2}(\dot{y}(p) + \dot{y}(q)) = \sin(\varphi + \theta) - \frac{1}{2} \sin \theta \cos \varphi. \end{aligned} \quad (5)$$

The solution to the ODE (3) shows that the point  $p$  moves around a circle at constant angular velocity as in the following picture:



The center  $o$  of the circle is the intersection of the lines through the axles. The point  $q$  moves along another circle with center  $o$  and the length  $\ell$  of the



A Tricycle and a Car

chord  $qp$  is constant. The wheels move around four other circles with this same center. All seven points move at the same angular velocity about  $o$  so the points that are farther from the center of the circle turn faster. For an explanation of how this works do a google search on *automobile differential*.<sup>2</sup>

**Remark.** On a real car the front axle does not rotate but remains parallel to the rear axle; only the front wheels rotate. However Equation (4) models both a tricycle and also a car where the perpendiculars to the wheels are concurrent.<sup>3</sup>

### Lie brackets.

The Lie algebra of vector fields on a manifold  $M$  is denoted by  $\mathcal{X}(M)$  and the ring of functions on  $M$  is denoted  $\mathcal{F}(M)$ . Each vector field  $\xi \in \mathcal{X}(M)$  determines a flow. This is the family  $(\varphi^t)_{t \in \mathbb{R}}$  of diffeomorphisms defined by the ODE

$$\frac{d}{dt} \varphi^t(x) = \xi(\varphi^t(x)), \quad \varphi^0(x) = x.$$

The group of diffeomorphisms acts on the ring  $\mathcal{F}(M)$  by composition on the right:

$$\varphi^*(f) = f \circ \varphi.$$

If we differentiate the function  $f$  in the direction  $\xi$  we get the derivation

$$(D_\xi f)(x) = df(x)\xi(x).$$

It is not hard to see that this equation defines a bijection between the space  $\mathcal{X}(M)$  of vector fields on  $M$  and the space of derivations of the ring  $\mathcal{F}(M)$ . Because of this, most geometers define a vector field to be a derivation of  $\mathcal{F}(M)$  but this can lead to confusion about signs as follows.

<sup>2</sup> See e.g. [http://en.wikipedia.org/wiki/Differential\\_\(mechanical\\_device\)](http://en.wikipedia.org/wiki/Differential_(mechanical_device)) and the links in the references at the end of that article.

<sup>3</sup>I don't know if any real car is designed this way.

In any group the commutator of two elements  $\varphi$  and  $\psi$  is defined to be the element  $\varphi\psi\varphi^{-1}\psi^{-1}$  and in a (noncommutative) algebra the commutator of two elements is defined by

$$[A, B] := AB - BA.$$

The Lie brackets  $[\xi, \eta]$  of two vector fields  $\xi$  and  $\eta$  is defined as the infinitesimal commutator

$$[\xi, \eta](p) = \left. \frac{d}{dt} \varphi^{\sqrt{t}} \circ \psi^{\sqrt{t}} \circ \varphi^{-\sqrt{t}} \circ \psi^{-\sqrt{t}}(p) \right|_{t=0}$$

where  $(\varphi^t)_t$  is the flow of  $\xi$  and  $(\psi^t)_t$  is the flow of  $\eta$ . (See [1] page 18.) For example, a square matrix  $A \in \mathbb{R}^{n \times n}$  determines a flow

$$\varphi^t(x) = e^{tA}x, \quad e^{tA} := \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$$

and for square matrices  $A$  and  $B$  we have

$$[A, B] = \left. \frac{d}{dt} e^{\sqrt{t}A} e^{\sqrt{t}B} e^{-\sqrt{t}A} e^{-\sqrt{t}B} \right|_{t=0} = AB - BA$$

as can easily be seen from the formula

$$\begin{aligned} & \left( I + tA + \frac{t^2 A^2}{2} \right) \left( I + tB + \frac{t^2 B^2}{2} \right) \left( I - tA + \frac{t^2 A^2}{2} \right) \left( I - tB + \frac{t^2 B^2}{2} \right) \\ &= I + 2t^2[A, B] + o(t^3). \end{aligned}$$

It is easy to see that the vector space of derivations of the ring  $\mathcal{F}(M)$  (in fact, of *any* ring) is closed under commutators. However the map  $\xi \mapsto D_\xi$  is an anti-homomorphism:

$$D_{[\xi, \eta]} = -[D_\xi, D_\eta].$$

This is because the action  $\varphi \mapsto \varphi^*$  is an anti-homomorphism of groups:

$$(\varphi \circ \psi)^* f = f \circ \varphi \circ \psi = \psi^*(f \circ \varphi) = \varphi^* \psi^* f.$$

### Back to the car parking problem

The state of the car (that is the positions of the points  $p$  and  $q$ ) is constrained by the equation

$$\|q - p\| = \ell.$$

With respect to rectangular coordinates  $(x, y)$  in the plane  $\mathbb{R}^2$  the constraint can be written as the three dimensional submanifold

$$M \subset \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$$

cut out by the quadratic equation

$$(x(p) - x(q))^2 + (y(p) - y(q))^2 = \ell^2.$$

By the above we can parameterize the constraint manifold via the equations

$$x(p) = x, \quad y(p) = y, \quad x(q) = x - \ell \cos \varphi, \quad y(q) = y - \ell \sin \varphi.$$

These equations show that the constraint manifold is diffeomorphic to  $\mathbb{R}^2 \times S^1$ .

Choose units so that  $\ell = 1$ . Introduce the abbreviation

$$\sigma(x, y, \varphi, \theta) = (x, y, \varphi).$$

The ODE (3) represents the motion of the car with the steering wheel held in a fixed position while the ODE

$$\dot{x} = 0, \quad \dot{y} = 0, \quad \dot{\varphi} = \cos \theta \tag{6}$$

represents the change in in the state when the car is at rest and the steering wheel is turned. (It is convenient for the calculations to take  $\dot{\varphi} = \cos \theta$  here because  $\cos \theta$  is the derivative of  $\sin \theta$  with respect to  $\theta$ . However the control  $\theta$  is restricted to an interval  $-\theta_{\max} < \theta < \theta_{\max}$  so using  $\sin \theta$  rather than  $\theta$  is nothing more than a change of variable.) Rewrite the ODEs (3) and (6) in vector field notation

$$\dot{\sigma} = \text{drive}(\sigma), \quad \dot{\sigma} = \text{steer}(\sigma).$$

These represent the “drive” and “steer” motions described above and the vector field  $\text{wriggle} := [\text{drive}, \text{steer}]$  represents the aforementioned “wriggle” motion. Let  $\text{slide} := [\text{wriggle}, \text{drive}]$ . By definition the three vector fields

$$\text{drive}, \quad \text{wriggle} = -\text{Ad}(\text{drive})\text{steer}, \quad \text{slide} = \text{Ad}(\text{drive})^2\text{steer}$$

all arise from  $\text{steer}$  by applying powers of  $\text{Ad}(\text{drive})$ . These vector fields all depend on  $\theta$ .

**1. Theorem.** *For sufficiently small  $\theta$ , the three vectors  $\text{drive}$ ,  $\text{wriggle}$ ,  $\text{slide}$  form a basis for the tangent space to  $M$  at each point.*

*Proof.* Let

$$X := \cos(\varphi + \theta)\partial_x + \sin(\varphi + \theta)\partial_y,$$

$$Y := -\sin(\varphi + \theta)\partial_x + \cos(\varphi + \theta)\partial_y.$$

Then by (6) and (3) we have

$$\begin{aligned} D_{\text{steer}} &= \partial_\theta, & \text{and} \\ D_{\text{drive}} &= X + \sin \theta \partial_\varphi, & \text{so} \\ D_{\text{wriggle}} &= Y + \cos \theta \partial_\varphi, & \text{and hence} \\ D_{\text{slide}} &= -\sin \theta X - \cos \theta Y. \end{aligned}$$

The assertions “so” and “hence” above are special cases of the following

**Lemma.** *The operators  $X, Y, \partial_\varphi, \partial_\theta$  satisfy*

- (i)  $[\partial_\theta, X] = [\partial_\varphi, X] = Y, \quad [\partial_\theta, Y] = [\partial_\varphi, Y] = -X, \quad [X, Y] = 0,$
- (ii)  $Xf = Yf = \partial_\varphi f = 0$  for any trigonometric polynomials  $f = f(\theta)$ .
- (iii) For any functions  $a = a(\theta), b = b(\theta),$  and  $c = c(\theta)$  (i.e. any functions which do not depend on  $x, y,$  and  $\varphi$ ) we have

$$\begin{aligned} [c\partial_\varphi, aX + bY] &= -cbX + caY, \\ [\partial_\theta, aX + bY] &= (a' - b)X + (b' + a)Y, \\ [\partial_\theta, c\partial_\varphi] &= c'\partial_\varphi. \end{aligned}$$

*Proof of Lemma.* Part (ii) holds because  $X$  and  $Y$  are linear combinations of  $\partial_x$  and  $\partial_y$  and  $f$  is assumed to be independent of  $x, y,$  and  $\varphi$ . Part (i) holds because  $X$  and  $Y$  are functions of  $\varphi + \theta$ . Part (iii) follows from parts (i) and (ii) and the product rule for differentiation.  $\square$

Theorem 1 now follows as

$$D_{\text{drive}} = X_0 \quad D_{\text{wriggle}} = Y_0 + \partial_\varphi, \quad D_{\text{slide}} = Y_0$$

at  $\theta = 0$  and  $X_0, Y_0$  are orthonormal.  $\square$

To park the car we must show that for every path  $\sigma(t)$  in  $M$  there is a function  $\theta = \theta(t)$  with  $-\theta_{\max} < \theta(t) < \theta_{\max}$  (the range of the steering wheel is limited) such that  $\sigma(t)$  solves the equation

$$\dot{\sigma} = \text{drive}(\sigma_{\theta(t)}).$$

Here’s the appropriate lingo from control theory.

**Definition.** Let  $M$  be a manifold,  $E$  be a topological space and  $\sigma : E \rightarrow M$  be a map. A **control problem** over  $\sigma$  is a **vector field along**  $\sigma$ , i.e. a map  $f : E \rightarrow TM$  such that  $f(w) \in T_{\sigma(w)}M$  for every  $w \in E$ . The problem is said to be **controllable** iff for every curve  $x = x(t) \in M$  there is a curve  $w = w(t) \in E$  such that

$$\sigma(w(t)) = x(t) \quad \text{and} \quad \dot{x}(t) = f(w(t)).$$

**Example.** Take  $E = TM, \sigma : E = TM \rightarrow M$  the projection, and  $f$  the identity map. This problem is controllable: given  $x = x(t)$  take  $w(t) = (x(t), \dot{x}(t))$ .

Usually we assume that  $E = M \times \mathcal{U}$  where  $\mathcal{U} \subset \mathbb{R}^n$  is an open set, that  $\sigma = \sigma(x, u) = x$  is projection on the first factor, and that  $f$  is a smooth function of  $(x, u)$ . The condition that  $f$  be a vector field along  $\sigma$  is  $f(x, u) \in T_x M$ .

## References

- [1] Richard L. Bishop & Richard L. Crittenden: *Geometry of Manifolds*, Academic Press Series in Pure and Applied Mathematics **XV**, 1964.
- [2] Ernest Bruce Lee & Larry Markus: *Foundations of Optimal Control Theory*, Siam Series in Applied Mathematics, Wiley, 1967.
- [3] Edward Nelson: *Tensor Analysis* (Mathematical Notes, Princeton University Press, 1967). <https://web.math.princeton.edu/~nelson/books/ta.pdf>.